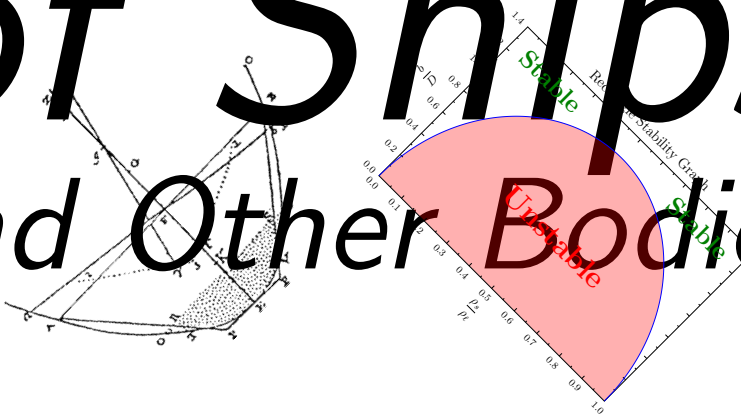


# *Stability of Ships and Other Bodies*



Vasa on August 10, 1628



by  
GENICK BAR-MEIR



# Stability of Ships and Other Bodies

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“We are like dwarfs sitting on the shoulders of giants”

from The Metalogicon by John in 1159



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# Abstract

## *Why Abstract*

The doi registration of the book forced the examination of what is written in this book. This abstract is the result of this examination.

## *Short Abstract*

This book contains innovative ideas. In fact, the book creates a revolution in the understanding of ship stability. One of the chief problems in ship stability is the rolling (ship rotation) which was treated in the past as a simple pendulum with damping (or without) under external forces (or without such as waves, wind etc). This common mistake in this approach is rooted in the assumption that the location of the rotating point. Mistakenly, it was assumed that the rotation point is at the metacenter, an imaginary point which measures the ship's stability. In fact, all the books on stability, that were examined, believe in this mistake. This assumption is wrong! Even for stationary conditions! Furthermore, the rotation point (line) is moving absolutely and relatively to the body (even for quasi-static stability). In other words, the rotation point is floating, depending on the geometry, a fact which was discovered by Bar-Meir. Even with all other mistakes infecting the ship stability, this error can reach to 400% (would you would like to be in ship build with design of error of 400%?). For any pendulum, the rotation point move relatively to the body, creates two effects: the moment of inertia changes and the pendulum becomes a double pendulum. The double pendulum has two rotating arms attached to each other. The double pendulum has no regular period and exhibits chaotic behavior. Large sums of money were spent on the finding the natural frequency something that does not exist consequence of erroneous belief. This book is the only book (currently) describing these phenomena. This book also describes the physical transfer mechanism from one rotation mode (like roll) to another (like pitch). The book also describes the transfer mechanism from a linear motion to a rotational motion and the reverse. To appreciate these points consider the status of knowledge of ship stability before this book. There was no physical transfer mechanism and it was supposed to be magic that results from wrong governing equations. This seminal work for the first time answer questions like how the energy and momentum are transferred from one mode to another and why (example roll to pitch or heave to roll). When you know why it something happened you more likely be able calculate it.

The reason that these developments, first depicted in this book, is because of

the new direct examination method. This innovative method also has a byproduct that shows a way to calculate the circular segment centroid. Nevertheless, the new technique to calculate the centroid is extremely far from being the main and only advantage. This technique examines the ship stability from a physical point of view which is the core usefulness of the technique. An example of the usefulness of this method is the stability dome. The metacenter technique (the old technique) was centered around the committee (IMO) without real understanding the physics of stability. In other words, the metacenter technique is only the beginning (realizing the movement of buoyancy centroid which was discovered by the French engineer Pierre Bouguer). The improvements to the metacenter technique which were added in this book like dimensional analysis are not sufficient enough to improve the metacenter technique to the point where it shows the physics clearly. These statements might look brazen yet, this undersign believe, they are not enough to show the advances have made in this book.

### *Long Abstract*

This book deals with stability of floating bodies. The approach taken in this book is that of looking at the physical point of view on the stability. This means that instead looking at regulations, the book examines why the things behave in a certain way. For example, the book does not start with the  $GM$  (metacenter approach) but deals with the direct approach (today standard approach **angle in angle out**). While this difference looks symbolic at first, the results are profoundly different. For example, in the common approach it was assumed (actually it is deeply entrenched belief "relic of the past") that a change in the cross section liquid line is irrelevant. As results, an upside triangle and downside triangle have the same stability. Or even more egregious is the fact, the believe in the metacenter approach which yield the "fact" that the moment of inertia of ship is fixed which could be 400% error. You do not want to be under or over bridge with this kind of error. Here in this book, it was found that during the floating body tilting, the rotation point changes according to the tilting angle. The rotation location is indirectly believed to be in the metacenter approach in  $M$  and  $M$  is assumed to be at a fixed point for small angles. These two assumptions are wrong, the rotation point is not around metacenter and the rotation is not fixed (only under certain circumstances). Even in the special cases where the rotation point is fixed, moment of inertia is not around the gravity centroid or metacenter but at point  $A$ . The discovery, that the rotation point (location and movement), provides the ability to describe and articulate how energy is transferred from one mode to another. For example, when floating body undergoes heave it changes the moment of inertia hence change the required calculation for roll mode.

This method, used in this book, demonstrates that the turning point of the floating body is at point  $A$  on the liquid line which can move around. Are these discoveries important? The answer is yes for most important aspects. Both methods yield the same results for the prediction of the floating bodies stability, Yet, the weakness of the metacenter method, is that even with the dimensional analysis improvement it miss major effects. For example, is a ship with 1[m]  $GM$  and a small boat same  $GM$  are

stable in the same way? Of course not! The most profound effect is the resonance of container ship (and other floating bodies). Ship is not a regular pendulum but is a double pendulum. In other words, Ship does not have a regular frequency and amplitude but chaotic behavior. This behavior becomes more extreme for medium and large wave. In fact, it is well documented that double pendulum strongly depend on initial conditions. While this book is in the forefront of knowledge in this area, there are topics in a needed to be improved.

Containers are preferred mode (in many cases) of delivery in marine transport which are in high demand now. This transport suffers from the resonance problem in which containers are damaged or failed of the ship. To rephrase it, the rolling motion (the rocking of the ship) inherently creates a problem to ship's cargo. This topic is related to the general resonance (rotation) which is impacted by the rotation point (see elementary physic book). A simple pendulum motion requires fix rotation point and rigid body. The metacenter does not yield or require a moving hinge and fluctuating of moment of inertia. The metacenter approach leads indirectly to the "fact" of fixed rotation point and wrong one (at the metacenter). These issues are presented in this book. The publication of the complex double pendulum of floating bodies is published in this book.

What should one do when ship is about to capsize? Add weight or reduce the weight of the ship? This book utilizes the stability dome (that was create by this author) to explain this phenomenon. For the first time the stability dome is displayed in the "main hall" of the book. Again the for the first time, the comparison between stability of body laying on solid and body put in liquid are compared.

Dimensional analysis was applied to the ship stability and now a direct equation or graphs shows the right results for your ship size. No tables are needed for various ship sizes (compared to other books). The standard material preceding the stability like centroid, moment of inertia, buoyancy, are discussed but at a higher level of the physics. For instance, this discuss coordinate transformation effect on the centroid location.

In this book, there is a tension among several forces or audiences for this book. On the one hand, the book is intended for individuals who have a strong background in mathematics, yet lack basic understanding the physics of (floating bodies) stability. The stability analysis attracted many mathematicians who thought that solving wrong governing equation will yield good physical situations. As anecdotal point, many the authors of books which dealt with rolling (which is kind of pendulum) are not aware that the moment of inertia depends on the rotation point (elementary high school physics). On the other hand, the book is intended for ship architects and ship operators like ship mates and engineers (this author was a ship engineer (third engine officer) in his youth). Personal knowledge in the last group lacks the mathematical background, but has the physical instinct and/or intuition. There is a third group who is made of individuals with invested interest so much in the relic "science" that cannot believe that there or should be other way. Hence, this book tries to answer the first two groups and admittedly some parts of the book should be avoid by the operators (too much theoretical math and physics such as dealing with Lagrangian physics) and some others should be avoid by other groups. As for the third group, in time this group will be diminished and they

will eventually read the material and will understand the new material. The reader can check how flat Earth believer are still there.

Even this author sinned with believing some (even many) misconceptions that the field of ship stability is infested with. Well, it is hard to clean so many entrenched "scientific" facts. Note four notable "facts": rotation point hinged point (of the ship) is fixed, added moment of inertia is fixed (not depended on geometry), and the transfer mechanisms between the floating body movement modes are magical, the location of the rotation (it is not at the metacenter). One immediate effect is the understanding that ships are actually double pendulum which is depends strongly on initial conditions. Additionally, one of strange finding is the transfer properties beside the added mass/moment of inertia known before. This book is addressing all these challenges and more.

The reader would find material in this book to be revolutionary which was not published before. Besides the direct stability concerns, issues like how many corners are immersed in a rectangular shape in liquid at different angles are analyzed. Comparison of stability on solid vs stability in liquid is presented, which is lack of all other books on stability. The methods developed in this book also spill into geometry in which new methods to calculate the centroid for a circular segment was developed. A test the author has been given questions to various experts in the field which they failed to pass answer correctly. Even the experts who wrote books about ship stability, can benefit from this book to provide a fresh perspective of the new advances. While these statements sound dramatic or brazen, it is believed that this book will transform how the stability field will be approached in the future.

# Prologue For This Book

*Version 0.8.2 April 13, 2022*

**pages 283 size 7.4M**

Again this version is mostly focus on the added properties. There several realizations in this respect. First there is only a single added mass which a scalar for any body which value depends on the acceleration direction. The added force is used in calculating added. The added mass can be broken into three components in any orthogonal coordinate system because the added force can be broken into components. As all the division of vector by a vector create a second order tensor. The added force (vector) was divided by the velocity (vector) which later divided the acceleration (or in different order). This create a scalar with value which depend on the direction.

*Version 0.8.0 Jan 17, 2022*

**pages 278 size 7.3M**

The common believe equations should describe the ship navigation (or any marine navigative body) in the most general form. These equations should describe all the situations and take into considerations all the effects including the added mass (not only at the initial position) and the transfer mechanisms. Yet, none of the transfer mechanisms appear in any of the formations of the general common believe equations. The damping appears in these equations as a constant. The only exception is exciting forces which described by various models. Now consider a ship that moves at constant velocity with a  $30^\circ$  to its ordinate ( $x$ ). According to these equations, no force due to the added mass (zero acceleration) exist. Furthermore, if the viscosity factors are negligible (or nonexistent), the ship will stay on course for ever without any additional force. Clearly, this physical picture intuitively is not correct, the ship will rotate not because the added mass but due to the transfer properties.

This book conflict with so much was common believe. For example, starting with the cross section does not affect the stability and continue with no transfer properties. This author to keep his sanity asked many experts industry to find flaw(s) with this explanation. There were several individuals who discussed this material with this author. Seven answered the request with explanations that book is not correct because the establishment does not except (good answer! Have not seen this kind answer was given before?). The most common response was simply to ignore it. The question is how

long it will be ignored?

Again this version is another dirty version not well organized and worded, but working on it. So many advances so little time to correct the English and provide examples.

### *Version 0.7.5 Jan 17, 2022*

**pages 261 size 7.0M**

This version is the dirty version. That is, it is not well organized since the amount of discoveries is overwhelming. To make the point, just before closing this version, it was realized that there is additional mechanism for transfer from heave to rolling mode discovered. It seem that the angle in angle out creating so much change that scientists, who worked in the area, will not recognized the field any more. Viva to open content! One scientist was asked about the added mass concept she was familiar with it and worked with it. However, she admitted that never hear or thought about the transfer properties or similar concept. The discovery of these properties of the body and surrounding liquid is remarkable which explains part of the mechanisms of transfer from mode to mode. One of the issue that was not addressed is the deviation from the standard like trim and list effect on the stability. At this stage, these issues are a black box.

### *Version 0.7.0 Dec 14, 2021*

**pages 237 size 6.7M**

The greatest fear that this author has been the fear to introduce errors into the blood stream of his writing. This statement not aim at typos, arithmetic mistakes, poor English, and layout issues, and  $\LaTeX$  (there are still many errors in the style file) but the big conceptional mistakes. This book embarks on the layout the foundation on the stability of floating bodies. Naturally, there will be errors which hopefully will be corrected during the author's life time. It already happened but it is not enough. This author would not like to see that his work introduce omissions (like Bouguer's omission of the effect of the cross area) or error on the calculations of added properties or solving the equations of rotating body (ship) without knowing where is the hinge point (this error so elementary that one wonder if, those who wrote it, skip the elementary physics class.)  $I$  depends on the hinge location. Thus, in writing this book, points not clear to the author are mentioned and left to other to contribute. If mistakes are found, the new version discuss it and try to fix it as soon as possible before they enter the blood and contaminate it. Thus, if you read the book, please try to get the latest version. In this undersign book on fluid mechanics several mistakes (in ship stability sections) are currently present (they will be corrected after this book improved) (well, they are commonly believed truths, sorry for blaming others). As contribution to contamination prevention, surface tension is not created by inter molecular forces (read about the book "basics of the fluid mechanics").

*Version 0.6.5 Oct 23, 2021*

**pages 223 size 5.3M**

After the rotation point of the floating body was established it must be followed by issue of the rolling (resonances). From the six possible motions only three have any potential drowning effect on the floating body (heave, pitch, roll). Of the three motions, the rolling is the most important not only because the droning but also the resonance (economical losses and comfort of passenger ship). Obviously, the change in the rotation location changes the moment of inertia which prompted this author to look into this area. In addition, while not expert in the area of added properties<sup>1</sup>, notice misconception about this issue in the research paper on this area. Hence, the rolling chapter was born. The experiments chapter was long in consideration and was inspired by Dr. Robert Ettema's work or others how have done it..

While the several topics that planed to be covered like liquid and characteristic time effect on stability and other the book approaching to the maintenance phase like a work on the tcolorbox (example environment).

*Version 0.6.0 Sep 27, 2021*

**pages 203 size 4.5M**

Again, it is hard to believe how much one can achieve in a single book. Beside the visual enhancement to the book the dimensionless presentation of what other call GZ was constructed. History like to repeats itself. The great scientist E.R.G. Eckert wrote his heat transfer book where he introduced dimensional analysis to the field. He was mocked and ridiculed, but after 20 years the Eckert's approached is the only used today. A similar thing happened to this author in the Pressure Die Casting field. It is a question what will in this field.

