International Journal of Current Science Research and Review
ISSN: 2581-8341
Volume 05 Issue 04 April 2022
DOI: 10.47191/ijcsrr/V5-i4-28, Impact Factor: 5.995
IJCSRR @ 2022
www.ijcsrr.org

A Comparative Study of Two-Sample Tests for High-Dimensional Covariance Matrices

Knavoot Jiamwattanapong[1], Nisanad Ingadapa[2], Piyada Phrueksawatnon[3]
[1,2]Department of General Education, Faculty of Liberal Arts, Rajamangala University of Technology Rattanakosin, Thailand
[3]Department of Statistics, School of Science, University of Phayao, Thailand

ABSTRACT: The equality of covariance matrices is an essential assumption in means and discriminant analyses for high-dimensional data. The performance of tests for covariance matrices may vary substantially depending on the covariance structure, so using inappropriate methods to verify the assumption will result in worse performance. The purpose of this study is to assess and compare the performance of three tests for two-sample high-dimensional covariance matrices: Schott's (2007), Srivastava and Yanagihara's (2010), and Li and Chen's (2012) under various covariance structures. A simulation study was conducted when the covariance structures were spherical, compound symmetric, block-diagonal, and first-order autoregressive with homogenous variances. The results show that Li and Chen's test outperforms the others with a sample size of at least 10 under particular covariance structures. When the number of variables is increased with a fixed sample size, Li and Chen's test still performs well, whereas Schott's performance deteriorates. Some recommendations for selecting appropriate tests are also provided in this paper.

KEYWORDS: Covariance matrix structure, Equality of covariance matrices, High dimensional data, Test for covariance matrices, Two-Sample Tests

I. INTRODUCTION

As measurement technology has advanced, high dimensional data have become more common in a variety of fields, including medical science, genomics, and economics. DNA microarrays, a powerful technology for studying gene expression on a genomic scale, are examples of high dimensional data in medical science since they involve thousands of variables of gene expression data with a very small sample size. In Alon et al. [1], the number of variables reached 6,500 while the sample sizes of the first and the second groups were 22 and 40, respectively, and despite the classification method being applied to reduce them into groups, there were still 2,000 variables left. The analytical methods used to deal with high-dimensional data differ from those used for low-dimensional data. When the number of variables exceeds the sample size, as in high dimensional data, statistical methods become very complicated, and in many cases, effective methods used in univariate and multivariate analyses are inapplicable. Two prominent instances are the Hotelling's test [2] for comparing two mean vectors and the likelihood ratio test for comparing two or more covariance matrices.

In means and discriminant analyses, testing the equality of two covariance matrices is an important method for data analysts to ensure that the data satisfy the assumption of homogeneous covariance matrices. However, methods for dealing with high-dimensional data are still limited and are dependent on covariance structure, the number of variables, and sample size. Among these were Schott [3], Srivastava and Yanagihara [4], Li and Chen [5], and Srivastava, Yanagihara, and Kubokawa [6], the latter of which was modified from Schott [3]. The goal of this study is to examine and compare the performance of three tests established by Schott [3], Srivastava and Yanagihara [4], and Li and Chen [5] for equality of two covariance matrices in high dimensional data under various covariance structures.

II. TESTS FOR COVARIANCE MATRICES IN HIGH-DIMENSIONAL DATA

Let $\mathbf{x_{ij}}$ be distributed as iid $N_p(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$, $i=1,2$ $j=1,2,...,n_i$ from population i and the two samples are assumed to be independent. In this study, we considered the case of high-dimensional data which is $p > n$, $n : n_1 + n_2 - 2$. One important obstacle that makes most statistical methods in multivariate cases inapplicable in high-dimension cases is the singularity of the high-dimensional sample covariance matrix. As a result, existing tests of high-dimensional covariance matrices such as those presented by Schott [3], Srivastava and Yanagihara [4], Li and Chen [5], and Cai, Liu and Xia [7] were developed without using the inverse

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IJCSRR @ 2022
www.ijcsrr.org

of the sample covariance matrix. Additionally, since the test proposed by Cai et al. [7] was based on a sparse matrix, which is narrower than the previous three techniques, it was not included in this study.

