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GROUPS IN TOPOLOGY

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Abstract:

The word "Mathematics" comes from Greek word "Mathema" which means science, knowledge or learning; mathematiko's means fond of learning. Today the term refers to specific body of knowledge - the deductive study of quantity structure, space and change. The basic problems of topology are to determine whether two given topological spaces are homomorphism or not. Two spaces are not homomorphism is a different matter. For that, one must show that a continuous function with continuous inverse does not exist.

Key Words: Topology, Homomorphism, etc.,

Introduction:

The exquisite world of algebraic topology came into existence out of our attempts to solve topological problems by the use of algebraic tools and this revealed to us the nice inter play between algebra and topology which causes each tore in force interpretations of the other there by braking down the often artificial sub division of mathematics into different branches and emphasizing the essential unity of all mathematics. The homology theory is the main branch of algebraic topology and plays the main role in the classification problems of topological spaces. The basic problems of topology are to determine whether two given topological spaces are homomorphism or not. Two spaces are not homomorphism is a different matter. For that, one must show that a continuous function with continuous inverse does not exist.

Basic Definitions and Examples:

Definition 1.1:

Anon-empty set G together with a binary operation$*: G \times G \to G$ is called a group if the following conditions are satisfied.

- $*$ is associative $ie)\ a * (b * c) = (a * b) * c\ \ for\ all\ a, b, c \in G$.
- There exists an element $e \in G$ such that $a * e = e * a\ \ for\ all\ a \in G$. e is called the identity element of G.
- For any element a in G there exists an element $a' \in G$ such that $a *\ a' = a' * a = e$. a' is called the inverse of a.

Example:

Consider the ordered pair $(Z, +)$, where Z is the set of all integers and + is the usual addition. We know that, $+$is associative. Now

$$0 \in Z\ and\ for\ all\ \ a\ \in Z$$
$$a + 0 = a = 0 + a$$

And so,

0 is the identity

Also for all $a\ \in Z, -a \in Z$ and

$$a + (-a) = 0 = (-a) + a$$

That is $-a$is the inverse of a
Hence $(Z, +)$ is a group

Definition 1.2:

A subset H of a group G is called a sub group of G if H forms a group with respect to the binary operation in G.

Example:

In the symmetric group S_3, $H_1 = \{e, P_1, P_2\};\ \ H_2 = \{e, P_3\};\ H_3 = \{e, P_4\};\ \ and\ H_4 = \{e, P_5\}$

Definition 1.3:

A group G is said to be abelian if ab = ba for all $a, b\ \in G$ a group which is not abelian is called a non-abelian group.

Definition 1.4:

The boundary operator ∂ on the free Abelian group $S_n(x)$ generated by all singular $n -$ chains as follows

$$\partial_n(\alpha) = \partial_n \left(\sum g_i \sigma_i\right)\ \ = \ g_i \sum \partial_n(\sigma_i)$$

It is easy to prove that $\partial_{n-1}\partial_n\ = 0$

$$\partial^2 = 0$$

Nothing that $S_n(x)$ is analogous to simplical complex $C_n(K)$ the graded complex

$$S_n(x) = \{S_n(x)\}$$

In a sequence $\ldots\ldots \xrightarrow{\partial_{n+2}} S_{n+1}(x) \xrightarrow{\partial_{n+1}} S_n(x) \xrightarrow{\partial_n} S_{n-1}(x) \xrightarrow{\partial_{n-1}}$

Singular Homology:

Theorem 2.1:

Prove that the set of all singular n – chains $S_n(x)$ is an abelian group.