Several examples were enhanced and correction of typos were done. Improvements of the visual will continue but mostly at this stage a discussion on the rolling frequency will be analyzed. At that stage the major work on this book will shift to maintenance (minor improvements). This author will (hopefully) shift focus to thermodynamics and heat transfer and especially focus on biological system.

*Version 0.5.5 Aug 22, 2021*

**pages 179 size 3.7M**

It is very proud moment to observe all the achievements that were done in writing this book. Normally, it is hard to see significance of the one book. However, it is moment one should be able to boast about amount of achievements. In fact, this is a field that is so outdated and with "facts" that are so entrenched with no challenges.

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<sup>1</sup>Added properties referred to the fact that bodies operating in heavier liquid (like a ship) cause resistance not by friction to the movement. This author never contributed to this area and simply took a class on this topic and was a Teaching Assistance for this class later.

For example, around what point does the ship roll? Fact, the entrenched answer is around the metacenter which is wrong. In this book, basically, the author discovered and demonstrated that the rotation depends on the cross section at the liquid line due to the new method developed in the book.

Furthermore, the fact that no one utilized Nusselt's technique is strange. Why did no one use the dimensional analysis in presenting the stability diagram? Is that presentation so hard? As a result, now the location of stable zones are clear. One indirect effect of the stability analysis provides straight and easy way to calculate the segment (part of circular) centroid. All the graphs showing the  $GZ$  in what is referred to sometimes as the right hand arm are smooth. This is another one of the entrenched misunderstanding of the stability field.

Still there are several topics that need clarifications that this book did not provide. It is hoped that others will start to provide significant work that merit inclusion in this book. For example, the dynamic stability analyzed in these days (2021) utilizing numerical calculations. It is not that the numerical calculations are wrong. It is said garbage in garbage out. For example, a paper (Hu, Ramlingam, Meyerson, Eckert, and Goldstein 1992) carried the author (previous) name (Meyerson, without permission). The correct thing in that paper is the experiments that this author solely had done and were stolen without permission. The model in that paper is wrong because it failed to take into account the right physical phenomenon. Is it the same for stability models present in the recent books? This author is not convinced, hence no section in this book can be presented on these topics. Yet, all the issues related to the safety regulations should appear in future.

*Version 0.5 June 4, 2021*

**pages 159 size 3.3M**

This author's introduction to ship stability was during 1974 where he was at Akko Nautical College a unique and special high school. The class was taught by a big man (size wise) as a main instructor who used to be captain and at time worked as psychologist. The first class was extremely interesting because his unique ability to handle boys<sup>2</sup>. However, because the class was after a war, it was carried in a condensed fashion (over 8 hours a day for two weeks) to fulfill missing material. The most of material taught was produced by the French Pierre Bouguer. For example, the numerical integration was done by the trapezoid method, the Metacenter etc. It is interesting that the instructor or his co-teachers did not know about the Bouguer himself.

Since that class, the author has worked on various stability<sup>3</sup> problems like the stability of thin film (small and large waves) in relationship to energy transfer in his Masters at Tel Aviv University. Later the author worked on various stability problems in compressible flow. Additionally, the author work on stability problems in multi-phase

<sup>2</sup>A very talented man in taking command of mischievous teen age boys. The author is ashamed of the things that his group used to do.

<sup>3</sup>Even actually doing every morning stability calculations for ship as part of his duty. Not really scientific but sheer fact of dealing with it.



flow. After the author went through these various stability problems, coming again to ship stability this author realized how old technology has been frozen in time (over 300 years). There were two attempts, that the author is aware, to bring to update the ship stability science. The first one from Paul Erdős and second from Lautrup. The first is a mathematician who did the mathematics correctly yet messed up some elements from a physics point of view. The second is by a Danish physicist who had too much respect to the old tradition, and yet he was able to renovate and focus on concepts like the stability diagram. This author will refer to diagram as Lautrup's diagram.

Perhaps the most amazing is the lack of dimensionless analysis of the stability of ships and other bodies. That is, what is the significance of  $GM=2m$ . Is it the same effect on a big ship as compared to a small ship? The ship size has a strong effect and dimensional analysis done only recently by several brave students from Egypt (Habib, Ali, and Nawar 2018). Nevertheless, proper analysis utilizing Nusselt's method was done in Fluid Mechanics book by this author for the first time.

This is the partial list of achievements in this book.

- The development of the angle in angle out approach for stability.
- The recognition that stability is a problem that deals solely with rotation.
- The proper usage of dimensional analysis within the stability analysis.
- The discovery of the effect of the change of cross area with the height on the stability.
- Construction of universal stability diagram for many fundamental geometrical shapes.
- Presentation of the  $GM$  in a dimensionless form for several geometrical shapes.
- Presentation of the correction gravity centroid needed to make the body stable.
- Discuss the differences between stability on solid and liquid
- Point to interdependence of the different rotations. In other words, the mechanism of the transfer for one motion to another was explained.
- Drive the relationship between the inclination angle and rotation point, and its derivative was obtained.

After reading this book one can answer questions that before one could not. For example, if you are on a boat which is about to be roll what should you do? Remove items and throw them overboard, or add more items like filled out buckets full of water, and why? If you were using all the books you could not answer this question.

This book is the fourth book in the series of POTTO project books (not counting the world large gas dynamics tables). POTTO project books are open content textbooks so everyone is welcome to join in. The topic of stability mechanics was chosen because the discovery or the development of the direct methods. And the fact, this author wanted to write a book that will be used by his old high school.

This book is written in the spirit of my adviser and mentor E.R.G. Eckert. Eckert, aside from his research activity, wrote the book that brought a revolution in the education of the heat transfer. Up until Eckert's book, the study of heat transfer was without any dimensional analysis. He wrote his book because he realized that the dimensional analysis utilized by him and his adviser (for his post doc), Ernst Schmidt, and their colleagues, must be taught in engineering classes. His book met with a strong criticism in which some called to "burn" his book. Today, however, there is no known place in world that does not teach according to Eckert's doctrine. It is assumed that the same kind of individual(s) who criticized Eckert's work will criticize this work. Indeed, the previous book, on compressible flow, met its opposition. These criticisms will not change the future or the success of the ideas in this work. As a wise person says "don't tell me that it is wrong, show me what is wrong"; this is the only reply. With all the above, it must be emphasized that this book is expected to revolutionize to some degree the field and change some of the way things are taught.

This book is written for several groups like high school students who study ship stability, sea officers (even skipper), and for university students, hence the additional material is required to satisfy these conflicting interest.

The book is organized into several chapters which, as a traditional textbook, deals with basic mathematics, than fundamental of mechanics, and fluid statics. Later several chapters will be dealing with various aspect of the floating stability.

# CONTRIBUTORS LIST

## *How to contribute to this book*

As a copylefted work, this book is open to revisions and expansions by any interested parties. The only "catch" is that credit must be given where credit is due. This *is* a copyrighted work: it is *not* in the public domain!

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## *Credits*

All entries have been arranged in alphabetical order of surname, hopefully. Major contributions are listed by individual name with some detail on the nature of the contribution(s), date, contact info, etc. Minor contributions (typo corrections, etc.) are listed by name only for reasons of brevity. Please understand that when I classify a contribution as "minor," it is in no way inferior to the effort or value of a "major" contribution, just smaller in the sense of less text changed. Any and all contributions are gratefully accepted. I am indebted to all those who have given freely of their own knowledge, time, and resources to make this a better book!

- **Date(s) of contribution(s):** 1999 to present
- **Nature of contribution:** Original author.
- **Contact at:** genick at potto.org

## **Steven from [artofproblemsolving.com](http://artofproblemsolving.com)**

- **Date(s) of contribution(s):** June 2005, Dec, 2009
- **Nature of contribution:**  $\LaTeX$  formatting, help on building the useful equation and important equation macros.
- **Nature of contribution:** In 2009 creating the `exEq` macro to have different counter for example.

**Dan H. Olson**

- **Date(s) of contribution(s):** April 2008
- **Nature of contribution:** Some discussions about chapters on moment of inertia and correction of English.

**Richard Hackbarth**

- **Date(s) of contribution(s):** April 2008
- **Nature of contribution:** Some discussions about chapter on moment of inertia and correction of English.

**Kosamo Mazer**

- **Date(s) of contribution(s):** June 2021
- **Nature of contribution:** found mistake in  $A - > G$  points, thank you
- **Contact at:** private

**John Motanizitz**

- **Date(s) of contribution(s):** August 2021
- **Nature of contribution:** major correction of typos, thank you
- **Contact at:** private

**Jimi Tan**

- **Date(s) of contribution(s):** Sep 2021
- **Nature of contribution:** questions about stability, thank you
- **Contact at:** private

**Ahsan Ibn Muhammed Ibn Yusuf**

- **Date(s) of contribution(s):** Oct 2021
- **Nature of contribution:** Clarification of the rolling chapter, and general discussion on what should be presented in a class on stability. Thank you very much
- **Contact at:** private

## Various Rolling researchers

- **Date(s) of contribution(s):** Sep-Oct 2021
- **Nature of contribution:** to all the researchers (over 10 individuals) who wrote a paper (or more) about the rolling rotations and who were approached by Potto Project and were asked about the ship rotations point. While none of them did considered an actual rotation point, the sheer fact that they were willing to communicate is greatly appreciated. This prompted the writing of the chapter on rolling. Again, sincerely thank you. Without these answers, it was not realized how important this issue.
- **Contact at:** private

## Galina Yakovlevna Dynnikova

- **Date of contribution:** March 25, 2022
- **Nature of contribution:** Ask questions about added mass which further focus the explanation.
- **Contact at:** researchgate.net

## Your name here

- **Date(s) of contribution(s):** Month and year of contribution
- **Nature of contribution:** Insert text here, describing how you contributed to the book.
- **Contact at:** my\_email@provider.net

## Typo corrections and other "minor" contributions

- **I. P. Loampo**, June 2021, pointed to specific typos and, helped with questioning the logic of the presentation.
- **Tousher Yang** April 2008, review of statics and thermo chapters (remain from fluid mechanics book)
- **Alessandra from Brazil** June 2021 questions several points.



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# Nomenclature

$\alpha$	The resulting angle for tilde, see equation (4.12), page 39
$\beta$	The large permanent angle, see equation (4.18), page 41
$A$	The rotation (fix?) point at liquid surface, see equation (4.18), page 41
$\ell$	Length, see equation (7.1), page 72
$\omega_0$	natural frequency, see equation (11.3), page 136
$\omega_W$	disturbing wave, see equation (11.3), page 136
$\rho$	Density of the body, see equation (4.4), page 29
$\rho_\ell$	The liquid density, see equation (8.4), page 97
$\rho_s$	The solid density, see equation (8.4), page 97
$\rho_{fresh}$	Fresh water density, see equation (2.2), page 10
$\rho_{sal}$	Sea water density, see equation (2.2), page 10
$\theta$	The imposed angle of tilde, see equation (4.12), page 39
$A$	Cross area or area in general, see equation (4.4), page 30
$A_M$	Midship wetted area of the cross section, see equation (2.7), page 12
$A_r, A_o$	Area removed and added, see equation (4.8), page 31
$A_{ls}$	Liquid surface area, see equation (2.11), page 14
$B$	Buoyancy centroid, see equation (4.18), page 41
$b$	The base of the body, see equation (4.9), page 31
$C_m$	Maximum section Area Coefficient, see equation (2.8), page 12
$C_{DW}$	Dead weight coefficient, see equation (2.4), page 12
$C_{ls}$	Liquid surface coefficient, see equation (2.11), page 14
$D$	the distance to point $G$ , see equation (8.4), page 97

$d$	Draft of the body normally underwater, see equation (4.9), page 31
$d$	the distance to point $B$ , see equation (8.4), page 97
$F$	The area of surface, see equation (6.1), page 61
$Fr_D$	Froude Displacement Coefficient, see equation (2.15), page 15
$g$	The gravity constant, see equation (7.2), page 72
$g(\theta)$	Added mass function based on the geometry., see equation (11.7), page 137
$GB$	The distance between the buoyancy centroid and the gravity centroid, see equation (8.4), page 96
$GM$	The distance between the gravity centroid to Metacenter point, see equation (9.6), page 114
$h$	Height in the pressure chapter, see equation (7.7), page 74
$I_{xx}$	Moment of inertia, see equation (5.1), page 49
$m$	Mass of the body, see equation (4.1), page 28
$M_D$	Damping function for rolling, see equation (11.2), page 135
$M_R$	Returning Moment for rolling, see equation (11.2), page 135
$M_W$	Exciting (Activating) Moment for rolling, see equation (11.2), page 135
$P$	The area of surface, see equation (6.1), page 61
$P_{atmos}$	Atmospheric Pressure, see equation (6.5), page 65
$R$	The big radius, see equation (4.9), page 31
$r$	Coordinate in $r$ direction, see equation (3.17), page 23
$t$	thickness, see equation (7.7), page 74
$V$	Volume of the body, see equation (4.1), page 28
$V_0$	The volume under the liquid, see equation (8.4), page 96
$V_d$	Encompassing volume, see equation (2.1), page 10
$x$	Coordinate in the base direction, see equation (3.0), page 18
$y$	Coordinate in the other direction, see equation (3.0), page 18

# The Book Change Log

## *Version 0.8.2*

### **April 13, 2022 (7.4M 283 pages)**

- add the added mass as scalar.
- Build/added subappendix
- make macro for emphasis.
- English and typo
- change many wording (either because poor english simply not correct).

## *Version 0.8.0*

### **March 17, 2022 (7.3M 277 pages)**

- Add the added mass chapter
- Several new examples.
- English corrections
- Some  $\text{\LaTeX}$  cleaning (not yet there more work is needed)

## *Version 0.7.5*

### **Jan 17, 2022 (7.0M 261 pages)**

- Add example of the density on ship's moment of inertia
- Start the explanation of the roll to pitch and heave to roll.
- Improve the introductions
- English corrections
- Fixed some typos and misconceptions that reflected in the introduction chapters
- Discover who introduce the error of the rotation point.

- Discovery of transfer properties.
- Discovery of transfer mechanism due to the stability from heave to rolling.

### *Version 0.7.0*

#### **Dec 08, 2021 (6.4M 225 pages)**

Not as usual the missing material will be mentioned which is the actual governing equation(s) of the rolling of the ship. After adding that material the book will be to maintenance only mode.