The hypothesis testing problem in this study is $H: \mathbf{\Sigma_1} = \mathbf{\Sigma_2}$ against $K: \mathbf{\Sigma_1} \neq \mathbf{\Sigma_2}$.

The sample mean vectors are $\bar{\mathbf{x}}_\mathbf{i} = \frac{1}{n_i}\sum_{j=1}^{n_i} \mathbf{x_{ij}}$, $i = 1, 2$,

and the sample covariance matrices are $\mathbf{S}_i = \frac{1}{n_i - 1}\sum_{j}^{n_i}(\mathbf{x_{ij}} - \bar{\mathbf{x}}_\mathbf{i})(\mathbf{x_{ij}} - \bar{\mathbf{x}}_\mathbf{i})'$, $i = 1, 2$.

Let $v_i = n_i - 1$ and $\mathbf{S}_p = \frac{\sum_i v_i \mathbf{S}_i}{\sum_i v_i}$, $i = 1, 2$.

A. Schott's Test

The test presented by Schott [3] is based on the square of the Frobenius norm $tr(\Sigma_1 - \Sigma_2)^2$ and the unbiased and consistent estimators of $tr(\Sigma_i)/p$ and $tr(\Sigma_i^2)/p$, $i = 1, 2.$

The Schott's test statistic, denoted by T_S , is

$$T_S = \frac{(n_1 - 1)(n_2 - 1)}{2n\hat{a}_2}\left[\hat{a}_{21} + \hat{a}_{22} - \frac{2}{p}tr(\mathbf{S_1S_2})\right] ,$$

where $n = n_1 + n_2 - 2$,

$$\hat{a}_{2i} = \frac{(n_i - 1)^2}{p(n_i - 2)(n_i + 1)}\left\{tr(\mathbf{S_i^2}) - \frac{1}{n_i - 1}[tr(\mathbf{S_i})]^2\right\}, i = 1, 2 ,$$

and $$\hat{a}_2 = \frac{n^2}{p(n-1)(n+2)}\left\{tr(\mathbf{S}_p^2) - \frac{1}{n}[tr(\mathbf{S}_p)]^2\right\}.$$

Under the null hypothesis $H: \mathbf{\Sigma_1} = \mathbf{\Sigma_2}$, the statistic $T_S \xrightarrow{L} N(0,1)$ when $(p, n_1, n_2) \to \infty$ and $p/n_i \to c_i \in (0, \infty)$, $i = 1, 2$. The null hypothesis would be rejected when $T_S > Z_{1-\alpha}$, where $P(Z > Z_{1-\alpha}) = \alpha$ and Z is a standard normal random variable.

The T_S test does not perform well when the data is right- skewed, particularly when the number of variables is increased while the sample size remains constant [6].

B. Srivastava and Yanagihara's Test

The test developed by Srivastava and Yanagihara [4] is based on the consistent estimator of the difference $\frac{tr(\Sigma_1^2)}{tr(\Sigma_1)^2} - \frac{tr(\Sigma_2^2)}{tr(\Sigma_2)^2} = \gamma_1 - \gamma_2$. Under the null hypothesis $H: \mathbf{\Sigma_1} = \mathbf{\Sigma_2}$, the term $\gamma_1 - \gamma_2 = 0$.

The test statistic, denoted by T_{SY} , is

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$$T_{SY}=\frac{\hat{\gamma}_1-\hat{\gamma}_2}{\sqrt{\hat{\xi}_1^2+\hat{\xi}_2^2}},$$

where $\hat{\gamma}_i=\frac{\hat{a}_{2i}}{\hat{a}_{1i}^2}$, $i=1,2$,

$$\hat{a}_{1i}=\frac{1}{p}tr(\mathbf{S}_i),\qquad \hat{a}_{2i}=\frac{(n_i-1)^2}{p(n_i-2)(n_i+1)}\left\{tr(\mathbf{S_i^2})-\frac{1}{n_i-1}[tr(\mathbf{S_i})]^2\right\},$$

$$\hat{\xi}_i^2=\frac{4}{(n_i-1)^2}\left\{\frac{\hat{a}_2^2}{\hat{a}_1^4}+\frac{2(n_i-1)}{p}\left(\frac{\hat{a}_2^3}{\hat{a}_1^6}-\frac{2\hat{a}_2\hat{a}_3}{\hat{a}_1^5}+\frac{\hat{a}_4}{\hat{a}_1^4}\right)\right\},$$