Proof:

To prove $S_n(x)$ is an abelian group. Let $S_n(x) = \sum_{j\in J} n_j \phi_j$

Define the addition of two elements of $S_n(x)$ by

$$\sum n_j \phi_j + \sum m_j \phi_j = \sum (n_j + m_j)\,\phi_j$$

The zero element of $S_n(x)$ is $\sum \phi_j$ and the inverse element of

$$\sum n_j \phi_j \quad as \; \sum (-n_j)\phi_j$$

The group axioms are satisfied the clearly the group is an abelian

Hence the proof

Theorem 2.2:

If ∂ is the boundary operator then the product of two boundary operators is zero.

i.e.) prove that $\partial\partial = 0$

Proof:

To prove $\partial\partial = 0$

Let ϕ be a singular n – simplex. It is enough to prove that $\partial\partial\phi = 0$

Where 0 stands for the identity "The boundary operator

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i$$

Using this condition

$$\partial\partial\phi = \partial \sum_{i=0}^{n} (-1)^i \partial_i \phi$$
$$= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} \partial_j \partial_i \phi$$

If $i \leq j$ then to prove

$$\partial_j \partial_i = \partial_i \partial_{j+1}$$

For this,

$$(\partial_j \partial_i \phi)(x_0 \ldots\ldots x_{n-2}) = \left(\partial_j (\partial_i \phi)\right)(x_0 \ldots\ldots x_{n-2})$$
$$= (\partial_i \phi)(x_0 \ldots\ldots x_{j-1}, 0, x_j \ldots\ldots x_{n-2})$$
$$= \phi(x_0 \ldots\ldots x_{i-1}, 0, x_i \ldots\ldots x_{j-1}, 0, x_j \ldots\ldots x_{n-2})$$
$$= (\partial_i \partial_{j+1} \phi)(x_0 \ldots\ldots x_{n-2})$$

And so $\partial_j \partial_i = \partial_i \partial_{j+1}$

Now

$$\partial_j \partial_i \phi = \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} \partial_i \partial_j \phi += \sum_{j=0}^{n+1} \sum_{i=j+1}^{j} (-1)^{i+j} \partial_i \partial_j \phi$$
$$= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} \partial_i \partial_{j+1} \phi += \sum_{j=0}^{n-1} \sum_{i=j+1}^{j} (-1)^{i+j} \partial_i \partial_j \phi$$
$$= \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j} \partial_i \partial_{j+1} \phi += \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} \partial_i \partial_j \phi$$
$$= \sum_{j=0}^{n-1} \sum_{i=j+1}^{n-1} (-1)^{i+j-1} \partial_i \partial_j \phi += \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} \partial_i \partial_j \phi$$
$$= 0$$

Hence, The product of two boundaries are zero.

Simplicial Homology Theory:

Theorem 3.1:

If I_m, I_{m-1} are the incidence number of m and $m-1$ simplexes respectively. Then the products are zero.

$$i.e)\; I_m I_{m-1} = 0$$

Proof:

Let I_m, I_{m-1} are incidence number. To prove: $I_m I_{m-1} = 0$

Here the products I_m, I_{m-1} exists. Because the number of columns of I_m is the number of m – simplexes which is equal to the number of rows of I_{m-1}

Let q be the number of m – simplexes then the elements in the i^{th} row and j^{th} column of I_m, I_{m-1} is defined by

$$\sum_{k=1}^{q} \eta_{ik}^{m} \eta_{jk}^{m-1} \ldots\ldots\ldots\ldots (1)$$

It is enough to prove that

$$\sum_{k=1}^{q} \eta_{ik}^{m} \eta_{jk}^{m-1} = 0$$

If t_{m+1}^{i} is not incident with S_m^k, η_{ik}^{m} is zero

If S_m^k is not incident with the $(m-1)$ simplex U_{m-1}^{j} then η_{jk}^{m-1} is zero

Since the term,

$$\sum_{k=1}^{q} \eta_{ik}^{m} \eta_{jk}^{m-1} \text{ is zero}$$

Unless S_m^k is incident with both t_{m+1}^{i} and U_{m-1}^{j}

If $U_{m-1}^{j} = (q_0, q_1 \ldots\ldots\ldots\ldots q_{m-1})$

Then $S_m^k = \eta_{jk}^{m-1}(PU_{m-1}^{j})$ and $t_{m+1}^{i} = \eta_{ik}^{m} \eta_{jk}^{m-1}(rPU_{m-1}^{j})$