- Added the Lagrangian calculations for double pendulum
- Added chapter on the rolling
- Added chapter on the experiments
- Change the moment of inertia to the longtable format to allow additional material without fighting the layout.
- English corrections
- Fixed some typos and misconceptions that reflected in the introduction chapters
- font change for  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{G}$ , and  $\mathcal{M}$  as  $\LaTeX$  macro regardless of math or text mode

### *Version 0.6.5*

#### **Sep 26, 2021 (5.3M 209 pages)**

- Some English corrections
- Build the base for the Rolling and experimental chapter
- Change of area derivative formulation base
- solutionM  $\LaTeX$  macro

### *Version 0.6.0*

#### **Sep 26, 2021 (4.5M 205 pages)**

- Many typos correction
- exBox programming almost completed (the color example now in the book with name of the example and difficulty level.)



### *Version 0.5.5*

#### **Aug 22, 2021 (3.9M 179 pages)**

- Added the chapter on the definitions.
- Added the section on  $GZ$  diagram.
- Added the explanation to change of coordinate system for finding centroid.
- Added the section dealing centroid of circle segment utilizing the stability new method direct investigation.

### *Version 0.5.2*

#### **July 1, 2021 (3.5M 167 pages)**

- English and layout corrections
- Add nomenclature.

### *Version 0.5.1*

#### **June 25, 2021 (3.5M 163 pages)**

- English and layout corrections
- More examples and some corrections to the content.

### *Version 0.5*

#### **June 15, 2021 (3.3M 159 pages)**

- Initial math/calculus geared to stability understanding,
- A small revolution in floating body stability)
- New change centroid location for added or subtracted masses
- Chapter on moment of inertia
- Chapter on Archimedes' Law
- Chapter on 2-D stability
- Chapter on GM
- Chapter on Variable Gravity Centroid



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# How This Book Has Been Written

## *How cooperation work*

Anonymous user contact this undersign and asked about question that appear in Rawson's book (Rawson and Tupper 2001). The question is a clear and standard material that should be discussed in the understating stability. The new thing is the fact how the new understating affects solution. The actual rotation point at point  $\mathcal{A}$  and not point  $\mathcal{M}$  and hence moment of inertia has error of up 400%. Therefore, obviously, Rawson's calculation of the frequency also has a significant error. This point is discussed here not to attack Rawson, the book which is the best according the what was believed at that time, but to demonstrate the change in the knowledge due to the current approach.<sup>4</sup>

## *The Smart-ass Questions*

As a typical phenomenon every time this author gets into a new field questions from individuals who try to challenge the intellect of this author are asked (Wow, why people repeat the same approach?). Basically the individuals claim that because it was done in a certain way for over 300 years and with million of dollars in research it must be correct. Similar things were said for the other advances that this author has made in the past. Thus, these remarks just increase the interest into the investigate. The investigation into the rolling was prompted indirectly by these kind of questions.

## *More info*

This author's initial research original interest was not focused on ship (bodies) stability. All colleagues of this author are not in the area of ship stability but multi-phase flow, heat transfer etc. Yet, when writing stated, this author approached several individuals who have worked in this area from U.S.A. None of them was cooperative but yet all have been polite (probably they heard about Bar-Meir), but strangely considerable email has come from the far east and the middle east (why?). This stream of questions led to the additional material which is about 40 pages.

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<sup>4</sup>This author was not groomed by the stability community and hence does not aware of many books and works. This work was "discovered" due to this question from anonymous user. There were other questions that this author cannot answer at this time.

## *Initial*

This book started because I found or developed a new way, after hundred years, to calculate the stability limits. While Simpson's role was penetrating to the stability field displacing the trapezoid integration there still ancient approach like lake of dimensional analysis approach which must be updated. Perhaps, this book is written by this author, who has non conforming and none traditional approach, will move the field to modernity (as 2021) as a result of writing this book.

The book started as to attempt to teach my kids experimental physics in hopes that they will like and want to be engineers. This attempt failed to get them to be interested in engineering yet this book was born.

The famous Eckert converted the field of heat transfer to a dimensional(less) science. Hence, it hope that tables (using big ships, small ship etc.) that appear in ship stability books like (Derrett and Barrass 1999; Biran and Pulido 2013) should be located only in library learning about the history of the issue.

There are several books on stability of floating bodies but none them is open content. The approach adapted in this book is practical, and with more hands-on approach. This book indented for many different and conflicting audiences and that creates a difficulty in writing a book. So, issue of proofs and so on are here only either to explain a point or have a solution of exams. Otherwise, this book avoids this kind of issue.

The structure developed in the author's book on fluid mechanics was adapted and used as a scaffolding for this book. This author was influenced by material that he was taught in high school, but this material will be ignored in this textbook. The chapters are written in an order that assume reader has little background knowledge.

The presentation of some of the chapters is different from other books because the new ideas and usability of the computers. The book does not provide the old style graphical solution methods, yet provides the graphical explanation of things.

Of course, this book was written on Linux (Micro\$oftLess book) using the vim editor for editing (sorry, never was able to be comfortable with emacs). The graphics were done by ipe, a new graphic program that this author is learning to use (and which is user friendly with poor documentation yet there no better package.). The figures were done by GLE. The figure in cover page was created by Genick Bar-Meir, and is copyleft by him.

# Preface

"In the beginning, the POTTO project was without form, and void; and emptiness was upon the face of the bits and files. And the Fingers of the Author moved upon the face of the keyboard. And the Author said, Let there be words, and there were words."

5

This book, *Stability of Ships and Other Bodies*, describes the fundamentals, when and why floating bodies are stable. In addition, it describes steps that transform unstable bodies to stable ones. This book is designed to replace all the other books and inseminate and integrate the recent developments, technology and advances. The material in standard books is so entrenched, archaic, and outdated material that one can be only amazed. For example, concepts like potential stability are not discussed or even mentioned in any of the books that this author reviewed. One of the recent subject study is the resonance of rolling. One should expect that at the very minimum the location of the rotation should be known or well defined. However, none those who work in the area and was asked about it, knew what is location of rotation point (hinge point). This misunderstanding has led to search work that study "pendulum" without knowing where the rotation point. The assumption is so cardinal that simple calculation leads to error of hundreds percents.

Stability naturally makes itself subject to dimensional analysis examination. There are no extraordinary situations like multi-phase flow. Yet, only one attempt was made to introduce it to stability analysis. This book has a whole chapter on dimensional analysis.

Stability analysis is a well established topic in engineering, physics, economics. The technique is done by introducing error and looking if the error grows or decay. Bouguer found obscure and a different technique because at his time the regular standard was not available or known. Now when the regular standard approach known, there is no reason to avoid use this well established technique. Especially when this "new" technique has captured major stability issues that were not known because of the metacenter technique.

This book was written from a physical point of view rather than of this what the safety committee decided like all the books reviewed by this author on stability. It is hoped that the book could be used as a reference book for people who have at least some basics knowledge of science areas such as algebra, basic physics, etc. Even without deep understanding, the graphs in this book can be used to find necessarily quantities needed to understand or to fix or operate your vessel.

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<sup>5</sup>To the power and glory of the mighty God. This book is only attempt to explain his power.

The structure of this book is such that many of the chapters could be usable independently. I hope this approach makes the book easier to use as a reference manual. However, this manuscript is first and foremost a textbook, and secondly a reference manual only as a lucky coincidence.

I have tried to describe why the theories are the way they are, rather than just listing "seven easy steps" for each task. This means that a lot of information is presented which is not necessary for everyone. Many in the industry have called and emailed this author with questions since this book is only source in the world of some of the information. These questions have led to more information and further explanations which are not available else where.

This book is written and maintained on a volunteer basis. Like all volunteer work, there is a limit on how much effort the undersigned was able to put into the book and its organization. Moreover, due to the fact that English is my third language and time limitations, the explanations are not as good as if I had a few years to perfect them. Nevertheless, I believe professionals working in other engineering fields could benefit from this information.

I have left some issues which have unsatisfactory explanations in the book, some marked with a Mata mark (still under work). I hope to improve or to add to these areas in the near future. Furthermore, I hope that other individuals will participated in this project and will contribute to this book (even small contributions such as providing examples or editing mistakes/typos are needed).

I have tried to make this text of the highest quality possible and am interested in your comments and ideas on how to make it better. Incorrect language, errors, ideas for new areas to cover, rewritten sections, more fundamental material, more mathematics (or less mathematics); I am interested in it all. I am particularly interested in the best arrangement of the book. If you want to be involved in the editing, graphic design, or proofreading, please drop me a line. You may contact me via email at "barmeir@gmail.com".

Naturally, this book contains material that never was published before (sorry cannot avoid it). Actually it is the main trust of the book showing the new way to do old things. Brazenly putting it in bold words, this book contain material that was never published before and does not adhere to social appeasing. This material never went through a close content review. While close content peer review and publication in a professional publication is excellent idea in theory, In practice, this process leaves large room to block of novel ideas and plagiarism. In fact, many ideas in the book intimidate many who has vast interest to keep them hidden. Image that you spend years and millions of dollars just to find that your research is based on erroneous concepts. It is naturally believed that those individuals will attempt block any regular channel publication (this author has experience it in the past). If you would like be "peer reviews" or critic to my new ideas please send me your comment(s). Or like many others, this author's book got many comments on many web sites. Some of these comments praise and detract from the books. All comments are welcome.

Several people have helped me with this book, directly or indirectly. I would like to especially thank to my adviser, Dr. E. R. G. Eckert, whose work was the inspiration

for this book. His approach to dimensional analysis was a key in understating concepts like what dimensional groups effect the stability but also the presentation.

I encourage anyone with a penchant for writing, editing, graphic ability,  $\text{\LaTeX}$  knowledge, and material knowledge and a desire to provide open content textbooks and to improve them to join me in this project. If you have Internet e-mail access, you can contact me at "barneir@gmail.com".





# 1

## Introduction to Stability

### *1.1 What is Stability of Floating Bodies?*

The prerequisite condition for stability of floating bodies regardless to usage is that the body (ship) has to float. This condition is controlled by Archimedes's law or principle. The stability of floating does not have vertical stability issue unless it is in a small scale or the density is almost transitional<sup>1</sup>. That is, once the body is floating, unless the body density is changed, there is no vertical stability concerns<sup>2</sup>. In other words, pushing or pulling the body in the vertical direction will not change the fact of the body's stability. The fact that ship has limitations on available flotation (how much ship can load before it will sink) is not related to stability. In fact, it will be shown later in the book that the ship become more stable when ship closer to fully loaded situation. This limit related how much liquid the ship can absorb before it is too heavy to float (see for more information in chapter 2). This discussion refers to the normal stability without exciting (exterior) forces. Yet in version 0.7.5 it was discovered that when the heave movement exceed the stability limit due to change in the density ratio, the body becomes unstable. This is a special case.

However, under what conditions the angle of a floating body will be in equilibrium is the question of ship or body stability. Floating bodies do not sink if their density is below the liquid density because lack of force or moment that can cause it. On the

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<sup>1</sup>Lautrup suggested this possibility.

<sup>2</sup>There are those who claim that there are issues with vertical stability. For example, an Australian mathematician (his name isn't mentioned) wrote a paper on rolling, that he alleged that it is clear all six motions are important. He claimed this strange "fact" because he does not understand physics. His paper is dealing with pendulum like situation/scenario yet he does know where is the rotation point of pendulum. This author would like see even one issue/case vertical stability issue.

contrary, when the body is pushed down below the stability point there is a net force that pushes the body up and vice versa. However, the body is placed on the liquid, only under certain conditions will the body remains in the initial position. It was shown in this book that floating body has to be above a certain width to be stable. For example, putting an extruded wood square in water straight up will results in a tilt to  $45^\circ$  (try it). Simply stating, the square is not wide enough.

A body placed over a solid surface will remain in the same conditions if the body is placed only, at minimum, when the center gravity is over the base. The situation with solid surfaces is very intuitive as most readers have experienced it for example, an extruded square has 4 possible angles of stability (for the 2D bodies). As almost always, the stability points (angles) are complicated to determine and to predict when placing bodies on liquid. The stability points (angles) on the solid surface are different from those for the same body on liquid. Even the number of stability points can be different between solid surface and liquid surface. For example, an extruded T-shape body can have 4 stability points on solid surface (orientation) while in liquid it can have only two stability points. As opposed to the solid case, in liquid the ratio of the densities affects the stability. The stability for a certain shape can be checked by modifying the geometrical parameters (or their ratio) of certain shapes for various density ratios. The experience has shown that the stability can be plotted as ratio of geometrical parameters for different density ratios as shown in a typical stability diagram (see Fig. 1.1).

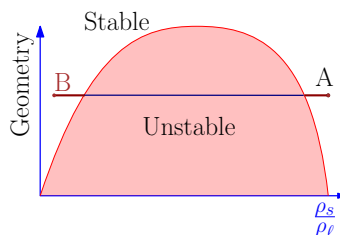


Fig. 1.1 – Typical stability diagram for solid bodies like extruded rectangular.

It is amazing that for a certain geometry the ship or floating body is stable (see point **A** in the Fig. 1.1) at one point and if the density ratio changes it becomes unstable (the line enters into the dome) and if further change in the density ratio occurs it becomes stable again (the line leaves the dome). This point is not obvious (at least it was not clear to this author when he took ship stability class in high school.<sup>3</sup>). In fact, ships have different scales for salt water and fresh water (they have different densities, and they should have different scales for different temperatures). This diagram is introduced (in Chapter 2). While this diagram is not related to the stability issue it provided the understanding of the power the density ratio has.

This book is organized by building on the foundations, like what is a centroid and how to calculate it. Additional important fundamental concepts will be introduced: the

<sup>3</sup>It is a good assumption that the teacher had no clue about this issue. In fact, a discussion with another teacher (head teacher) simply demonstrates it is the prevailing belief.

moment of inertia, pressure. The buoyancy will be followed and analyzed. Later, these ideas will be used in the calculation of the ship stability (and floating body and in this book the word "ship stability" should be interpreted as floating bodies as well.) As all the books by this author, the books starts with history of ship stability.

Like the die casting industry, in the maritime industry the problems looks simple but it is deceiving. Furthermore, it is full with many "scientific" facts. In the course of the year 2021 writing this book, this undersigned change his opinions on many topics in ship stability. The stability criterion and understanding are not intuitive, clearly it was not for this author especially after his class in high school on ship stability. Ships have a moving rotation point which creates a giant double pendulum. Furthermore, the ship's added moment of inertia depends on the angle (the added moment of inertia) and because the ship's "hinge" is floating. These facts make the situation very complicate that even the governing equations were not established. In turn these complications make the writing very complicated. In actuality, there are so many discoveries in this book that it make difficult for one who found them see the simplicity. For instance, the discovery of the transfer properties that could affect other area as well. On one hand, basic topics like centroid have to established and other hand topics like Lagrangian calculations have to be presented. Thus, even in this chapter, complications presets difficulties especially with varied audience. Thus, if you read something that is out of scope for you just skip that section (high school should skip the chapter on rolling).

