

On Robin's criterion for the Riemann Hypothesis

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Abstract Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^\gamma \times n \times \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We call this inequality, the Robin inequality. In 2022, Vega stated that the possible existence of the smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$ and $(\log n)^\beta < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$. Based on that result, we obtain a contradiction just assuming the existence of such possible smallest counterexample $n > 5040$ for the Robin inequality. By contraposition, we show that the Riemann hypothesis should be true.

Keywords Riemann hypothesis · Robin inequality · Sum-of-divisors function · Prime numbers · Counterexample

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1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n :

$$\sum_{d|n} d$$

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where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n . Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robins(n) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and \log is the natural logarithm. The importance of this property is:

Theorem 1.1 Robins(n) holds for all natural numbers $n > 5040$ if and only if the Riemann hypothesis is true [3].

It is known that Robins(n) holds for many classes of numbers n . We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$.

Theorem 1.2 Robins(n) holds for all natural numbers $n > 5040$ that are square free [1].

Let $q_1 = 2, q_2 = 3, \dots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^m q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [1]. Now, we are able to use this recently result:

Theorem 1.3 The possible existence of the smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$ and $(\log n)^\beta < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$ [5].

Putting all together yields a proof for the Riemann hypothesis using the Theorem 1.3 as the principal argument.

2 Known Results

These are known results:

Lemma 2.1 For every $x > -1$ [2]:

$$\log(1+x) \geq \frac{x}{x+1}.$$

Lemma 2.2 For every real number x [2]:

$$e^x \geq 1+x.$$

Lemma 2.3 For every $x > -1$ [2]:

$$\log(1+x) \geq -0.9 + \frac{2 \times x}{x+2}.$$

Lemma 2.4 For every $x > 1$ [4]:

$$\sum_{q \leq x} \frac{1}{q} > \log \log x.$$

3 A Central Lemma

The following is a key Lemma.

Lemma 3.1 *If the natural number $n > 5040$ is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$, then $\beta \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$ where $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.*

Proof If we apply the logarithm to the value of

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$$

then we obtain that

$$\sum_{i=1}^m \log\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}\right).$$

For every $1 \leq j \leq m$, we know that

$$\frac{q_j^{a_j+1}}{q_j^{a_j+1}-1} = 1 + \frac{1}{q_j^{a_j+1}-1}.$$

We use the Lemma 2.1 to show that

$$\begin{aligned} \log\left(1 + \frac{1}{q_j^{a_j+1}-1}\right) &\geq \frac{\frac{1}{q_j^{a_j+1}-1}}{\frac{1}{q_j^{a_j+1}-1} + 1} \\ &= \frac{1}{(q_j^{a_j+1}-1) \times \left(\frac{1}{q_j^{a_j+1}-1} + 1\right)} \\ &= \frac{1}{1 + (q_j^{a_j+1}-1)} \\ &= \frac{1}{q_j^{a_j+1}}. \end{aligned}$$

So,

$$\sum_{i=1}^m \log\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}\right) \geq \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$$

and thus,

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1} \geq e^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}}.$$

Using the Lemma 2.2, we have that

$$e^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$$

and therefore,

$$\beta \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}.$$

4 Main Insight

This is the main insight.

Lemma 4.1 *Suppose that $n > 5040$ is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$ and $q_m > e^{31.018189471}$. Then $(\log n)^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \geq 1.03352795481$.*

Proof If we apply the logarithm to the both sides of the inequality, then

$$\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times \log \log n \geq \log(1.03352795481).$$

Let's multiply the both sides of the inequality by e^γ ,

$$\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times e^\gamma \times \log \log n \geq e^\gamma \times \log(1.03352795481).$$

From the Theorem 1.2, we know that

$$\begin{aligned} e^\gamma \times \log \log n &\geq e^\gamma \times \log \log N_m \\ &> f(N_m) \\ &= \prod_{i=1}^m \left(1 + \frac{1}{q_i}\right) \end{aligned}$$

since $n > 5040$ is an Hardy-Ramanujan integer, $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , $n \geq N_m$ and N_m is square free. Hence, we would have that

$$\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times \prod_{i=1}^m \left(1 + \frac{1}{q_i}\right) \geq e^\gamma \times \log(1.03352795481).$$

If we apply the logarithm to the both sides again, then

$$\log \left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) + \sum_{i=1}^m \log \left(1 + \frac{1}{q_i}\right) \geq \log(e^\gamma \times \log(1.03352795481)).$$

We use the Lemma 2.3 to show that

$$\begin{aligned}
\log\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right) &= \log\left(1 + (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}})\right) \\
&\geq -0.9 + \frac{2 \times (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}})}{(-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}) + 2} \\
&= -0.9 + \frac{2 \times (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}})}{1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \\
&> -0.9 + 2 \times (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}) \\
&= -0.9 - 2 + 2 \times \left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right) \\
&> -2.9
\end{aligned}$$

since

$$-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} > -1.$$

For every $1 \leq j \leq m$, we know that

$$\begin{aligned}
\log\left(1 + \frac{1}{q_j}\right) &\geq \frac{\frac{1}{q_j}}{\frac{1}{q_j} + 1} \\
&= \frac{1}{q_j \times (\frac{1}{q_j} + 1)} \\
&= \frac{1}{1 + q_j}
\end{aligned}$$

according to the Lemma 2.1. However, we note that

$$-2.9 + \sum_{i=1}^m \frac{1}{1 + q_i} > 0$$

when $q_m > e^{31.018189471}$. Certainly, we have that

$$\begin{aligned}
\frac{1}{2} + \sum_{i=1}^m \frac{1}{1 + q_i} &> \sum_{i=1}^m \frac{1}{q_i} \\
&> \log \log q_m
\end{aligned}$$

using the Lemma 2.4. It is enough to see that

$$-2.9 - \frac{1}{2} + \log(31.018189471) > 0.03457378953 > 0.$$

In addition, we have that

$$0 > \log(e^\gamma \times \log(1.03352795481))$$

and finally, the proof is complete.

5 Main Theorem

We conclude with the following statement:

Theorem 5.1 *The Riemann hypothesis is true.*

Proof Suppose that $n > 5040$ is the possible smallest number such that $\text{Robins}(n)$ does not hold. By the Theorem 1.3, we know that $q_m > e^{31.018189471}$ and $(\log n)^\beta < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$. From the Lemma 3.1, we know that

$$(\log n)^\beta \geq (\log n)^{\left(1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right)}$$

and therefore, we would have that

$$(\log n)^{\left(1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right)} < 1.03352795481 \times \log(N_m)$$

when $n > 5040$ is the possible smallest number such that $\text{Robins}(n)$ does not hold. Thus, we would obtain that

$$(\log n)^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} < 1.03352795481$$

since n must be an Hardy-Ramanujan integer and $\log n \geq \log N_m$. However, we know the previous inequality cannot be satisfied because of the Lemma 4.1 when $q_m > e^{31.018189471}$. By contraposition, we show that the Riemann hypothesis is true, since we obtain a contradiction just assuming the possible smallest counterexample for the Robin inequality greater than 5040. Certainly, this is a direct consequence of the Theorem 1.1.

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