$$\hat{a}_1=\frac{1}{p}tr(\mathbf{S}_p),\ \hat{a}_2=\frac{n^2}{p(n-1)(n+2)}\left\{tr(\mathbf{S}_p^2)-\frac{1}{n}[tr(\mathbf{S}_p)]^2\right\},\quad n=n_1+n_2-2,$$

$$\hat{a}_3=\frac{1}{n(n^2+3n+4)}\left[\frac{1}{p}tr(nS_p)^3-3n(n+1)p\hat{a}_1\hat{a}_2-np^2\hat{a}_1^3\right],$$

$$\hat{a}_4=\frac{1}{c_0}\left[\frac{1}{p}tr(nS_p)^4-c_1p\hat{a}_1\hat{a}_3-c_2p^2\hat{a}_1^2\hat{a}_2-c_3p\hat{a}_2^2-np^3\hat{a}_1^4\right],$$

where the constants are

$c_0=n(n^3+6n^2+21n+18)$, $c_1=2n(2n^2+6n+9)$, $c_2=2n(3n+2)$, and $c_3=n(2n^2+5n+7)$.

When $(p,n)\to\infty$, the statistic $T_{SY}\xrightarrow{L}N(0,1)$. The null hypothesis would be rejected when $T_{SY}>Z_{1-\alpha}$, where $P(Z>Z_{1-\alpha})=\alpha$ and Z is a standard normal random variable. In addition, when $Q^2=T_{SY}^2$, then $Q^2\xrightarrow{L}\chi_1^2$, where χ_1^2 is the chi-squared distribution with 1 degree of freedom. The null hypothesis would be rejected when $Q^2>\chi^2_{1,1-\alpha}$, where $P(\chi^2>\chi^2_{1,1-\alpha})=\alpha$. When the number of variables was 200, it was shown that the test statistic Q^2 performed well [4].

C. Li and Chen's Test

The test developed by Li and Chen [5] is based on unbiased and consistent estimator of $tr(\Sigma_1-\Sigma_2)^2$ and Schott's [3] method. The test statistic, denoted by T_{LC}, is

$$T_{LC}=\frac{A_{n_1}+A_{n_2}-2C_{n_1n_2}}{\hat{\sigma}_{0,n_1,n_2}},$$

where
$$A_{n_i}=\frac{1}{n_i(n_i-1)}\sum_{j\neq k}^{n_i}(\mathbf{x}'_{ij}\mathbf{x}_{ik})^2-\frac{2}{n_i(n_i-1)(n_i-2)}\sum_{j\neq k\neq m}^{n_i}(\mathbf{x}'_{ij}\mathbf{x}_{ik}\mathbf{x}'_{ik}\mathbf{x}_{im})$$
$$+\frac{1}{n_i(n_i-1)(n_i-2)(n_i-3)}\sum_{j\neq k\neq m\neq n}^{n_i}(\mathbf{x}'_{ij}\mathbf{x}_{ik}\mathbf{x}'_{im}\mathbf{x}_{in}),\ i=1,2,$$

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www.ijcsrr.org

$$C_{n_1 n_2} = \frac{1}{n_1 n_2}\sum_{j}^{n_1}\sum_{k}^{n_2}(\mathbf{x}'_{1j}\mathbf{x}_{2k})^2 - \frac{1}{n_1 n_2(n_1-1)}\sum_{j\neq m}^{n_1}\sum_{k}^{n_2}(\mathbf{x}'_{1j}\mathbf{x}_{2k}\mathbf{x}'_{2k}\mathbf{x}_{1m}) - \frac{1}{n_1 n_2(n_2-1)}\sum_{j\neq m}^{n_2}\sum_{k}^{n_1}(\mathbf{x}'_{2j}\mathbf{x}_{1k}\mathbf{x}'_{1k}\mathbf{x}_{2m})$$

$$+\frac{1}{n_1 n_2(n_1-1)(n_2-1)}\sum_{j\neq m}^{n_1}\sum_{k\neq n}^{n_2}(\mathbf{x}'_{1j}\mathbf{x}_{2k}\mathbf{x}'_{1m}\mathbf{x}_{2n}),$$

$$\hat{\sigma}_{0,n_1,n_2} = \frac{2}{n_2}A_{n_1} + \frac{2}{n_1}A_{n_2} .$$

Under the null hypothesis, the statistic $T_{LC} \xrightarrow{L} N(0,1)$ when $(p,n_1,n_2)\to\infty$. The null hypothesis would be rejected when $T_{LC} > Z_{1-\alpha}$, where $P(Z > Z_{1-\alpha}) = \alpha$ and Z is a standard normal random variable.