## 1.2 History of Ship Stability

### 1.2.1 Original History

The history of the stability analysis is reflective of general physics and fluid mechanics science in particular. A good summary is given by (Nowacki and Ferreiro 2003) which lacks major developments that have occurred in the last 30 years. Another source on history of ship stability was written from a stronger historian point of view even though the first author has undergraduate engineering background (Ferreiro and McGee 2006). Some technical details are missing (historians care about other things than engineers). The highlights of stability analysis research show that it was an important topic for a long time. Clearly having a ship that does not flipped at sea (or other body of water) was important since the early times. The test was done by some individuals moving on the floating body to examine how the ship stable has. The real understanding to more advanced mathematics of the stability is tied and fluid mechanics, without which there was no ability to examine this issue. For example, Archimedes did not know about the concept of pressure, hence he lacked a major tool in his understanding. The early work was done by Huygens (1629-1695), (Huygens

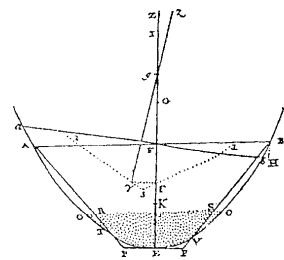


Fig. 1.2 – Bouguer Showing Metacenter.

1967) by that time he knew about the concept of pressure and some knowledge of early calculus. Even the concept of “specific gravity” (density) was introduced by that time (density was introduced 1586 by Simon Stevin). Stevin also discovered that the forces (gravity and buoyancy) have to act in the same line as a prerequisite for stability. French mathematic Paul Hoste, (1652-1700) made attempt to tackle the stability problem, but failed because he did not know calculus. The famous case of the Swedish flag ship **Vesa** shown on the cover page during her maiden voyage was turned to a no-return-point and eventually sank. The king's, Gustavus Adolphus, demands on ship's cannons were so high and heavy that it was ridiculously absurd showing the contempt people had to science.

Euler was requested by the Russians (at the time he was Russian Tzer children's tutor, what a lucky students) to review the work of La Croix work (Euler 1735; Euler 1736). As usual, money was the reason pushing the science forward. That was the age of discovery and the ability to project power especially with a marine power was essential. Ferreiro claims that research was driven by the academia without a reason. There are evidences that scientists were in a stiff competition for fame and believing that these topics will be important (meaning money). Thus, it seems that financial reasoning was the motivation. During that era the ship's gunport was developed. The need to find the water (liquid) line and maximum turning point before water got into the ship were important. Hence the importance of developing the science behind the stability.

Pierre Bouguer French hydrologist (fluid mechanics) got his father royal professor post at the age of 15 after his father pass away (must have been a very smart kid). He improved the numerical integration methods (he improved the trapezoid method that other suggested)<sup>4</sup>. Later he derived the Metacenter concept (Bouguer 1746) see Fig. 1.2. This Metacenter method is the most used method today. Yet, when one tries to use it, it is found to be complicated and graphical representation (or numerical modeling) is commonly required. Perhaps, the intrinsic deficiencies of this method is that it is confusing as to where the rotational point is and fails to capture the effect of liquid line cross section area change on the stability.

As a result, another method namely, the potential/energy principle is or could be used. In this method the energy or the potential of the system is written and utilized to find stability points. This technique was first proposed by Huygens and again because lack of calculus developed at that time he failed to work out the technical details.

Paul Erdős et al was the first (this author is not aware of any who else worked to solve this problem utilizing potential energy) to have used this approach successfully (Erdős, Schibler, and Herndon 1992). Amazingly the authors were not aware the centroid calculations were a well established topic and used complex integral calculations to find the centroid of trapezoid (and these calculations were done in 1992!). Additionally they have made some nonessential assumptions which Mohammad Abolhassani was able to fix. The calculations of centroid were not explained in the last paper (Abolhassani 2004). The potential method will be explained briefly later on. This approach

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<sup>4</sup>This method is widely used in stability study even though there are simpler and better methods like Simpson's rule or by its other name?

utilizes mathematics without the ability to see or examine what cause what and why. In a way the method abstracts the physics and converts it into a pure mathematical creation. The method seeks to find the angle(s) in which the vertical distance between buoyancy centroid and the gravity/mass centroid is shortest. Numerous mathematical papers (dealing with the mathematics) were published later dealing with abstract. It is the opinion of this undersign that many of these papers are without any real meaning to the stability of floating body. It is interesting to point out that because the lack of physical observation ability or because the underlined the equilibrium analysis it was assumed that it is a dimensionally compartmental. For floating bodies the stability is compartmental under very unique cases where the body is symmetrical and extruded. For example, using marine terminology, roll rotation creates yaw and pitch rotations because of the change of centroid location in  $x, y$ , and  $z$  directions.

The newest approach is the Direct Examination approach and it is suggested by this undersign. The Direct Examination method is probably the closest to Metacenter method.

**Example 1.1: Why  $G$  is above  $B$** **Level: Easy**

In the illustration 8.2 depicts  $G$  above  $B$ . Explain why at equilibrium stage the  $G$  and  $B$  must be in the same vertical action line.

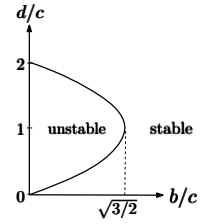
**Solution**

One of the favorite question that this undersign brings to engineers. Assume that  $G$  is not the same vertical action line as the  $B$ . In that case, a moment is created and the body will rotate until  $G$  and  $B$  do so on the vertical action line.

### 1.2.2 Recent Developments

Some developments have occurred recently will be presented in this section. Among these recent developments of the ship stability is the

stability diagram/dome. The stability diagram is an important tool showing the floating stability of a geometrical body. The investigation of history this diagram is not conclusive. While it is too personal for this author, as he built these diagrams and he was sure that he is the first (it is a disappointment to find that someone invented them head of you). However, it seems that the credit should go toward to Lautrup (Lautrup 2011). While the coordinate system presented in Lautrup book is different (and there is only one diagram) but the concept is somewhat similar. Lautrup did not realized that this is a typical diagram and it can be built for different geometries and maybe he did not attempt to build one for other geometries. Another advance Lautrup should be credited for is the solution of the principle axis.



Stability diagram for the floating block

Fig. 1.3 – Lautrup diagram with strange notations. Notice that coordinate should switch. Also instead of the density ratio, the total  $d$  over  $D$  is presented. (in his terms  $d/c$ .)

Almost the investigation (research) on the ship stability was under the assumption of compartmental effect<sup>5</sup>. Perhaps, the only transfer mechanism discussed in the literature is the mechanism of the moving wave along  $x$  coordinate which makes rolling motion (but these special cases that are discussed later). For example, the potential energy implemented by Erdős (Erdős, Schibler, and Herndon 1992) worked under the assumption of compartmental effect. This separation is not correct and was demonstrated in (Bar-Meir 2021). It is interesting to point out that Bouguer, Bernoulli attempted to solve the roll, pitch, and heave iteration (Ferreiro and McGee 2006). They raised the white flag and claim science has failed them. Later, the connection of the roll–pitch was investigate (Sayed and Hamed 2011). However in the said investigation presents no physical mechanism presented show how a disturbance transferred. Influence of Bouguer, Euler etc was so great that error that they introduced of rotation around the metacenter has remained for 300 years. Without knowing the rotation point of the ship, it is impossible to write the governing equation the floating body. Thus, the chief problem of the metacenter is abstraction of the rotation point. While it can be understood why Bouguer, etc, were confused, it bewildering why it is still to this day the prevailing opinion. The only one beside this author realize it was Lautrup. Yet Lautrup did not appreciate the meaning and furthermore he also have written things that conflict this statement.<sup>6</sup> In all these works there is imaginary mechanism which responsible the transfer from pitch to roll and vice versa. In this book for the first time (at least the author is not aware about anyone else) describe these mechanisms and

<sup>5</sup>If you aware of such non compartmental work, please notify this author.

<sup>6</sup>This author was involved several times in areas that were plagued with misconceptions. For example, the critical velocity shot sleeve in die casting was solved by the author 1996 but remained hidden until recently because commercial reason. Over 100 papers, since then, have been published. None of these papers solved the problem and the papers were plagued with errors like assuming turbulent flow where slip flow exist (no turbulence what so ever) and wrong boundary and initial conditions. It is to teach you that even common “true” have to be questioned.

methods to calculate them.

Beside the potential energy, this undersigned developed the Direct Examination method which is simpler to understand and implement (Bar-Meir 2021). Additionally, the author introduces the dimensional analysis to this area. There are quite new advances in this book and recently developed and these advances will be discussed below. The main advantage of the direct method is that it force to examine all the major conventions.

### 1.3 Rolling Rotation

Any body has six possible movements which have special names relative to the body (ship) communally used in the marine industry. The most dangerous and important because commercial damages is the rolling which refers to rotation around the small dimension of the ship (in other words rotation around the long axis). The complicate question what happened to a body under rotation not in the principle axis really was not addressed yet because lack of understanding (angle to  $x$  coordinate, for example.).

The damages created by rolling rotation are exhibits in the example of Fig. 1.4.



Fig. 1.4 – Containers falling from a their location on a ship after rolling rotation.

These damages are very substantial (according many sources). According to research conducted by the World Shipping Council, on average 1,390 shipping containers are lost at sea each year, whilst in 2019 there were approximately 16,000 lost cargo containers. These numbers are only an approximation and the number of damaged (damaged is more than lost) containers are higher. Regardless, of this specific information (it is not scientific and has a very small amount of material to verify facts), it is estimated the majority are due to the rolling rotation (not clear as of writing this section). This information is considerable enough to warrant an examination of lost and damaged containers.

Rolling is extension of static stability which was covered earlier. The metacenter method is almost exclusively used by researchers in this area (this author did observe even a single publication utilizing any other method). The weaknesses of the metacenter method are abstracting and the prevailing belief that the ship's rotation point is at the metacenter which can be believed to be fixed for small angles. That is, the rolling rotation is like a giant pendulum movement. Clearly, if this assumption was correct then at first glance the equation used in pendulum should be applied. Even this author sinned with this erroneous assumption. The literature survey shows that the oldest

publication found is from 1939 (Rahola 1939)<sup>7</sup> and the work referred to as Rahola's criteria. These publications are relatively recent (Francescutto 2007).

The next issue is to find the energy source or in other words the external (exciting) force or moment that causes the resonance (Spyrou and Thompson 2000; Bulian, Francescutto, and Fucile 2009). This physical situation had to be converted to a mathematical model. As always engineers when the situation cannot be modeled or solved, they invent a safety factor relative to something that is known at the time. Thus, the weather conditions are based simply on the  $\mathbf{GM}$  with a coefficient that is obtained from the formula Eq. (1.1) (Francescutto, Serra, and Scarpa 2001). The roll-back angle is written as,

$$\phi = 109 k X_1 X_2 \sqrt{W_e W_s} \quad (1.1)$$

The roll-back angle is a kind of ship twisted angle. Where the terms  $k$  and  $X_1$  and  $X_2$  represent the damping (not clear how? it is based on Newman book Marine Hydraulic, which are based on Buckingham Dimensional Analysis which makes this author very suspicious of their scientific validity.). While  $W_e$  is the waves effective wave slope,  $W_s$  is the regular wave steepness at resonance frequency. The 109 coefficient was chosen to match the data. This information associated with the Fig. 1.5. This weather criteria was improved but the change does not seem to be significant and the main point demonstrates that rotating point was ignored.

Galeazzi et al (Galeazzi, Blanke, and Poulsen 2012) have noticed that there is relationship between the various six movements of the ship<sup>8</sup>. They also speculated that it results from the change of the buoyancy centroid which was corrected but the explanation did not appear. However, they did not connect the change to any physical effects. This oversimplification lead to improvements which depend on the dynamic conditions (in the literature it is referred as operational conditions) (Petacco and Gualeni 2020). One of the typical theory was a wave propagates along ship ( $x$  coordinate) and hence changes the buoyancy point and hence makes the ship vulnerable at specific frequencies. According to this idea, waves with a wave ship length (or some fraction (1/2, 1/4, etc) of the ship in such away that a wave crest pears one per ship length cause a change in  $\mathbf{GM}$  (change in the metacenter). The change in  $\mathbf{GM}$  causes the ship to have a varied stiffness depending on the wave crest location. The arrangement is such that ship will over respond in part of the cycle and will under respond in the

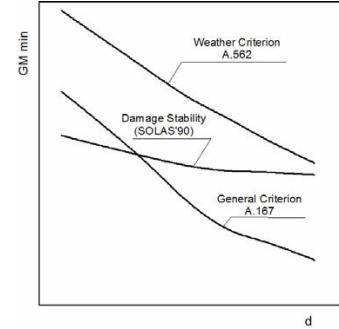


Fig. 1.5 – The Correction for the weather based on  $\mathbf{GM}$ . The references to other papers that appear on the exhibits are leftover the paper by Francescutto.

<sup>7</sup>The first curious thing about this work is who were the sponsors? Were they Nazis or Nazi sympathizers. If you have any information, please provide it to this author.

<sup>8</sup>It seems to based on Newman (Newman 2018).



other part the ship situation that can create capsizing. One can wonder why not slightly change the ship's velocity to solve the problem. The new term that used to describe this interaction is heave-roll dependency.

In this book, ship rolling is recognized as a special pendulum movement (actually special double pendulum) where the rotation point is fluctuating. The additional complication of varying moment of inertia of the ship changes during the rotation. Additionally, in this book it was identified that added moment of inertia is a strong function of the angle. The fact the ship has moving hinge point that changing means that the moment of inertia is changing during the rotation.

### *1.4 Summary*

It interesting to observe how the profound Bouguer affected the field of ship stability. The realization of the change of buoyancy centroid is the critical point in the understanding. Later it can be observed that this great personalty plus chilling effects the Buckingham's method had on the stability understanding.



# 2

## Marine Bodies Definitions

### *2.1 Opening Remarks on Definitions*

Marine nomenclature dominates (obviously) the ship or the floating bodies stability. While trying to find what is known and what is still a black art, this author concluded that the terminology is used to complicate the topic. Furthermore, it seems that many books on ship stability written is such jargon that it is hard to read them especially if the reader is out of the field. In a way the chapter is a kind of dictionary or key that is provided to be able to read books in the area.

