# Odd perfect numbers do not exist

**Tony Kuria Kimani**

**tkimani19@gmail.com**

**Student, Msc. Research Methodology**

**Jomo Kenyatta University of Agriculture and Technology**

**Nairobi, Kenya**

**Abstract**

An odd perfect number N is a number whose sum of divisors is equal to 2N. Euler proved that if an odd perfect number exists then it must have the form $p^x q^2$. Where p is a prime number and p and x are of the form 1 mod 4. This paper proves than an odd perfect number does not exist.

**Introduction**

Nielsen(2003) improved the upper bound of an odd perfect number to

$$N < 2^{4^k}$$

Where N is an odd perfect number and k is the number of unique prime divisors of an odd perfect number.

Nielsen[2007] further proved that an odd perfect number has at least 9 unique prime divisors.

Goto and Ohno[2008] proved that an odd perfect number has a prime divisor exceeding $10^8$

I will write these three results as lemmas 1,2 and 3 respectively as follows:

**Lemma 1**

$$N < 2^{4^k} \tag{1}$$

**Lemma 2**

*An odd perfect number has at least 9 unique prime divisors*

**Lemma 3**

*An odd perfect number has a prime divisor exceeding* $10^8$

**Theorem 1**

*Odd perfect numbers do not exist*

**Proof**

Using lemma 1, we can obtain the following:

$$N < 2^{4^k} = 16^k \tag{2}$$

$$N < 16^k \tag{3}$$

Suppose that N has exactly 9 unique prime divisors, then that means that:

$$N < 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \tag{4}$$

However, since we do not know the actual number of unique prime divisors of an odd perfect number, we can rewrite equation 4 as follows:

$$N < 16 \times 16 \times 16 \times 16 \times 16 \times \cdots \times 16$$

Where the ellipsis represents $16^t$ where t is the unknown number of unique prime divisors of an odd perfect number in the inequality above. For example if an odd perfect number has exactly 50 unique prime divisors then the ellipsis would represent $16^{44}$ because $16^{44}$ is the unknown number of unique prime divisors of an odd perfect number in the inequality above.

Therefore we get $16^6 \times 16^{44} = 16^{50}$.

Likewise if an odd perfect number has exactly 90 unique prime divisors then the ellipsis would represent $16^{84}$ so that we get $16^6 \times 16^{84} = 16^{90}$. It is very important that the reader understands the definition of the ellipsis in this paper because it will be used multiple times later in the proof.

Now let us look at the following comparison:

$$16 \times 16 \times 16 \times 16 \times 16 \times \cdots \times 16 \tag{5}$$

$$3 \times 5 \times 7 \times 11 \times 13 \times \cdots \times 10^8 \tag{6}$$

While the ellipsis for equation 5 has already been defined above, the ellipsis for equation 6 has not yet been defined and I will hereby proceed to define it. For equation 6 the ellipsis represents:

$$\prod_{i=1}^{t} p_i$$

Where p is a prime number and $p_1 = 17, p_2 = 19, p_3 = 23$ and so on. Where t is still the unknown number of unique prime divisors of an odd perfect number in the expression in equation 6. For example if an odd perfect number has exactly 9 unique prime divisors then $t = 3$ and the ellipsis in equation 6 will represent $17 \times 19 \times 23$ and equation 6 will becomes:

$$3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 10^8$$

Now let us look at equation 6. We can see that $3 \times 5 \times 7 \times 11 \times 13$ is the product of the 5 smallest possible odd prime numbers. Notice that the next prime number which is 17 is greater than 16. This fact will be useful in our analysis. Courtesy of Goto and Ohno, we know that there is a unique prime divisor of N greater than $10^8$. So $10^8$ can act as a lower bound of this unique prime divisor. In equation 6 we are assuming that $10^8$ is the largest unique prime divisor of N. However, if N has other unique prime divisors that are greater than $10^8$, that does not affect the logic of the argument. So it doesn't matter whether the largest unique prime divisor of N is of the size $10^8$ or even greater than that because the argument remains the same.

Notice that equation 5 and 6 must have t of the same value. For example if an odd perfect nummber has exactly 9 unique prime divisors then the value t in the ellipses of both equations 5 and 6 will have the value of 3. The ellipsis in equation 5 will be of the form $16^3$ while the ellipsis in equation 6 will be of the form $17 \times 19 \times 23$.

Equation 5 and 6 can be rearranged as follows

$$16 \times 16 \times 16 \times 16 \times 16 \times 16 \times \cdots \tag{7}$$

$$3 \times 5 \times 7 \times 11 \times 13 \times 10^8 \times \cdots \tag{8}$$

Where the definition of the ellipsis remains the same as defined previously.

$$16 \times 16 \times 16 \times 16 \times 16 \times 16 \times \cdots = 16,777,216 \times \cdots \tag{9}$$

$$3 \times 5 \times 7 \times 11 \times 13 \times 10^8 \times \cdots = 1,501,500,000,000 \times \cdots \tag{10}$$

For equation 9 above, every new unique prime divisor will be represented by a new 16 that will be multiplied by the number on the right hand side of equation 9.