To obtain the value of the test statistic T_{LC}, it needs to expand the terms in the summations to the order of n^4; this complicates the calculation and takes a long time if the sample sizes are very large [5].

III. SIMULATION PROCEDURE

To evaluate the performance of the tests created by Schott [3], Srivastava and Yanagihara [4] and Li and Chen [5], four structures of covariance matrices: sphericity, compound symmetry (CS), block diagonal structure (BD), and first-order autoregressive structure with homogenous variances, or AR(1), were formed in a simulation study. The simulation study was conducted using R version 4.1.0. Let the first random sample $\mathbf{x}_{11},\mathbf{x}_{12},...,\mathbf{x}_{1n_1}$ come from a p-variate normal population $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma}_1)$ and the other random sample $\mathbf{x}_{21},\mathbf{x}_{22},...,\mathbf{x}_{2n_2}$, being independent of the first sample, come from $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma}_2)$, where $p > n$, $n : n_1 + n_2 - 2$ and $n_1 = n_2$. We set $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$ in the null hypothesis, and in the alternative, we set the first population covariance matrix $\boldsymbol{\Sigma}_1$ to be the same as in the null hypothesis, but the second population covariance matrix $\boldsymbol{\Sigma}_2$ to be different from $\boldsymbol{\Sigma}_1$ but with the same structure as follows:

A. Sphericity

Under the null hypothesis, set $\boldsymbol{\Sigma}_i = \mathbf{I}_p$, $i = 1,2$ and under the alternative hypothesis, set $\boldsymbol{\Sigma}_2 = 3\mathbf{I}_p$.

B. CS

Under the null hypothesis, set $\boldsymbol{\Sigma}_i = 0.5\mathbf{I}_p + 0.5(\mathbf{1}_p\mathbf{1}'_p)$, $i = 1,2$ and 1_p is a vector of 1's. Under the alternative hypothesis, set $\boldsymbol{\Sigma}_2 = 0.9\mathbf{I}_p + 0.1(\mathbf{1}_p\mathbf{1}'_p)$.

C. BD

Under the null hypothesis, set $\boldsymbol{\Sigma}_i = diag(\mathbf{B}_1,\mathbf{B}_2,...,\mathbf{B}_{(m-1)(m-1)},\mathbf{B}_{mm})$, $i = 1,2$, where $\mathbf{B}_j = 0.5\mathbf{I}_p + 0.5(\mathbf{1}_p\mathbf{1}'_p)$, $j = 1,2,...,m$, where the first $m-1$ blocks contain block size of 3, so $p = 3(m-1)+m$. Under the alternative hypothesis, set $\boldsymbol{\Sigma}_2 = diag(\mathbf{B}_1,\mathbf{B}_2,...,\mathbf{B}_{(m-1)(m-1)},\mathbf{B}_{mm})$, where $\mathbf{B}_j = 0.9\mathbf{I}_p + 0.1(\mathbf{1}_p\mathbf{1}'_p)$ $j = 1,2,...,m$, where the first $m-1$ blocks contain block size of 3.

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www.ijcsrr.org

D. AR(1)

Under the null hypothesis, set $\boldsymbol{\Sigma}_k = (\rho^{|i-j|}\sigma_{ij}),\ k=1,2,\ \rho=0.5$, and $\sigma_{ij} \sim U(2,3)$, where $U(2,3)$ is a continuous uniform distribution on the interval [a,b]. Under the alternative hypothesis, set $\boldsymbol{\Sigma}_2 = (\rho^{|i-j|}\sigma_{ij}),\ \rho=0.5$ and $\sigma_{ij} \sim U(5,6)$. The sample sizes (n_i) were set as $n_1 = n_2$ and the number of variables (p) was set as $p=\{20,30,40,50,100\}$ for $n_i=5$; $p=\{40,60,80\ 100,200\}$ for $n_i=10$; $p=\{60,\ 80,100,150,200\}$ for $n_i=15$; and $p=\{80,\ 100,150,200,400\}$ for $n_i=20$.