Hence, this chapter presents a literature survey on the nomenclature. In this book, a minor adaptation of some of this terminology is reused. This old terminology is entrenched and appear as ancient relic of outdated an science frozen for 300 years. Some terms were changed to reflect the advances in science or specifically the new technology developed here. Furthermore, this material is backported into other books (for example, "Basics of Fluid Mechanics") and hence the terminology of the wider field is preferred. In a typical stability analysis (in fluid mechanics and solid mechanics) there is a typical disturbance which result in change of the field (where the disturbance grows or decayed). In ship stability, the metacenter method does not have such concept. For example, for a stability indication is whether **GM** positive or negative but further more the analysis assumes that the metacenter is fixed regardless to the disturbance  $\theta$ . Therefore, for this author, who started his career a thermofluid with a ship stability class, is forced to modify many terms.

Yet, the terms like left or right of the ship are not used (but rather port (left) and starboard (right)), are used adapted here because they do not conflict with the standard fluid dynamics field. These terms can trace their root to the Vikings (Hendrickson 2008).

Later term like draft (depth) and freeboard (height about liquid line) were added.

The main definitions are dealing with sizes and not naming. The first definition is dealing with volume of a ship. The width (some called beam) of ship is denoted as  $b$  and height of ship is  $D$  and the length of the ship is  $s$ . Notice the lower case  $d$  is only part of the  $D$  and it is the distance from bottom to the liquid level<sup>1</sup>. In some books this quantify is denoted as  $T$  (not in this book). The displacement is volume displaced by the ship or the floating body denoted as  $V_0$ . The encompassing volume

$$V_d = d b s \quad (2.1)$$

Note that some books refers to the displacement as the displaced mass or the displaced weight (yes, these strange definitions are common).

### 2.1.1 Salinity and temperature effects

In the ship industry the minute change of density due to salinity and/or temperature (about 2%) appear in approximation as

$$\frac{d_{sal}}{d_{fresh}} \sim \frac{\rho_{fresh}}{\rho_{sal}} \quad (2.2)$$

The fresh water density is approximated as 1000 [ $kg/m^3$ ] and the density of sea water is approximated as 1025 [ $kg/m^3$ ]. The temperature has small effect and yet also appear as seasonal. In other words, the seasonal refers to Summer, Winter, etc. Eq. (2.2) can be derived from Archimedes's formula (will be covered in Buoyancy chapter). It has pointed that temperature play role nevertheless the changes are negligible.

### 2.1.2 Load Lines

Load lines are marked on a vessel's side for various conditions water salinity, and temperature. Yet, the common in the industry to make them on the ship

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<sup>1</sup>Notice that in this book the term liquid is used and not water. The reason for this is to indicate that apply to any kind of liquid and not limited to fresh or sea water.

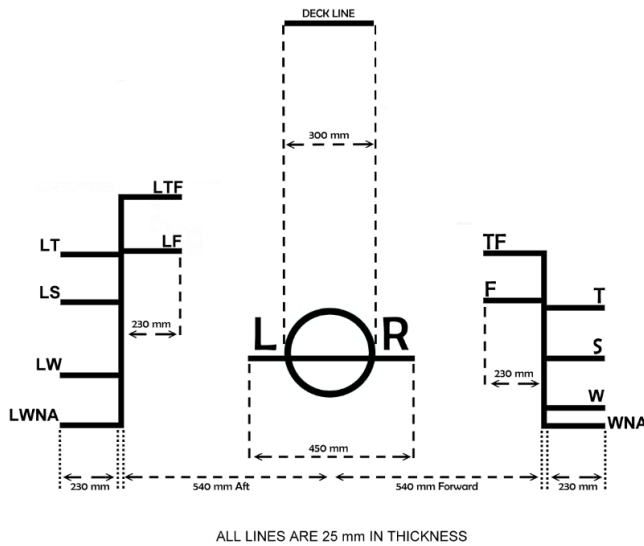


Fig. 2.1 – The first International Load Line Convention took place in 1930 after which, continue to amended to latest in 2003. Load lines based international Load Line Convention after wikimedia.org.

0.9 From the top with smaller density from the top: TF is tropical fresh water, F for fresh water, T is for tropical seawater, S is based and the summer seawater, W is the winter seawater, and WNA is winter North Atlantic. The major part of the standard is dealing the thickness of the lines etc. One can wonder how this relic still in use. The answer to that has to do with historical validation and people respect things that have been used for long.

### 2.1.3 Tonne Per Centimeter

This definition deals with the amount weight that increase the weight of the ship per centimeter. It is strange that this definition is not defined in a scientific way like the following

$$TPC_{proper} = \frac{dW}{dh} \quad (2.3)$$

In this case,  $dh$  refers to the infinitesimal height element and  $dW$  is the ship weight element. Regardless the definition, the assumption is that the cross area (parallel liquid surface) does not change much with the height (in other words, cross area is almost constant). In fact, this definition should be a function (example of how outdated material is currently in marine industry).

There two extreme cases of displacements one: Light Displacement two: Load Displacement is the weight of the liquid displaced by a ship when at summer time in salt

water for maximum load (conflicting requirements, salt and temperature are pushing to different directions.). The ship's displacement lay between these two extremes. The light displacement included fuel, stores, ballast etc which referred as "deadweight." Deadweight Coefficient describes the effectiveness of the ship. It is defined as

$$C_{DW} = \frac{\text{Light Displacement}}{\text{Displacement}} \quad (2.4)$$

This value is a dimensionless number. Basically it describes the useless fraction of the ship weight. The typical values are

Table 2.1 – Typical values for dead weight coefficient.

Vessel Type	$C_{DW}$	Vessel Type	$C_{DW}$
General Cargo Ship	0.7	Carry Wheeled Cargo	0.3
Oil Tanker	0.83	Container Ship	0.6
Bulk Carrier	0.82	Passenger Liners	0.37
Gas Tanker	0.62	Cross-channel Ferry	0.2

Some of these vessels type carry heavy weight and it is very wasteful.

The block coefficient is defined as the ratio between the actual volume to encompassing rectangular (extruded rectangular).

$$C_b = \frac{V_0}{D b s} \quad (2.5)$$

$C_b$  is used for application and regulations of vessels. The coefficient indicates how "smooth" the ship is. The ship displacing volume is denoted  $V_0$  and is defined as

$$V_0 = C_b D b s \quad (2.6)$$

Another relic from the past is the Prismatic Coefficient which is the ratio of

$$C_p = \frac{V_0}{s A_m} \quad (2.7)$$

This coefficient represents the width of the ship if the mid-section was expended to full length of the ship. Notice that the shape encompassing the ship is not rectangular. The Maximum Sectional Area Coefficient is defined as

$$C_m = \frac{A_m}{b D} \quad (2.8)$$

**Example 2.1:  $C_d$  calculations****Level: Easy**

A general cargo ship has a dead weight of 50k tonnes. Estimate the fully loaded displacement ( $W$ ).

**Solution**

The value of  $C_d$  is given the table 2.1 is 0.7. Hence, the displacement is

$$\text{Displacement} = \frac{\text{Light Displacement}}{C_d} \quad (2.1.a)$$

Applying the data to the equation Eq. (2.1.a)

$$\text{Displacement} = \frac{50000}{0.7} \sim 71428.6 \text{ [tonnes]} \quad (2.1.b)$$

Basically, the ship can carry about 21,428 [tonnes].

The bilge region is at the bottom and there are several definitions for it.

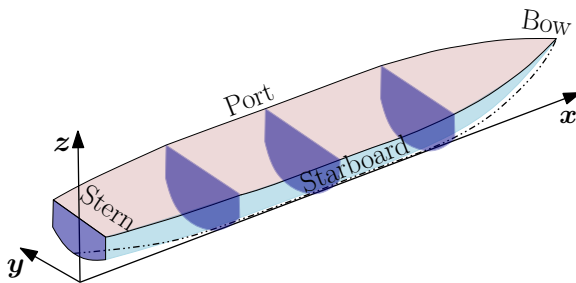


Fig. 2.2 – Ship Coordinates System suggested by DIN 81209-1.

Strangely there is a recommendation for coordinate system. However there are differences between the various standards. For example, ISO 7460 and 7463 defined different origin as compared to DIN 81209-1. In this book, this recommendation is not followed because it does not compatible with presentation of the material. The concept of the system used by the context. It is assumed that this principle foreign to those who write the standards. For example, in two dimensions, it is convenient to use  $x, y$  coordinate while in three dimensions it should be  $x, z$ . This kind of system is adapted in this book.

General coordinate system is attached to the floating body or ship. So, ship with fixed  $z$ -coordinate fixed to liquid  $z$ -coordinate is referred to as upright condition. A rotation of the  $z, y$  plane are referred to as heel or roll depending on the variance with time. Even keel is referred to the situation in which  $x$ -coordinate is parallel to the liquid level. Nautical terminology depreciates pitch and trim (see Fig. 8.3).

### 2.1.4 Deviations From the Design

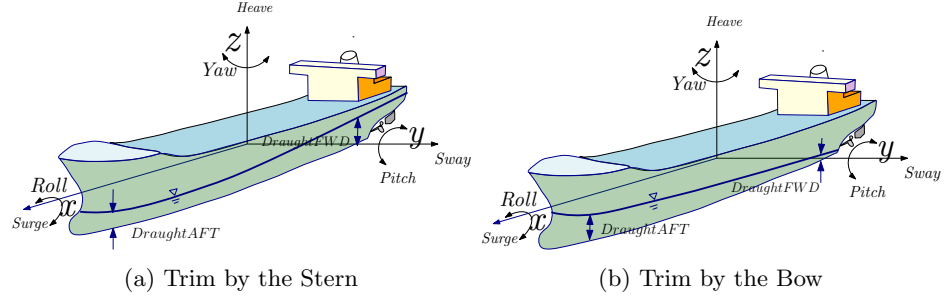


Fig. 2.4 – Trim in two possibilities which are to opposite of each other. The trim is results from unbalance loading more on stern or the bow.

Ideally ship and other floating bodies are designed to move relative to “center” coordinate system. A deviation from the “center” coordinate exist and increase the resistance to the flow and increase the cost operation. For example, the trim is pitch rotation to new stability point. The trim is defined positive when the ship bow is up and referred as trimmed by the head. The trim is the deviation from symmetry of the ship which desired in certain situations ( like speedboat) more often is a problem.

The ship’s list is deviation form “zero” of the rolling movement. This change is static. There is a dynamic component what have the zigzag movement which is results from the non–symmetrical ship design and it is different from the ship’s list. There is another deviation that is not very common or significant is referred as deviation offset. As oppose to the zigzag in the list direction the deviation is more important and this author estimate that ship because this movement travel additional 2% of the total road or more. This occur as the other zigzag because symmetry considerations and waves. This 2% additional road means additional 2% in fuel cost.

### 2.1.5 Ship Length

While this books does not deal with ship design several points has to mention about ship’s dimensions. The ship width is limited by the cost and the space available (canal limitations). The ship length is determined by cost and need but limited by space at designated ports. In same way the draft (depth) depends on the limited space with the exception of the large tanker which can load and unload outside the port.

The measuring how much ship is way from the “box” shape is

$$C_M = \frac{A_M}{bD} \quad (2.9)$$

The center cross section is referred as midship-section  $C_M$  is referred as midship coefficient. The compactness of ship is defined as

$$C_P = \frac{V_0}{s A_M} \sim \frac{C_b}{C_M} \quad (2.10)$$



Where  $A_M$  is the wetted area of the midship cross section. Another definition to measure compactness of the surface area at the liquid level as

$$C_{ls} = \frac{A_{ls}}{s b} \quad (2.11)$$

Where Liquid Surface Coefficient is the measure of compactness relative recommendation area at liquid surface and some define denotes as  $C_w$ . The liquid cross area,  $A_{ls}$ , is the area at the surface of the liquid. For completeness, addition has be mentioned as Prismatic Coefficient.

### 2.1.6 Ship Length

The design of the ship/boat is art more than science as indication shown that a new design is based on the old and tested design rather design totally from fresh. yet, some engineering principles can be utilized. The displacement is related to the ship length. Yet, ships especially smaller one heaving a complex shape which the relationship has to account for these effects.

$$s \sim \sqrt[3]{V_0} \longrightarrow s = f(\dots) \sqrt[3]{V_0} \quad (2.12)$$

The logical assumption is that the correction factor which is related to deviation factors from the extruded rectangular. These deviation factors  $C_b$ ,  $C_M$ ,  $C_P$  were defined earlier. The value of these is given by experience and no science but black art is used. Notice that in these calculations the actual ship width and the ship depth are unknown. The relationship The ratio of the length,  $s/b$  is determined from the requirement on the resistance to the flow (ship speed). The ratio of  $b/D$  is dictated by the stability consideration (more this issue in the book). With deviation factors given above the length of a ship can be estimated for given dead weight,  $DW$  (which related to  $V_0$ ) as

$$s = \sqrt[3]{\frac{DW}{\rho g}} f(C_b, C_D) \quad (2.13)$$

There are several formulas suggested in the literature. No information on the scientific nature of the development of these equations was provided and hence none can be recommended. An example of such formula is

$$s = \sqrt[3]{\frac{DW \left(\frac{s}{b}\right)^2 \left(\frac{b}{D}\right)}{\rho g C_b C_D}} \quad (2.14)$$

A dimensionless parameter known as Froude Displacement Coefficient to measure the stretchiness (resistance to the flow) of the vessel and is

$$Fr_D = \frac{s}{\sqrt[3]{V_0}} \quad (2.15)$$

Froude number The Froude is range from 17 for racing boat, racing yacht 7, container ship about 6.5 and Seventeenth-century first rate of 4. Beside this Froude number there additional numbers have similar name and such example of definition is

$$Fr = \frac{U}{\sqrt{gs}} \quad (2.16)$$

Heaving is a vertical movement of the ship. Sway is the linear movement in the  $y$  direction. Surge is the linear movement in the  $x$  direction. If the body is symmetrical than these movements do not affect each other (and also the other rotations). However, if the body is not a symmetrical they affect each other. Furthermore, all these moments creates all the rotational moments. The base for the interaction between these movements is clear but the calculation is beyond scope of this book. Estimate this author done on this effect show that it about percent of order of magnitude. While this number sound insignificant, in real life this can be a significant amount.

## 2.2 Special Dimensions of the Ships

This section is tied with historical naming and such depend on school of thought and no real uniform system is agreed. The length of the ship is defined by some as the length at the water line. The same can said on the ship width. The curvature of ship like round shape of the deck (to make sure that the water are deflected to the sea) is referred as camber. The arc created the ship floor is referred as "bilge." The bilge exist for the several reasons, like collection the water (or liquids), strength, and etc. The other curvature surfaces are created for structural reasons. However, the flare can help with the stability.

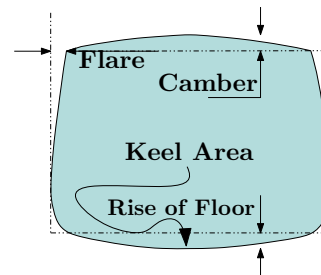


Fig. 2.5 – Mid ship cross section to illustrate various definitions.