For equation 10, every new unique prime divisor will be represented by a new prime which will be bigger than 16 that will be multiplied by the number on the right hand side of equation 10. If the new unique prime is 17 and if it is the Euler prime then it could be represented as $17^1, 17^5$ and so on. If 17 is a non-Euler prime then it can be represented as $17^2, 17^4$ and so on. The main point is that $17^x > 16$ for all values of x greater than 0, where x is an integer. However to simplify the argument, I will just assume that all unique prime divisors of N are raised to the power of 1.

Therefore we can conclude that each unique prime divisor of N that we multiply by the figure in equation 10 will only make the figure in equation 10 grow bigger than the one in equation 9. This is because we are multiplying equation 9 by 16 for every new unique prime divisor while multiplying equation 10 by a prime number greater than 16 for every new unique prime divisor. Therefore the more unique prime divisors an odd perfect number has, the bigger the difference will be between the figures in equation 9 and equation 10.

Since $1,501,500,000,000 > 16,777,216$ and since any new unique prime divisor will only increase the difference between these two numbers, we can therefore conclude that the figure to the right of equation 10 will always be bigger than the figure to the right of equation equation 9 regardless of how many unique prime divisors an odd perfect number has.

We therefore get the final equation as follows:

$$16,777,216 \times \cdots < 1,501,500,000,000 \times \cdots \tag{11}$$

$$N < 16,777,216 \times \cdots \tag{12}$$

Therefore:

$$N < 1,501,500,000,000 \times \cdots \tag{13}$$

Therefore an odd perfect number N does not exist because an odd perfect number N will always be smaller than the figure to the right of equation 9, regardless of the number of unique prime divisors of N. However, the figure to the right of equation 9 will always be smaller than the figure to the right of equation 10, regardless of the number of unique prime divisors of N. Therefore N will always be smaller than the figure to the right of equation 10, regardless of the number of unique prime divisors of N. Notice that the figure to the right of equation 10 is the Eulerian form an an odd perfect number. We can state this as, since N is always smaller than the Eulerian form of an odd perfect number, regardless of the number of unique prime divisors of N, therefore N does not exist. That is the end of the proof.

Q.E.D

**Tables**

The following 3 tables are used to reinforce the idea behind the proof. They are meant as examples or visual aids for those who haven't understood the proof.

| Label | Value |
|---|---|
| a1 | $16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16$ |
| a2 | $3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 10^8$ |

Table 1: The case where N has exactly 9 unique prime divisors

| Label | Value |
|---|---|
| b1 | $16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16$ |
| b2 | $3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 10^8$ |

Table 2: The case where N has exactly 11 unique prime divisors

| Label | Value |
|---|---|
| c1 | $16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16 \times 16$ |
| c2 | $3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41 \times 10^8$ |

Table 3: The case where N has exactly 13 unique prime divisors

**Analysis**

Table 1: $a2 > a1$. Since $a1 > N$ therefore $a2 > N$

Table 2: $b2 > b1$. Since $b1 > N$ therefore $b2 > N$

Table 3: $C2 > C1$. Since $C1 > N$ therefore $C2 > N$

This table can be replicated for any number of unique prime divisors of an odd perfect number. The value in a2, b2 and c2 represent a simplified version of the Eulerian structure of N where each unique prime divisor is raised to the power of 1. In reality all but 1 unique primes divisors should be raised to the power of at least 2. However, I am trying to be as conservative as possible in formulating the Eulerian structure of N to keep the argument as simple as possible. Of course there is nothing stopping anyone from raising all the unique prime divisors except one of them to the power of 2 or 4 or any other value. However, doing so doesn't change the logic of the argument and that is why I have decided to raise all the unique prime divisors to the power of 1. Also note that Ianucci(1999) proved that the second largest unique prime divisor of N is at least $10^4$ and this fact could be added to the Eulerian structures a2,b2 and c2 to strengthen the argument. However, whether we add it or not will not change the conclusion. Therefore it is just extra work that leads to the same result and that is why I have left it out in order to make the argument as simple as possible. This is the end of the paper.

## References

1. Goto, T., and Ohno,Y.(2008). Odd perfect numbers have a prime factor exceeding $10^8$. *Mathematics of Computation.* 77(263), pp.1859-1868.

2. Ianucci,D.E.(1999). The second largest prime divisor of an odd perfect number exceeds ten thousand. *Mathematics of Computation.* 68(228), pp. 1749-1760.

3. Nielsen,P.P.(2003). An upper bound for odd perfect numbers. *Integers: Electronic Journal of Combinatorial Number Theory.*3(A14)

4. Nielsen,P.P.(2007). Odd perfect numbers have at least nine distinct prime factors. *Mathematics of Computation.* 76(260). pp.2109-2126.