For each condition, 1,000 samples were generated at a nominal significance level of 0.05. The performance of the tests was assessed using the attained significance level (ASL) and empirical power. Under the null hypothesis, the ASL was calculated as the proportion of the number of times the calculated test statistics fell inside the critical region in 1,000 times. The ASL was evaluated using Bradley's liberal criterion [8]. When the ASLs of a test fall within the interval [0.04, 0.06], the test was regarded as acceptable. To obtain the empirical power, the simulation study was conducted under the alternative hypothesis of unequal covariance matrices but with the same covariance structure.

IV. RESULTS

The simulation results were presented in terms of the attained significance level (ASL) and empirical power under four covariance matrix structures: sphericity, CS, BD, and AR(1), as shown in Tables 1-4, respectively.

Table 1. ASL and Empirical power of the tests with spherical covariance structure at nominal level 0.05

n_i	p	ASL			Empirical power		
		T_S	T_{SY}	T_{LC}	T_S	T_{SY}	T_{LC}
5	20	0.0350	0.0120	0.0240	0.0450	0.0070	0.1840
	30	0.0390	0.0050	0.0290	0.0180	0.0020	0.1670
	40	0.0330	0.0030	0.0310	0.0040	0.0010	0.1550
	50	0.0380	0.0030	0.0310	0.0030	0.0000	0.1740
	100	0.0330	0.0000	0.0180	0.0000	0.0000	0.1620
10	40	0.0430	0.0270	0.0410	0.3940	0.0140	0.6370
	60	0.0520	0.0280	0.0540	0.2310	0.0070	0.6150
	80	0.0400	0.0180	0.0370	0.1270	0.0010	0.6390
	100	0.0580	0.0090	0.0540	0.0770	0.0000	0.6270
	200	0.0370	0.0090	0.0380	0.0010	0.0000	0.6380
15	60	0.0510	0.0260	0.0470	0.7990	0.0130	0.9310
	80	0.0510	0.0280	0.0400	0.7100	0.0100	0.9210
	100	0.0470	0.0390	0.0450	0.6460	0.0070	0.9350
	150	0.0560	0.0240	0.0660	0.3827	0.0002	0.9320
	200	0.0600	0.0240	0.0590	0.1838	0.0000	0.9370
20	80	0.0520	0.0440	0.0510	0.9470	0.0100	0.9970
	100	0.0570	0.0380	0.0560	0.9640	0.0060	0.9980
	150	0.0370	0.0300	0.0390	0.9000	0.0040	0.9960
	200	0.0500	0.0320	0.0480	0.8230	0.0010	0.9940
	400	0.0460	0.0130	0.0390	0.1690	0.0000	0.9970

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www.ijcsrr.org

Table 1 shows that when the sample size is at least 10, both the tests T_S and T_{LC} are acceptable under the spherical covariance structure, while T_{SY} is not. When the number of variables is increased while the sample size remains constant, the test T_{LC} still performs better than T_S considering from the empirical power of the tests. Overall, it can be concluded that the test T_{LC} outperformed the others when the covariance structure is spherical.

Table 2 shows that all three tests in the study, T_S, T_{SY}, and T_{LC}, performed unacceptably under the compound symmetric covariance matrix.

Table 3 illustrates that the statistic T_S performs acceptably, and the test T_{LC} performs well with the sample size of at least 10, but the test T_{SY} does not.

Table 2. ASL and Empirical power of the tests with compound symmetric covariance structure at nominal level 0.05