## 2.3 Summary

This chapter deals with topics related to ship structure and design. For example, different ships were introduced and various international standards discussed. These definitions are indirectly related to ship stability. This discussion was add to the book because there was a demand for it. Moreover, the main usage of the stability is for the marine industry. Some of the material related to dimensional analysis but it is covered because the dimensionless aspects are not the main issue.

# 3

## The Mathematical Background for Stability

In this chapter is review for most readers and new for others and intend to cover minimum topics related to stability of floating body. These topics are present so that one with some background in trigonometry could deal with the mathematics that encompass in this book. Hence without additional reading, this book could be used by most readers.

### 3.1 Differentiation

What is differentiation and why it is important? In trigonometric there is a function call tangent for short  $\tan$ , that reader should know, the ratio the sides of triangle. A parabolic function shown Fig. 3.1 has a point  $x, x^2$  denoted by the red point. To find the tangent to the function a triangle is build starting at red point. The second point is at  $(x + \Delta x, x^2)$  and the third is on the graph vertically from second point denoted by green point. Clearly the line connecting the second and third point is above the graph (the purple line). To minimize the distance between the purple line (see the figure)

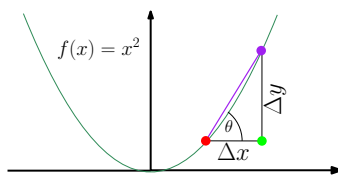


Fig. 3.1 – Parabola to explain derivative. Not to scale and point the triangle starts is at  $(x, x^2)$  (red point)

and the graph, the triangle has to shrink. The tangent of the angle of the triangle at it is

$$\tan \theta = \frac{\Delta y}{\Delta x}$$

The value of  $\Delta y$  can be ascertained the difference between the function value at  $x$  and  $x + \Delta x$  as

$$\Delta y = f(x + \Delta x) - f(x) = (x + \Delta x)^2 - x^2$$

The first term on right hand side can be expand as

$$\Delta y = \cancel{x^2} + 2x\Delta x + (\Delta x)^2 - \cancel{x^2} = 2x\Delta x + (\Delta x)^2$$

The tangent in this case is

$$\tan \theta = \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$$

When  $\Delta x$  become smaller and smaller (about zero) the value of the tangent is  $2x$ . This example of differentiation explains the concept and it has been extend. In this book the concept was explained and other derivative of other functions should obtained from table or other sources. There is no need to emphasis here. The reason that differentiation is extremely important because almost all the calculations in stability requires understanding this concept. Ultimately, most of the calculations ended to simple arithmetic. The derivative are denote as  $dy/dx$  and they are actually  $\tan \theta$ .

$$\tan \theta = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (3.1)$$

The tangent can be used to evaluate the at some distance from point  $x$  as

$$f(x + \Delta x) \approx f(x) + \Delta x \overbrace{\frac{dy}{dx}}^{\tan \theta} \quad (3.2)$$

It must be note that a short notation of derivative is  $f'(x)$  or even shorter hand is simply  $f'$ .

### 3.2 Integration

The integration is calculation of the area under the curve. The area is made of many trapezoids as shown in Fig. 3.2. Without any proof but simply using the intuition integration is opposite of differentiation. The explain to that is based on the following. Note the explanation is not rigorous but just the underline. The integration is denoted by the symbol  $\int$ . For example, the function ( $F$ ) can be defined as

$$F(x) = \int_0^x f(t)dt$$

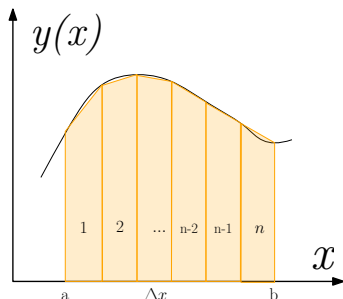


Fig. 3.2 – The trapezoid method to explain the numerical integration.

In this case, the notation of  $dt$  and  $t$  is just a dummy variable and the integral from point 0 to point  $x$ . Integration is done by adding small rectangular shape thus,

$$F(x + \Delta x) = \int_0^x f(t) dt + \overbrace{\int_x^{x+\Delta x} f(t) dt}^{\sim \Delta x f(x)} \tag{3.3}$$

For small distance of  $\Delta x$  the value of the function is constant. Hence, the second integral can be written as over the brace was the width and height given.

$$F(x) + \Delta x F'(x) \approx \int_0^x f(t) dt + \int_x^{x+\Delta x} f(t) dt \tag{3.4}$$

The first term on the left is equal to first term on the right. Thus, the second the term on left has to equal to second of the left

$$\Delta x F'(x) \approx \Delta x f(x) \tag{3.5}$$

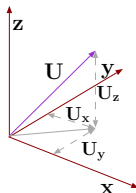


Fig. 3.3 – Vector in Cartesian coordinates system.

Thus  $F'(x) \approx f(x)$  and this consider fundamental equation or law of calculus. To find the integral of function is done by "guessing." For example, it was found before that the derivative of  $x^2$  is  $2x$ . Hence, the integral of  $2x$  is  $x^2$ .

This a brief introduction to calculus and no mean it meat to be complete and comprehensive. Yet, the reader should be able to understand some of the mathematical explanation.

### 3.3 Vectors and Coordinates System

The are many coordinate systems which are used to describe location or direction. In this book two systems will be discussed, Cartesian, Cylindrical coordinate. The Cartesian system describes the location with three linear coordinate normally call,  $x$ ,  $y$ ,  $z$ . The

Cylindrical coordinate measures with a distance from the center on the plane and the angle and  $z$ .

Vector is a quantity with direction as oppose to scalar. The length of the vector in Cartesian coordinates (the coordinates system is relevant) is

$$\|\mathbf{U}\| = \sqrt{U_x^2 + U_y^2 + U_z^2} \quad (3.6)$$

Vector can be normalized and in Cartesian coordinates depicted in Figure 3.3 where  $U_x$  is the vector component in the  $x$  direction,  $U_y$  is the vector component in the  $y$  direction, and  $U_z$  is the vector component in the  $z$  direction. Thus, the unit vector is

$$\hat{\mathbf{U}} = \frac{\mathbf{U}}{\|\mathbf{U}\|} = \frac{U_x}{\|\mathbf{U}\|} \hat{\mathbf{i}} + \frac{U_y}{\|\mathbf{U}\|} \hat{\mathbf{j}} + \frac{U_z}{\|\mathbf{U}\|} \hat{\mathbf{k}} \quad (3.7)$$

and general orthogonal coordinates

$$\hat{\mathbf{U}} = \frac{\mathbf{U}}{\|\mathbf{U}\|} = \frac{U_1}{\|\mathbf{U}\|} \mathbf{h}_1 + \frac{U_2}{\|\mathbf{U}\|} \mathbf{h}_2 + \frac{U_3}{\|\mathbf{U}\|} \mathbf{h}_3 \quad (3.8)$$

Vectors have some what similar rules to scalars which will be discussed in the next section.

### 3.3.1 Vector Algebra

Vectors obey several standard mathematical operations which are applicable to scalars. The following are vectors,  $\mathbf{U}$ ,  $\mathbf{V}$ , and  $\mathbf{W}$  and for in this discussion  $a$  and  $b$  are scalars. Then the following can be said

1.  $(\mathbf{U} + \mathbf{V}) + \mathbf{W} = (\mathbf{U} + \mathbf{V} + \mathbf{W}) = \mathbf{U} + (\mathbf{V} + \mathbf{W})$
2.  $\mathbf{U} + \mathbf{V} = \mathbf{V} + \mathbf{U}$
3. Zero vector is such that  $\mathbf{U} + \mathbf{0} = \mathbf{U}$
4. Additive inverse  $\mathbf{U} - \mathbf{U} = \mathbf{0}$
5.  $a(\mathbf{U} + \mathbf{V}) = a\mathbf{U} + a\mathbf{V}$
6.  $a(b\mathbf{U}) = ab\mathbf{U}$

The multiplications and the divisions have somewhat different meaning in a scalar operations. There are two kinds of multiplications for vectors. The first multiplication is the “dot” product which is defined by equation (3.9). The results of this multiplication is scalar but has no negative value as in regular scalar multiplication.

$$\mathbf{U} \cdot \mathbf{V} = \overbrace{|\mathbf{U}| \cdot |\mathbf{V}|}^{\text{regular scalar multiplication}} \cos \overbrace{(\angle(\mathbf{U}, \mathbf{V}))}^{\text{angle between vectors}} \quad (3.9)$$

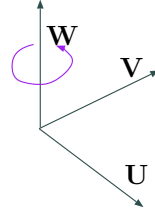


Fig. 3.4 – The right hand rule, multiplication of  $\mathbf{U} \times \mathbf{V}$  results in  $\mathbf{W}$ .

The second multiplication is the “cross” product which in vector as opposed to a scalar as in the “dot” product. The “cross” product is defined in an orthogonal coordinate  $(\hat{h}_1, \hat{h}_2, \text{ and } \hat{h}_3)$  as

$$\mathbf{U} \times \mathbf{V} = |\mathbf{U}| \cdot |\mathbf{V}| \sin \overbrace{(\angle(\mathbf{U}, \mathbf{V}))}^{\text{angle}} \hat{\mathbf{n}} \quad (3.10)$$

where  $\theta$  is the angle between  $\mathbf{U}$  and  $\mathbf{V}$ , and  $\hat{\mathbf{n}}$  is a unit vector perpendicular to both  $\mathbf{U}$  and  $\mathbf{V}$  which obeys the right hand rule. The right hand rule is referred to the direction of resulting vector. Note that  $\mathbf{U}$  and  $\mathbf{V}$  are not necessarily orthogonal. Additionally note that order of multiplication is significant. This multiplication has a negative value which means that it is a change of the direction.

One of the consequence of this definitions in Cartesian coordinates is

$$\hat{\mathbf{i}}^2 = \hat{\mathbf{j}}^2 = \hat{\mathbf{k}}^2 = 0 \quad (3.11)$$

In general for orthogonal coordinates this condition is written as

$$\hat{\mathbf{h}}_1 \times \hat{\mathbf{h}}_1 = \hat{\mathbf{h}}_1^2 = \hat{\mathbf{h}}_2^2 = \hat{\mathbf{h}}_3^2 = 0 \quad (3.12)$$

where  $\hat{\mathbf{h}}_i$  is the unit vector in the orthogonal system.

In right hand orthogonal coordinate system

$$\begin{aligned} \hat{\mathbf{h}}_1 \times \hat{\mathbf{h}}_2 &= \hat{\mathbf{h}}_3 & \hat{\mathbf{h}}_2 \times \hat{\mathbf{h}}_1 &= -\hat{\mathbf{h}}_3 \\ \hat{\mathbf{h}}_2 \times \hat{\mathbf{h}}_3 &= \hat{\mathbf{h}}_1 & \hat{\mathbf{h}}_3 \times \hat{\mathbf{h}}_2 &= -\hat{\mathbf{h}}_1 \\ \hat{\mathbf{h}}_3 \times \hat{\mathbf{h}}_1 &= \hat{\mathbf{h}}_2 & \hat{\mathbf{h}}_1 \times \hat{\mathbf{h}}_3 &= -\hat{\mathbf{h}}_2 \end{aligned} \quad (3.13)$$

The “cross” product can be written as

$$\mathbf{U} \times \mathbf{V} = (U_2 V_3 - U_3 V_2) \hat{\mathbf{h}}_1 + (U_3 V_1 - U_1 V_3) \hat{\mathbf{h}}_2 + (U_1 V_2 - U_2 V_1) \hat{\mathbf{h}}_3 \quad (3.14)$$

Equation (3.14) in matrix form as

$$\mathbf{U} \times \mathbf{V} = \begin{pmatrix} \widehat{h}_1 & \widehat{h}_2 & \widehat{h}_3 \\ U_2 & U_2 & U_3 \\ V_2 & V_2 & V_3 \end{pmatrix} \quad (3.15)$$

The most complex of all these algebraic operations is the division. The multiplication in vector world have two definition one which results in a scalar and one which results in a vector. Multiplication combinations shows that there are at least four possibilities of combining the angle with scalar and vector. The reason that these current combinations, that is scalar associated with  $\cos \theta$  vectors is associated with  $\sin \theta$ , is that these combinations have physical meaning. The previous experience is that help to define multiplication help to definition the division. The number of the possible combinations of the division is very large. For example, the result of the division can be a scalar combined or associated with the angle (with  $\cos$  or  $\sin$ ), or vector with the angle, etc. However, these above four combinations are not the only possibilities (not including the left hand system). It turn out that these combinations have very little<sup>1</sup> physical meaning. Additional possibility is that every combination of one vector element is divided by the other vector element. Since every vector element has three possible elements the total combination is  $9 = 3 \times 3$ . There at least are two possibilities how to treat these elements. It turned out that combination of three vectors has a physical meaning. The three vectors have a need for additional notation such of vector of vector which is referred to as a tensor. The following combination is commonly suggested

$$\frac{\mathbf{U}}{\mathbf{V}} = \begin{pmatrix} \frac{U_1}{V_1} & \frac{U_2}{V_1} & \frac{U_3}{V_1} \\ \frac{U_1}{V_2} & \frac{U_2}{V_2} & \frac{U_3}{V_2} \\ \frac{U_1}{V_3} & \frac{U_2}{V_3} & \frac{U_3}{V_3} \end{pmatrix} \quad (3.16)$$

One such example of this division is the pressure which the explanation is commonality avoided or eliminated from the fluid mechanics books including the direct approach in this book.

This tensor or the matrix can undergo regular linear algebra operations such as finding the eigenvalue values and the eigen "vectors." Also note the multiplying matrices and inverse matrix are also available operation to these tensors.

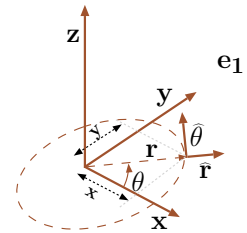


Fig. 3.5 – Cylindrical Coordinate System.

<sup>1</sup>This author did find any physical meaning these combinations but there could be and those the word "little" is used.



