A Note on Multiple Integrals of 1

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Abstract. In this paper, we consider and provide a generalization of multiple integrals of 1.

Keywords. Multiple integrals.

We can easily know that

$$\int \mathrm{d}x = x + c_1 \tag{1}$$

where c_1 is a constant of integration.

Also, we can easily know that

$$\iint \mathrm{d}x\mathrm{d}x = \frac{x^2}{2!} + c_1 x + c_2 \tag{2}$$

and

$$\iiint \mathrm{d}x\mathrm{d}x\mathrm{d}x = \frac{x^3}{3!} + \frac{c_1 x^2}{2!} + c_2 x + c_3 \tag{3}$$

where c_1, c_2, c_3 are constant of integration.

Since it is $\frac{x^1}{1!} = x$ and $\frac{x^0}{0!} = 1$,

$$\int \mathrm{d}x = \frac{x^1}{1!} + \sum_{k=0}^{1-1} \frac{c_{1-k} x^k}{k!} = \frac{x^1}{1!} + \frac{c_1 x^0}{0!}, \tag{4}$$

$$\iint \mathrm{d}x\mathrm{d}x = \frac{x^2}{2!} + \sum_{k=0}^{2-1} \frac{c_{2-k} x^k}{k!} = \frac{x^2}{2!} + \frac{c_1 x^1}{1!} + \frac{c_2 x^0}{0!}, \tag{5}$$

$$\iiint \mathrm{d}x\mathrm{d}x\mathrm{d}x = \frac{x^3}{3!} + \sum_{k=0}^{3-1} \frac{c_{3-k} x^k}{k!} = \frac{x^3}{3!} + \frac{c_1 x^2}{2!} + \frac{c_2 x^1}{1!} + \frac{c_3 x^0}{0!}. \tag{6}$$

Here, we can know that the equation (4), (5), and (6) is same as following formula

$$\underbrace{\int \cdots \int}_{n \text{ times}} \overbrace{\mathrm{d}x \cdots \mathrm{d}x}^{n \text{ times}} = \frac{x^n}{n!} + \sum_{k=0}^{n-1} \frac{c_{n-k} x^k}{k!} \tag{7}$$

where $n = 1$, $n = 2$, $n = 3$ and c_k are constant of integration for every $k \in \{1, 2, \cdots, n\}$.

Proposition. *Let $n \in \mathbb{N}$ and let c_k be constant of integration for every $k \in \{1, 2, \cdots, n\}$. Then,*

$$\underbrace{\int \cdots \int}_{n \text{ times}} \overbrace{\mathrm{d}x \cdots \mathrm{d}x}^{n \text{ times}} = \frac{x^n}{n!} + \sum_{k=0}^{n-1} \frac{c_{n-k} x^k}{k!}. \tag{8}$$

Proof. For $n \in \mathbb{N}$, we have

$$I = \underbrace{\int \cdots \int}_{n \text{ times}} \overbrace{\mathrm{d}x \cdots \mathrm{d}x}^{n \text{ times}} = \underbrace{\int \cdots \int}_{n-1 \text{ times}} (x + c_1) \overbrace{\mathrm{d}x \cdots \mathrm{d}x}^{n-1 \text{ times}} \tag{9}$$

$$= \underbrace{\int \cdots \int}_{n-2 \text{ times}} \left(\frac{x^2}{2} + c_1 x + c_2 \right) \overbrace{\mathrm{d}x \cdots \mathrm{d}x}^{n-2 \text{ times}} \tag{10}$$

$$= \underbrace{\int \cdots \int}_{n-3 \text{ times}} \left(\frac{x^3}{3!} + \frac{c_1 x^2}{2!} + c_2 x + c_3 \right) \overbrace{\mathrm{d}x \cdots \mathrm{d}x}^{n-3 \text{ times}} \tag{11}$$