n_i	p	ASL			Empirical power		
		T_S	T_{SY}	T_{LC}	T_S	T_{SY}	T_{LC}
5	20	0.0780	0.1250	0.0600	0.2260	0.4900	0.1520
	30	0.0630	0.1290	0.0530	0.2500	0.6090	0.1970
	40	0.0800	0.1470	0.0640	0.2810	0.6380	0.2180
	50	0.0650	0.1280	0.0610	0.2700	0.6580	0.2260
	100	0.0710	0.1570	0.0600	0.3320	0.7080	0.2540
10	40	0.0000	0.0000	0.0000	0.6220	0.9260	0.5890
	60	0.0820	0.1160	0.0840	0.6460	0.9470	0.6160
	80	0.0400	0.0180	0.0370	0.6670	0.9490	0.6490
	100	0.0900	0.1470	0.0870	0.6990	0.9620	0.6820
	200	0.0380	0.0070	0.0480	0.7080	0.9640	0.6387
15	60	0.1060	0.0960	0.0970	0.8630	0.9990	0.8470
	80	0.0920	0.0940	0.0880	0.8530	0.9900	0.8490
	100	0.0980	0.0914	0.0990	0.8770	0.9950	0.8720
	150	0.1010	0.0820	0.1020	0.8890	0.9950	0.8810
	200	0.0910	0.0750	0.0890	0.8790	0.9920	0.8710
20	80	0.1000	0.0740	0.0870	0.9360	0.9980	0.9320
	100	0.0920	0.0650	0.0850	0.9510	1.0000	0.9480
	150	0.0990	0.0830	0.0990	0.9550	1.0000	0.9440
	200	0.0950	0.0870	0.0980	0.9510	1.0000	0.9500
	400	0.0400	0.0230	0.0360	0.9600	1.0000	0.9600

Table 3. ASL and Empirical power of the tests with block diagonal covariance structure at nominal level 0.05

n_i	p	ASL			Empirical power		
		T_S	T_{SY}	T_{LC}	T_S	T_{SY}	T_{LC}
5	20	0.0510	0.0200	0.0390	0.0760	0.0460	0.0580
	30	0.0350	0.0050	0.0290	0.0600	0.0420	0.0440
	40	0.0410	0.0080	0.0330	0.0590	0.0330	0.0390
	50	0.0440	0.0060	0.0270	0.0660	0.0210	0.0440
	100	0.0450	0.0050	0.0270	0.0720	0.0090	0.0580
10	40	0.0560	0.0410	0.0530	0.1420	0.2670	0.1400
	60	0.0420	0.0320	0.0410	0.1540	0.2860	0.1400
	80	0.0490	0.0150	0.0430	0.1440	0.2510	0.1280
	100	0.0340	0.0240	0.0420	0.1500	0.2200	0.1340
	200	0.0550	0.0130	0.0490	0.1470	0.1420	0.1330

 *Corresponding Author: Knavoot Jiamwattanapong

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15	60	0.0480	0.0460	0.0430	0.2360	0.5990	0.2150
	80	0.0570	0.0400	0.0500	0.2470	0.5860	0.2390
	100	0.0550	0.0350	0.0590	0.2210	0.5590	0.2150
	150	0.0440	0.0230	0.0360	0.2380	0.5260	0.2150
	200	0.0380	0.0290	0.0350	0.2460	0.4770	0.2190
20	80	0.0490	0.0390	0.0430	0.3560	0.8030	0.3420
	100	0.0480	0.0380	0.0470	0.3370	0.8170	0.3240
	150	0.0680	0.0290	0.0610	0.3820	0.8050	0.3490
	200	0.0580	0.0340	0.0490	0.3620	0.7830	0.3420
	400	0.0500	0.0230	0.0480	0.3580	0.7680	0.3310

Table 4. ASL and Empirical power of the tests with covariance structure of AR(1) at nominal level 0.05

n_i	p	**ASL**			**Empirical power**		
		T_S	T_{SY}	T_{LC}	T_S	T_{SY}	T_{LC}
5	20	0.0450	0.0190	0.0430	0.0740	0.0310	0.0990
	30	0.0450	0.0120	0.0350	0.0470	0.0070	0.0970
	40	0.0390	0.0120	0.0310	0.0320	0.0080	0.0980
	50	0.0530	0.0070	0.0410	0.0240	0.0090	0.0840
	100	0.0420	0.0010	0.0360	0.0010	0.0010	0.0800
10	40	0.0580	0.0270	0.0510	0.2490	0.0420	0.3070
	60	0.0430	0.0360	0.0410	0.2160	0.0270	0.3160
	80	0.0630	0.0400	0.0620	0.2020	0.0200	0.3490
	100	0.0410	0.0130	0.0460	0.1360	0.0120	0.3180
	200	0.0370	0.0150	0.0400	0.0250	0.0020	0.3320
15	60	0.0640	0.0380	0.0590	0.4540	0.0380	0.5050
	80	0.0600	0.0370	0.0500	0.4940	0.0320	0.6050
	100	0.0700	0.0350	0.0630	0.4270	0.0360	0.5730
	150	0.0410	0.0310	0.0360	0.3470	0.0110	0.5850
	200	0.0530	0.0250	0.0510	0.2760	0.0050	0.6070
20	80	0.0650	0.0440	0.0590	0.7560	0.0330	0.8000
	100	0.0580	0.0480	0.0570	0.7120	0.0380	0.7710
	150	0.0580	0.0360	0.0580	0.6790	0.0190	0.7910
	200	0.0470	0.0340	0.0460	0.6190	0.0150	0.7960
	400	0.0500	0.0150	0.0440	0.2650	0.0010	0.7480