### 3.3.2 Cylindrical Coordinates

The cylindrical coordinates are commonly used in situations where there is a line of symmetry or kind of symmetry. This kind of situation occurs in pipe flow even if the pipe is not exactly symmetrical. These coordinates reduce the work, in most cases, because the problem is reduced to two dimensions. Historically, these coordinates were introduced for geometrical problems about 2000 years ago<sup>2</sup>. The cylindrical coordinates are shown in Figure 3.5. In the figure, it shows that the coordinates are  $r$ ,  $\theta$ , and  $z$ . Note that unit coordinates are denoted as  $\hat{r}$ ,  $\hat{\theta}$ , and  $\hat{z}$ . The meaning of  $\vec{r}$  and  $\hat{r}$  are different. The first one represents the vector that is the direction of  $\hat{r}$  while the second is the unit vector in the direction of the coordinate  $r$ . These three different  $r$ s are somewhat similar to any of the Cartesian coordinate. The second coordinate  $\theta$  has unit coordinate  $\hat{\theta}$ . The new concept here is the length factor. The coordinate  $\theta$  is an angle. In this book, the dimensional chapter shows that in physics, derivatives have to have the same units in order to compare them or use them. Conversion of the angle to units of length is done by the length factor which is, in this case,  $r$ . The conversion between the Cartesian coordinate and the Cylindrical is

$$r = \sqrt{x^2 + y^2} \quad \theta = \arctan \frac{y}{x} \quad z = z \quad (3.17)$$

The reverse transformation is

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z \quad (3.18)$$

The line element and volume element are

$$ds = \sqrt{dr^2 + (r d\theta)^2 + dz^2} \quad dr r d\theta dz \quad (3.19)$$

## 3.4 Numerical Analysis

The integration and the derivation sometimes cannot or hard to perform. In fact, the first integration method was introduced by the father of modern, Bouguer, Pierre, namely the trapezoid method. First, the differentiation will be introduced<sup>3</sup>. The concept of numerical differentiation is direct as

$$\frac{\Delta y}{\Delta x} \sim \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{\Delta x} \quad (3.20)$$

<sup>2</sup>Coolidge, Julian (1952). "The Origin of Polar Coordinates". *American Mathematical Monthly* 59: 7885. [http://www-history.mcs.st-and.ac.uk/Extras/Coolidge\\_Polars.html](http://www-history.mcs.st-and.ac.uk/Extras/Coolidge_Polars.html). Note the advantage of cylindrical (polar) coordinates in description of geometry or location relative to a center point.

<sup>3</sup>It is hard to discuss intelligently the issue of differentiation without understanding Taylor series. Taylor series will be presented hopefully in the next version. It is assumed that they are known to the reader.

This method is called a forward differentiation or one-sided differentiation. The similar method is backward differentiation is

$$\frac{\Delta y}{\Delta x} \sim \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{\Delta x} \quad (3.21)$$

If the distance between the point is uniform then improvements can be made. Another approach is to take the two sides which is sometimes referred to as centered differentiation

$$\frac{\Delta y}{\Delta x} \sim \frac{y_2 - y_0}{x_2 - x_0} = \frac{y_2 - y_0}{2 \Delta x} \quad (3.22)$$

This equation is more precise under certain or even (most) certain circumstances.

### 3.5 Integration

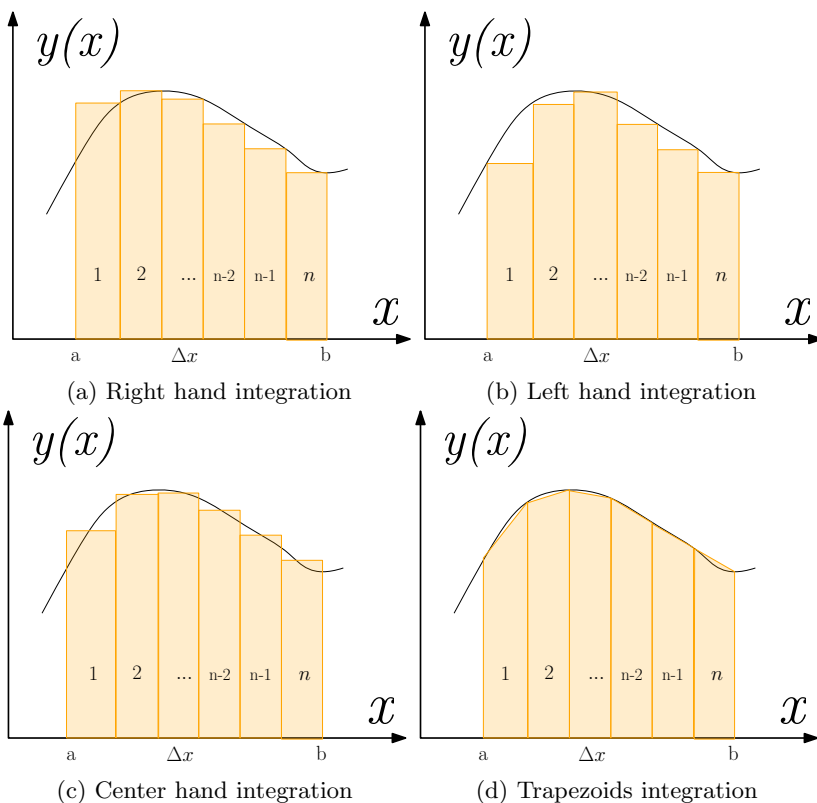


Fig. 3.7 – Four different integration methods showing the difference graphically between the actual function to the integrated area. It can be seen that Bouguer's method (trapezoid) is much better technique than these methods.

Integration is calculating the area under the "graph." The simplest method is left or right rectangular which is basically the value of the function (data point) times

*Delta x.* Better technique was invented by Bouguer, Pierre. In this methods it is assumed that the area can be calculated by a trapezoid. For practical purpose if distance between the data points is equal, the calculation is

$$A = \left( \frac{y_0 + y_n}{2} + \sum_{i=1}^{n-1} y_i \right) \Delta x \quad (3.23)$$

### 3.6 Simpson's Rule

A better method is what is called Simpson Rule and it should be called Kepler rule. The idea behind that methods is that the function can be represented by parabola so it close the actual function. In this case the area is even equal parts (intervals) because the nature of the method.

Simpson's Rule is just a tool and the derivation can be found on numerous places. However, actual procedure is presented here. For example if the interval is split into two sub intervals than the equation is

$$\text{Area} = (2 \Delta x) \left( \frac{y_0 + 4y_1 + y_2}{6} \right)$$

The minimal larger influence on the results. Simpson's rule gives you the following estimate for the area under the curve:

$$\text{Area} = (2 \Delta x) \left( \frac{1}{6} \right) [(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (\cdots y_n)] \quad (3.24)$$

We can combine terms here by exploiting the following pattern in the coefficients:

$$\begin{array}{cccccc} 1 & 4 & 1 & & & \\ + & & 1 & 4 & 1 & \\ + & & & & 1 & 4 & 1 \\ \hline 1 & 4 & 2 & 4 & 2 & 4 & 1 \end{array}$$

Simpson's rule get the final form

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \quad (3.25)$$

#### Example 3.1: Simpson Rule

Level: Easy

A function  $y(x)$  is given by the table of values. Approximate the area under the function between point  $x = 0$  and and the point  $x = 4$  using Simpsons Rule with 4 subintervals.

x	0	1	2	3	4
y	2	7	12	10	5

**Solution**

For  $n = 4$  subintervals, Simpsons rule is reduced into

$$A = \frac{\Delta x}{3} [y(0) + 4y(1) + 2y(x_2) + 4y(x_3) + y(x_4)]$$

Every subintervals is

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

Using the values from the table and calculate the approximate value of the area

$$A \approx \frac{1}{3} [2 + 4 \cdot 7 + 2 \cdot 12 + 4 \cdot 10 + 5] = \frac{1}{3} [2 + 28 + 24 + 40 + 5] = \frac{1}{3} \cdot 99 = 33 \quad (3.1.a)$$

### 3.7 Summary

Mathematics is the language of science. It is hard to see how one can understand concept as stability without knowledge of calculus. While the mathematics in stability does not expand to two dimensions or even three dimensions. In the chapter dealing moment of inertia and centroid of area or volume or mass requires knowledge integration. This author experience was such that once intro was given and taken seriously the material could be understood. Mathematics is only the tool and not core what this book is focus on.

# 4

## Mass Centroid

The mass (or gravity) centroid is divided into two sections, first, the mass centroid and two, area centroid (two-dimensional body with equal distribution mass). Additionally, the change of center of mass due to addition or subtraction of mass plus discrete areas are presented. The reason for the last addition, is that explain core explanation of the stability analysis.

### 4.1 *Actual Mass Centroid*

In many engineering problems, the knowledge of mass centroid is required to make the calculations especially for stability. This concept is derived from the fact that a body has a centroid (mass/gravity) which interacts with other bodies and that this force acts on the center (equivalent force)<sup>1</sup>. It turns out that this concept is very useful in calculating rotations, moment of inertia, etc. The mass centroid doesn't depend on the coordinate system and on the way it is calculated. The physical meaning of the mass centroid is that if a straight line force acts on the body in away through the center of gravity, the body will not rotate. In other words, if a body will be held by one point it will be enough to hold the body in the direction of the mass centroid. Note, if the body isn't be held through the mass centroid, then a moment, in additional to the force, is required (to prevent the body for rotating). It is convenient to use the Cartesian system to explain this concept. Suppose that the body has a distribution of the mass (density,  $\rho$ ) as a function of the location. The density "normally" defined as mass per volume.

---

<sup>1</sup>(Lautrup 2011) has suggested when a body is long this statement needs corrections in variable gravity field. To encourage participation of students, it is suggested here small projects. If you a student or someone how interested in the field, and like your name mentioned, contact this author for a project discussion.

Here, the line density is referred to density mass per unit length in the  $x$  direction.

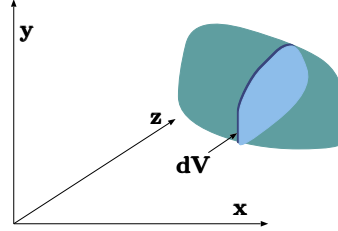


Fig. 4.1 – Description of how the center of mass is calculated. In  $x$  coordinate, the center will be defined as

$$\bar{x} = \frac{1}{m} \int_V x \overbrace{\rho(x) dV}^{dm} \quad (4.1)$$

Here, the  $dV$  element has finite dimensions in  $y$ - $z$  plane and infinitesimal dimension in  $x$  direction see Figure 4.1. Also, the mass,  $m$  is the total mass of the object. It can be noticed that center of mass in the  $x$ -direction isn't affected by the distribution in the  $y$  nor by  $z$  directions. In same fashion the center of mass can be defined in the other directions as following

$x_i$  of Center Mass

$$\bar{x}_i = \frac{1}{m} \int_V x_i \rho(x_i) dV \quad (4.2)$$

where  $x_i$  is the direction of either,  $x$ ,  $y$  or  $z$ . The density,  $\rho(x_i)$  is the line density as function of  $x_i$ . Thus, even for solid and uniform density the line density is a function of the geometry. When finite masses are combine the total mass Eq. (4.2) converted into

$$\bar{x} = \frac{\sum x_i m_i}{\sum m_i} \quad (4.3)$$

where  $i$  denotes every mass in the system.

**Example 4.1: Calculation of ship centroid**

**Level: Easy**

An example of ship' weights is given in the following table.

End of Ex. 4.1

Table 4.1 – Ship Data for typical calculation of gravity centroid. Notice that it done in this example for z direction only

Your Ship			
Item	$z_i [m]$	$W_i(Weight) [ton]$	$z_i W_i [ton \times m]$
Main engines	7	2,000	14,000
Anchors and cable	12	150	1,800
Lifeboats	10	20	200
Fuel	0.75	120	80
Stores	150	15	2,250
Hull structure	9	18,000	162,000
Cargo	8	20,000	160,000
Smuggled material	15	10	150
<b>Total</b>	n/a	40,315	340,490

Solution

The solution is obtained by utilizing Eq. (4.1) and it is

$$\bar{z} = \frac{\sum z_i W_i}{\sum W_i} = \frac{340,490}{40,315} \sim 8.446[m] \tag{4.1.a}$$

### 4.2 Approximate Centroid of Area

The previous section dealt with the body was a three dimensional characteristics. There are cases where the body can be approximated as a two-dimensional shape because the body is with a thin with about uniform density. Consider a uniform thin body with a constant thickness shown in Figure 4.2 which has density,  $\rho$ . Thus, equation (4.1) can be transferred into

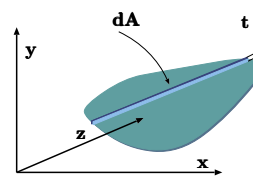


Fig. 4.2 – Thin body centroid of mass/area schematic.

$$\bar{x} = \underbrace{\frac{1}{tA}}_V \rho \int_V x \overbrace{\rho t dA}^{dm} \tag{4.4}$$

The density,  $\rho$  and the thickness,  $t$ , are constant and, therefore, can be canceled. Thus

equation (4.4) can be transferred into

Approximate  $x_i$  of Center Mass

$$\bar{x}_i = \frac{1}{A} \int_A x_i dA \quad (4.5)$$

when the integral now is over only the area as oppose over the volume. Eq. (4.5) can also be written for discrete areas as

$$\bar{x}_i = \frac{\sum x_i A_i}{\sum A_i} \quad (4.6)$$

It must be noted that area  $A_i$  can be positive or negative. The meaning of negative area in this context is subtraction of area.

### 4.3 Change of Centroid Location Due to Area Change

Solid body with one area added and one area removed. The old centroid marked "o" the new centroid marked "n" and area removed "r" and area added "a." This section deals with a change centroid location when an area is add or subtracted from a given area with a know centroid (or unknown). This topic is important when a centroid of area was found or previously calculated and

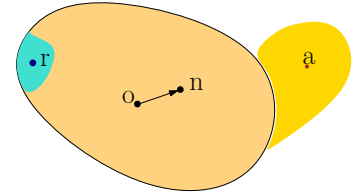


Fig. 4.3 – Solid body with removed and added area.

hence only change needed. Furthermore, for some problems, the change or its direction has more importance as it will be discussed in greater detail in, chapter on stability of floating bodies on page 91. The centroid of body in Fig. 4.3 is denoted at point "o" (old). The centroid of the added and removed areas are at points "a" (added) and "r" (removed), respectively. The point "n" (new) is the centroid after modification. A special case when the added area is equal to the subtracted area and its application will be discussed in an example below. It has to be noted that added and subtracted areas do not have to be continuous. Utilizing Eq. (4.6) for the identical areas reads for this case as

$$x_n = \frac{x_o A_o + x_r A_r - x_a A_a}{A_o + A_r - A_a} \quad (4.7)$$

In a special case where subtracted area is equal to added area ( $A_r = A_a$ ) Eq. (4.7) is reduced to

$$x_n = x_o + x_r \frac{A_r}{A_o} - x_a \frac{A_a}{A_o} \quad (4.8)$$



## Change of Centroid

$$\bar{x} - x_o = \frac{A_r}{A_o} (x_r - x_o) \quad (4.9)$$

Finding the centroid location should be done in the most convenient coordinate system since the location is coordinate independent. There should be a sign convention to determine the centroid direction movement so that the direction should be immediately expressed in the result. However, faults were found in several options that were considered<sup>2</sup>.