$$\tag{12}$$

Repeating the integration as (9), (10), and (11),

$$I = \iiint \left(\frac{x^{n-3}}{(n-3)!} + \frac{c_1 x^{n-4}}{(n-4)!} + \cdots + c_{n-4} x + c_{n-3} \right) \mathrm{d}x \mathrm{d}x \mathrm{d}x \tag{13}$$

$$= \iint \left(\frac{x^{n-2}}{(n-2)!} + \frac{c_1 x^{n-3}}{(n-3)!} + \cdots + c_{n-3} x + c_{n-2} \right) \mathrm{d}x \mathrm{d}x \tag{14}$$

$$= \int \left(\frac{x^{n-1}}{(n-1)!} + \frac{c_1 x^{n-2}}{(n-2)!} + \cdots + c_{n-2} x + c_{n-1} \right) \mathrm{d}x \tag{15}$$

$$= \frac{x^n}{n!} + \frac{c_1 x^{n-1}}{(n-1)!} + \cdots + c_{n-1} x + c_n \tag{16}$$

Here, since it is $\frac{x^1}{1!} = x$ and $\frac{x^0}{0!} = 1$,

$$I = \frac{x^n}{n!} + \frac{c_1 x^{n-1}}{(n-1)!} + \cdots + \frac{c_{n-1} x^1}{1!} + \frac{c_n x^0}{0!} = \frac{x^n}{n!} + \sum_{k=0}^{n-1} \frac{c_{n-k} x^k}{k!}. \tag{17}$$

Therefore,

$$I = \underbrace{\int \cdots \int}_{n \text{ times}} \overbrace{\mathrm{d}x \cdots \mathrm{d}x}^{n \text{ times}} = \frac{x^n}{n!} + \sum_{k=0}^{n-1} \frac{c_{n-k} x^k}{k!}. \tag{18}$$

This completes the proof of (8). □

Example 1. Evaluate $\iiiint \mathrm{d}x\mathrm{d}x\mathrm{d}x\mathrm{d}x$.

$$\iiiint \mathrm{d}x\mathrm{d}x\mathrm{d}x\mathrm{d}x = \frac{x^4}{4!} + \sum_{k=0}^{4-1} \frac{c_{4-k}x^k}{k!} \tag{19}$$

$$= \frac{x^4}{24} + \sum_{k=0}^{3} \frac{c_{4-k}x^k}{k!} \tag{20}$$

$$= \frac{x^4}{24} + \frac{c_1x^3}{3!} + \frac{c_2x^2}{2!} + \frac{c_3x^1}{1!} + \frac{c_4x^0}{0!} \tag{21}$$

$$= \frac{x^4}{24} + \frac{c_1x^3}{6} + \frac{c_2x^2}{2} + c_3x + c_4 \tag{22}$$

Example 2. Calculate $f(19) - f(13)$ where $f(x) = \iiint \mathrm{d}x\mathrm{d}x\mathrm{d}x$. However, ignore constant of integration.

$$\iiint \mathrm{d}x\mathrm{d}x\mathrm{d}x = \frac{x^3}{3!} + \sum_{k=0}^{2} \frac{c_{3-k}x^k}{k!} \tag{23}$$

$$= \frac{x^3}{6} + \frac{c_1x^2}{2} + c_2x + c_3 \tag{24}$$

$$\therefore f(x) = \frac{x^3}{6} \tag{25}$$

$$f(19) - f(13) = \frac{19^3}{6} - \frac{13^3}{6} \tag{26}$$

$$= \frac{6859 - 2197}{6} = \frac{4662}{6} = 777 \tag{27}$$

$$\therefore f(19) - f(13) = 777 \tag{28}$$

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Reference

[1] "Multiple integral". Wikipedia. Wikimedia Foundation, August 23, 2021. https://en.wikipedia.org/wiki/Multiple_integral.

[2] "Integral". Wikipedia. Wikimedia Foundation, March 25, 2022. https://en.wikipedia.org/wiki/Integral.