It can be seen, from Table 4, that under the covariance structure of AR(1), the tests T_S and T_{LC} perform well, whereas T_{SY} performs poorly. In addition, when the number of variables (p) is increased with a fixed sample size, the test T_{LC} performs slightly better than T_S.

V. CONCLUSION AND DISCUSSION

The purpose of this study is to assess and compare the performance of tests for equality of covariance matrices in high-dimensional data. The tests considered in this study were Schott's [3], Srivastava and Yanagihara's [4], and Li and Chen's [5] and the simulation study was conducted under four structures of covariance matrices: sphericity, compound symmetry, block diagonal structure, and first-order autoregressive structure with homogenous variances, or AR(1).

Conclusion

When the data are multivariate normal distributed with the covariance matrix structures of sphericity, block diagonal matrix, or AR(1), the tests presented by Schott [3], Srivastava and Yanagihara [4], and Li and Chen [5] perform differently. Overall, Li and

Chen's test outperforms the others; actually, it is slightly better than Schott's test. For sample sizes of at least 10, Li and Chen's and Schott's tests can be used effectively. When the number of variables is increased with a fixed sample size, Li and Chen's test performs better, while Schott's test performs worse. In addition, when the covariance matrix structure is compound symmetry, none of the three tests in this study performs well. To test the equality of covariance matrices in high-dimensional data, the structures of the covariance matrix should be examined first, which can be done by considering the pattern of the sample covariance matrix. When the covariance matrix structure is spherical, block-diagonal, or first-order regressive, the guidelines are:

Case 1**:** When the sample size (n_i) is at least 10 and the number of variables (p) is substantially greater than the sample size, such as $p > 7n_i$, Li and Chen's test should be applied.

Case 2**:** When the sample size (n_i) is at least 10 and the number of variables (p) is not substantially greater than the sample size, such as $n_i < p < 7n_i$, either Li and Chen's or Schott's test should be used.

Case 3**:** When the sample size is smaller than 10, Schott's test is recommended.

Discussion

The findings from this study collaborated by Li and Chen [5], which a simulation study was conducted under spherical and block-diagonal covariance structures, and it was found that the ASLs were close to the nominal significance level. From the results, it was shown that the test by Srivastava and Yanagihara [4] did not perform well under all structures of covariance matrices in this study, the test might be only suitable for particular covariance structures. For instance, the covariance structure, determined by Srivastava and Yanagihara [4], was that under the null hypothesis of $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \mathbf{D}\Delta_0\mathbf{D}$, where $\mathbf{D}$ was a diagonal matrix and $\Delta_0 = (\delta_{ij})$, $\delta_{ij} = (-1)^{i+j}(0.4)^{|i-j|^{0.1}}$, and under the alternative hypothesis of $\mathbf{\Sigma}_2 = \mathbf{D}\Delta_2\mathbf{D}$, where $\Delta_2 = (\delta_{ij})$, $\delta_{ij} = (-1)^{i+j}(0.8)^{|i-j|^{0.1}}$ and $\mathbf{\Sigma}_1$ was the same as in the null hypothesis. This result leads to a suggestion for developing a new test, i.e. if a developed test performs acceptably with a variety of covariance structures, analysts and researchers will find it easier to select a test for equality of covariance matrices.

ACKNOWLEDGEMENT

This research was supported by research funds from Rajamangala University of Technology Rattanakosin (RMUTR).

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