

Bound State Entropy as Mutual Information?

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In a previous note (1) we suggested using $P(x)P(p/x)$ where $P(x)=W(x)W(x)$ and $P(p/x) = a(p)\exp(ipx)/W(x)$ for a bound state (where $W(x)$ is the wavefunction) in Shannon's entropy expression - $\sum_i P(i) \ln(P(i))$. This yields a bound state entropy of $S = .5S_p + .5S_x$. Historically (2) $S_p + S_x$, called entropic uncertainty, has been of interest as an alternative to the Heisenberg uncertainty relationships for p and x .

In this note we argue that this sum of momentum and spatial entropies is directly linked to the idea of mutual information from information theory. In (3) it is noted that mutual information is "the amount of information that can be obtained about one random variable by observing another". Thus this seems to be directly related to uncertainty in two p and x .

Bound State Entropy

Bound state entropy is usually either given in terms of either a momentum Shannon's entropy - $\int dp a(p)a(p) \ln[a(p)a(p)]$ or a spatial entropy - $\int dx W(x)W(x) \ln[WW]$, but not the sum of both. The two are summed only in order to compare their values to a generalized quantum uncertainty relationship (see (2)).

In (1) we suggested that $P(x)P(p/x)$ may serve as the overall probability in Shannon's entropy - $\sum_i P(i) \ln(P(i))$. This leads to an entropy density of:

$$S \text{ density} = W(x)W(x) \ln(W(x)) + a(p)a(p) \ln(a(p)) + ipx W(x)a(p)\exp(ipx) \quad ((1))$$

$$\text{Integrating over } p \text{ and } x \text{ yields: } S = .5S_x + .5S_p \quad ((2))$$

In another note we saw that for a particle in a box with infinite potential walls, the length of the box cancels if one uses a sum of S_x and S_p . Thus if the box is extended in length quantum adiabatically (i.e. the particle remains in the same eigenstate) then the process is isentropic.

Entropic Uncertainty

The Heisenberg uncertainty relationship for p and x is well known i.e.

$$\sqrt{\text{Var}(x)} \sqrt{\text{Var}(p)} \geq .5 \hbar \quad ((2))$$

In 1957 it was suggested there might be a more general quantum uncertainty principle for p and x and this was fully confirmed in 1975 (2). In particular a function $f(x)$ and its Fourier transform $g(p)$ were considered. This is linked to quantum mechanics as $W(x)=\text{wavefunction} = \sum_p a(p)\exp(ipx)$. The generalized uncertainty principle (2) is given in terms of Shannon's entropy $H(p)$ and $H(x)$ i.e.

$$H(W(x)W(x)) + H(a(p)a(p)) \geq \log(e/2) \quad ((3))$$

As a result, one may find calculations in the literature of Shannon's p and x entropy for various quantum systems such as a particle in a box or an oscillator (5). These are then compared with the limit ((3)).

Mutual Information

In this note, we wish to compare $.5S_p + .5S_x = S$ to the idea of mutual information from information theory. This quantity is said (3) to represent the information about one random variable which may be obtained by observing another. This seems to be linked to the idea of uncertainty in p and x.

Mutual information is defined (3) as:

$I(x,p) = H(p) + H(x) - H(x,p)$ ((4)) where H is Shannon's entropy and:

$H(x,p) = - \sum_{x,p} P(x,p) \ln(P(x,p))$ where $P(x,p) = P(x)P(p/x)$ ((5))

In (1) we used $P(x,p) = P(x)P(p/x)$ where $P(x) = W(x)/W(x)$ and $P(p/x) = a(p)\exp(ipx)/W(x)$ to find that $H(x,p) = \text{entropy of a bound state} = .5 H(p) + .5 H(x)$.

As a result, our definition of entropy of a bound state is equivalent to $H(x,p)$, but is also linked to mutual information through:

$I(x,p) = H(p) + H(x) - .5H(p) - .5H(x) = H(x,p)$ ((6))

In other words $I(x,p)$ is our bound state entropy. Mutual information is used as a measure of uncertainty of two variables in a system and so there is also a link to entropic uncertainty.

Conclusion

In conclusion, we argue that using $P(x)P(p/x)$ as probability in Shannon's entropy - $\sum_i P(i) \ln(P(i))$ leads to a bound state entropy $S = .5S_x + .5S_p = H(x,p)$ where H is Shannon's entropy which uses $P(x,p) = P(x)P(p/x)$. More interestingly, mutual information $= H(x) + H(p) - H(p,x)$ is also equal to $H(p,x)$ so $S = .5S_p + .5S_x$ is both mutual information $I(p,x)$ and $H(p,x)$. $I(p,x)$ is said to yield information about one random variable based on an observation of another. This is of the sense of the uncertainty principle and thus mutual information which we argue is bound state entropy is also linked to entropic uncertainty.

References

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