Where P is the additional vertex of S_m^k and r is the

Additional vertex of t_{m+1}^{i} this gives

$$\begin{aligned} t_{m+1}^{i} &= -\eta_{ik}^{m} \eta_{jk}^{m-1}(PrU_{m-1}^{j}) \\ &= -\eta_{jk}^{m-1}(PU_m^h) \end{aligned}$$

Where U_m^h is taken to be the m – simplex $\eta_{ik}^{m}\ (r\,u_{m-1}^{j})$

This shows that there exists an m – simplex U_m^h incident with both t_{m+1}^{i} and u_{m-1}^{j} having incidence numbers $-\eta_{kj}^{m-1}$ and η_{ik}^{m} respectively. Clearly, $\pm U_m^h$ is a face of the simplex t_{m+1}^{i} of the complex K and so $\pm U_m^h$ is a simplex of K. Moreover, S_m^k and $\pm U_m^h$ are the only two simplexes of K which are incident with both t_{m+1}^{i} and u_{m-1}^{j}

Hence to every non – zero term in (1) there corresponds another non – zero term equal in magnitude but opposite in sign.

Thus,

$$\sum_{k=1}^{q} \eta_{ik}^{m} \eta_{jk}^{m-1} = 0$$

Which implies that $I_m I_{m-1} = 0$.

Fundamental Group:

Theorem 4.1.1:

A connected, locally path – connected space X is path connected

Proof:

Let $a \in X$ and

Let H be the set of all points of X which can be joined by a path to a. As $a \in H$, H is non – empty

If H is closed and open, $H = X$. But H is open. For b$\in H$.

Let U be a path – connected neighborhood of b. Then any point $z \in U$ can be joined by a path to b and hence can be joined to a by adding the path from b to a.

Also H is closed. For if $b \in H$, let U be any path connected neighborhood of b.

Then $U \cap H \neq \phi$; say $Z \in U \cap H$.

Now b can be joined to z by a path and z can be joined to a by a path, so by addition of paths again $b \in H$.

Corollary,

An open connected subset of R is path – connected. we are now in a position to use the ‘addition’ of paths to associate with any topology space a group [actually, several groups] .

Theorem 4.1.2:

Every continuous mapping $f : (x, x_0) \to (y, y_0)$ induces a homomorphism $f^* : \pi, (x, x_0) \to (y, y_0)$.

Proof:

For each loop g at x_0 in X, let $f'(g)$ be the loop at y_0 in y, defied by $f'(g)(t) = f[g(t)]$

This defines a mapping $f^{'}$ from $L\ [X, X_0]\ to\ \ L\ [Y, Y_0]$ which in turn induces a mapping $f^{*}:\ \pi,(x, x_0) \to \pi,(y, y_0)$ as follows

$$f^{*}([g]) = [f^{'}(g)]$$

To see that f is well – defined, not that if H is homotopy between $g_1\ and\ \ g_2\ in\ \ L\ [X, X_0],\ then\ \ foH$ is a homotopy between $[f^{'}(g_1)]$ and $[f^{'}(g_2)]\ in\ \ L\ [Y, Y_0]$

It remains to shows that f is a homomorphism, for which it suffices to establish. The necessary algebraic property for f, but

$$f^{'}(g * h) = \begin{cases} f\ [g(2x)] = f^{'}(g)(2x) & when\ \ 0 \le x \le 1/2 \\ f[h(2x-1)] = f^{'}(h)(2x-1) & when\ \ 1/2 \le x \le 1 \end{cases}$$
$$= f^{'}(g) * f^{'}(h)\ ,$$

Hence the proof.

Conclusion:

In this paper, “groups in topology” briefly discussed about the homology groups and its computations. Also it discusses about axioms for the homology groups and some theorem related with homology groups. Also this paper provides some concept of simpilical homology theory and singular homology. And then it gives some concept of exact sequence pair and relative homology theory. This paper concluded that the briefly explained about homology groups and singular homology and also its used in the fields are particularly mathematics, biology, science and engineering and etc…

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