## Example 4.2: Center of Circle inside Circle

Level: Easy

A circle with a radius,  $r$  has a cut out from a larger circle with radius,  $R$  where  $R > r$  (see Fig. 4.4). The distance between the center of the larger circle and the small circle is  $x$ . Calculate the centroid of the circle that a smaller circle was cut out of it. Assume that  $x$  is small enough so that the small circle is whole.

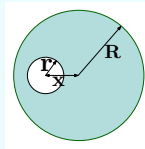


Fig. 4.4 – Subtraction of circle from a large circle for calculating the new center.

## Solution

The change in the centroid is only the direction of  $x$ . It should be noted that for  $x = 0$ , the centroid is at  $x = 0$  and  $y = 0$  that is the centroid is at the center of the larger circle. For larger distance up to the  $x = R - r$  the centroid can be calculated utilizing Eq. (4.7) reads

$$\Delta x = \frac{\cancel{r} r^2}{\cancel{r} R^2} (x - 0) = x \left( \frac{r}{R} \right)^2 \quad (4.2.a)$$

Notice that  $x$  is the distance between the two centers while  $\Delta x$  is the change in the centroid location. Additionally, if the removed circle is not on the  $x$  coordinate then these calculations can be reused. For instance, if the cut is at angle,  $\theta$ , the change will be along straight line from the center of the large circle at the distance that was obtained in Eq. (4.2.a). The conversion to a regular coordinate system could be done by utilizing simple trigonometric functions.

<sup>2</sup>If you have a good method/technique please consider discussing it with this author.

**Example 4.3: Centroid of Trapezoid****Level: Complex**

In some situations a rectangle shape is transferred to a trapezoid shape while the area size remain constant. For example, a body floating (in liquid) has a rectangle shape. Rotation (hypothetical or real) of the floating body will result in a change to a trapezoid. This situation is used to check whether a body is stable. Assume that the rectangle body is transferred into a trapezoid with the exact same area see Fig. 4.5. Calculate the change of the centroid in the  $y$  and  $x$  directions, the total change, and the angle of change  $\alpha$ . Note that  $\alpha$  is measured from point  $(b/2, d)$ .

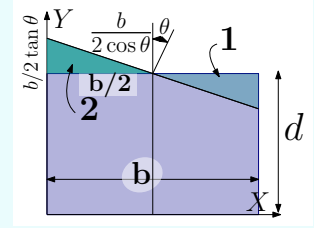


Fig. 4.5 – Trapezoid created from a tilted rectangle shape.

**Solution**

The rectangular centroid is located at  $b/2, d/2$  (see table 4.2). The two triangles must be identical to keep the condition of constant area. It also implies that two triangles must start from the same point at the “center” (point  $x = b/2$  and  $y = d$  to be denoted Point A). Eq. (4.9) dictates that the change is due to the addition and subtraction of the triangles. The change can be expressed as

$$\Delta(A_i y_i) = \overbrace{\left(d + \frac{b \tan \theta}{6}\right)}^{y_2} \overbrace{\frac{b^2 \tan \theta}{8}}^A - \overbrace{\left(d - \frac{b \tan \theta}{6}\right)}^{y_1} \overbrace{\frac{b^2 \tan \theta}{8}}^A \quad (4.3.a)$$

continue Ex. 4.3

It can be noticed that the two triangles contribute identical value in the direction of raising the center. After simplifications and canceling Eq. (4.3.a) reads

$$\Delta(A_i y_i) = \cancel{2} \left( \frac{b \tan \theta}{3} \right) \times \frac{b^2 \tan \theta}{8} = \frac{b^3 \tan^2 \theta}{24} \quad (4.3.b)$$

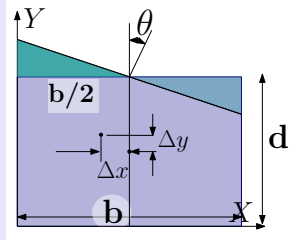


Fig. 4.6 – Mass Center Moving to New Location center. Notice that the actual size is exaggerated.

It can be observed that the net change in the  $y$  direction is

$$\Delta y = \frac{b^3 \tan^2 \theta}{\underbrace{24}_{bd}} = b \left( \frac{b}{d} \right) \frac{\tan^2 \theta}{24} \quad (4.3.c)$$

The change depends on the rectangle width and the ratio of the height to the width. The change in the  $x$  direction is

$$\Delta(A_i x_i) = \frac{\overbrace{x_1=b-b/6}^{5b}}{6} \frac{\overbrace{A_1}{b^2 \tan \theta}}{8} - \frac{\overbrace{x_2=1/3 \times a/2}{b}}{6} \frac{\overbrace{A_2}{b^2 \tan \theta}}{8} \Bigg|_{A_1=A_2=A} = \frac{b^3 \tan \theta}{12} \quad (4.3.d)$$

This change is to the left of the old centroid. Or the net change is

$$\Delta x = \frac{b^3 \tan \theta}{12} \frac{\overbrace{1}^{1/A}}{bd} = b \frac{b \tan \theta}{d} \frac{1}{12} \quad (4.3.e)$$

Note that for small angle  $\theta \sim \tan \theta$  therefore  $\theta \gg \theta^2$  and hence,  $\Delta x \gg \Delta y$ . The total change  $\Delta r$  can be expressed as a function of  $\Delta y$  and  $\Delta x$  to read

$$\Delta r = \sqrt{\Delta y^2 + \Delta x^2}$$

or in a dimensionless form as

$$\frac{\Delta r}{b} = \sqrt{\frac{\Delta y^2}{b^2} + \frac{\Delta x^2}{b^2}} \quad (4.3.f)$$

Substituting Eq. (4.3.e) and Eq. (4.3.c) into Eq. (4.3.f) reads

$$\frac{\Delta r}{b} = \sqrt{\left( \frac{b}{d} \right)^2 \frac{\tan^2 \theta}{12^2} + \left( \frac{b}{d} \right)^2 \frac{\tan^4 \theta}{24^2}} \quad (4.3.g)$$

rearranging the Eq. (4.3.g) provides

$$\frac{\Delta r}{b} = \frac{b \tan \theta}{d} \sqrt{1 + \frac{\tan^2 \theta}{4}} \quad (4.3.h)$$

The angle is measured from a specific point. In this case the point is the connection between the two triangles. The reason for this selection is that this point will be used in stability issue. It can be noticed that angle,  $\alpha$ , of the new center can be obtained from the straight triangle with one side of  $(d/2 - \Delta y)$  and  $\Delta x$ .

$$\alpha = \tan^{-1} \frac{\Delta x}{d/2 - \Delta y} \quad (4.3.i)$$

Substituting the the values for  $\Delta x$  and  $\Delta y$  yields

$$\alpha = \tan^{-1} \frac{b \frac{b \tan \theta}{d} \frac{12}{12}}{\frac{d}{2} - b \frac{b \tan^2 \theta}{24}} \quad (4.3.j)$$

with additional simplifications

$$\alpha = \tan^{-1} \left( \frac{\frac{b^2 \tan \theta}{d} \frac{12}{12}}{\frac{b^2}{d} \left( \frac{d^2}{2b^2} - \frac{\tan^2 \theta}{24} \right)} \right) \quad (4.3.k)$$

It is interesting to point out that the  $\alpha$  is related to the ratio of  $d/b$  and to the rotating angle,  $\theta$ . For small angle,  $\tan \theta \cong \theta$  and  $d/b \geq 1/8$  (or even smaller) Eq. (4.3.k) becomes

$$\alpha = \tan^{-1} \left( \frac{2 \tan \theta}{12 \left( \frac{d}{b} \right)^2 - \tan^2 \theta \sim 0} \right) \cong \tan^{-1} \left( \frac{\tan \theta}{6} \left( \frac{b}{d} \right)^2 \right) \quad (4.3.l)$$

Thus under the condition of  $(b/d)^2 = 6$ , the change in  $\alpha$  is equal  $\theta$ . This is a limiting case which determines in certain situations whether a floating body is stable and it will be discussed in Chapter 8 on page 91.

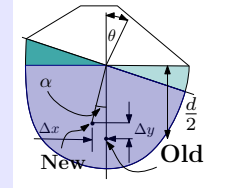


Fig. 4.7 – Movement of area center to new location due to the change of shape.

**Example 4.4: Centroid of Triangle**

**Level: Complex**

Triangle turns to a trapezoid as shown in Fig. 4.8. Calculate the change in the centroid (in the directions of  $x, y, r, \alpha$ ) due to change of angle  $\theta$ . Hint due to the symmetry, it does not matter rectangular turns to left or the right.

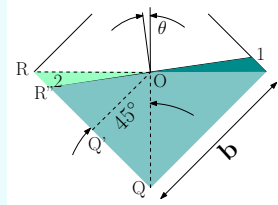


Fig. 4.8 – Triangle turns to trapezoid.

**Solution**

The added and subtracted areas are shown in Fig. 4.8.

$$\Delta(A_i x_i) = A (x_1 - x_2) \tag{4.4.a}$$

The triangle  $\triangle ROQ$  is an isosceles triangle with  $45^\circ$ . The angle  $\angle RR''O$  is  $(180^\circ - 45^\circ - \theta)$ . Law of sines can be written for  $\triangle RR''O$  as

$$\frac{RR''}{\sin \theta} = \frac{b/\sqrt{2}}{\sin(135^\circ - \theta)} \tag{4.4.b}$$

$\sin(135^\circ - \theta)$  according to the trigonometry identities equal to  $(\sin \theta + \cos \theta)/\sqrt{2}$  and for small angle of  $\theta$  Eq. (4.4.b) becomes

$$\overline{RR''} = \frac{b/\sqrt{2} \sin \theta}{1/\sqrt{2} (\sin \theta + \cos \theta)} = \frac{b \sin \theta}{\cos \theta + \sin \theta} \Big|_{\theta \sim 0} = b \sin \theta \tag{4.4.c}$$

Note that for small  $\theta$  the standard approximation for  $\sin \theta$  and  $\cos \theta$  were used. The height of  $\triangle ROR''$  is  $b/2$  (it is the same height as for  $\triangle ROQ$  line  $Q'O$  see Fig. 4.8). Hence the area of  $\triangle ROR''$  is

$$A_{\triangle ROR''} = \frac{\overline{RR''} \times \frac{b}{2}}{2} \sim \frac{b^2 \sin \theta}{4} \tag{4.4.d}$$

$x_2$  can be ascertained from table 4.2 and other data (geometry) from Fig. 4.9. The dashed lines in the Fig. 4.9 are parallel to their respective base. Further, these lines are at one third of the distance.

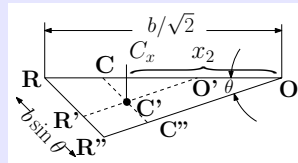


Fig. 4.9 – The centroid of the small triangle. Point  $C'$  is the centroid location. Line  $\overline{R'O'}$  is parallel to base Line  $\overline{R''O}$ .

The distance  $\overline{OR} = b/\sqrt{2}$  (It can be verified by looking at the isosceles triangle  $\triangle RQ'O$  that has  $\overline{OR}$  against  $\pi/2$  angle.). The distance  $\overline{OC} = 2/3 \times \overline{OR} = \sqrt{2} b/3$ . The distance  $\overline{C'C''} = 2/3 \times \overline{R'R''}$ . Hence,  $\overline{C'C''} = 2b \sin \theta/3$ . The line  $\overline{R'O'}$  split the line  $\overline{C'C''}$  exactly in the middle (why? because  $\overline{C''C'}$  is two third of  $\overline{R'R''}$ ). Thus,

continue Ex. 4.4

$$\overline{C'C'} = \frac{b \sin \theta}{3} \quad (4.4.e)$$

The triangle  $\triangle C'C_xC$  is an isosceles triangle with  $\pi/2$  angle. Hence, the line  $\overline{C'C_x} = \overline{C'C'}$  is  $b \sin \theta / (3\sqrt{2})$ .  $x_2$  can be found by

$$x_2 = \overline{OC'} - \overline{C'C_x} = \frac{\sqrt{2}b}{3} - \frac{b \sin \theta}{3\sqrt{2}} \quad (4.4.f)$$

after simplification, Eq. (4.4.f)

$$x_2 = \frac{\sqrt{2}b}{3} \left( 1 - \frac{\sin \theta}{2} \right) \cong \frac{2b}{3\sqrt{2}} \quad (4.4.g)$$

The distance in the  $y_2$  direction is  $\Delta y = \frac{a \sin \theta}{3\sqrt{2}}$ . With the values of  $x_2$  and  $y_2$  the values of  $x_1$  and  $y_1$  are also known. Utilizing Eq. (4.9) it can be written that

$$\Delta(A_i x_i) = \underbrace{\frac{1}{A} \left( \frac{b^2 \sin \theta}{1} \right)}_A \underbrace{\left( \frac{\frac{2b}{3\sqrt{2}}}{x} \right)}_x = \frac{b^3 \sin \theta}{3\sqrt{2}} \quad (4.4.h)$$

The change in the  $x$  direction is

$$\Delta x = \frac{1}{\underbrace{b \times b/2}_A} \frac{b^3 \sin \theta}{\underbrace{3\sqrt{2}}_{\Delta(A_i x_i)}} = \frac{\sqrt{2}b \sin \theta}{3} \quad (4.4.i)$$

The other direction can be obtained in similar fashioned

$$\Delta(A_i y_i) = \underbrace{\frac{1}{A} \left( \frac{b^2 \sin \theta}{1} \right)}_A \underbrace{\left( \frac{b \sin \theta}{3\sqrt{2}} \right)}_y = \frac{b^3 \sin^2 \theta}{3\sqrt{2}} \quad (4.4.j)$$

and the net change in  $\Delta y$  is

$$\Delta y = \frac{2}{\underbrace{b^2}_{1/A}} \frac{b^3 \sin^2 \theta}{3\sqrt{2}} = \frac{\sqrt{2}b \sin^2 \theta}{3} \quad (4.4.k)$$

As it was shown earlier, the change in angle,  $\alpha$  is from a straight triangle with one side is  $\Delta x$  and the other side  $\frac{b}{3\sqrt{2}} - \Delta y$  which is

$$\frac{b}{3\sqrt{2}} - \Delta y = \frac{b}{3\sqrt{2}} (1 - 2 \sin^2 \theta)$$

#### 4.4. CHANGE OF MASS CENTROID DUE TO ADDITION OR SUBTRACTION OF MASS IN 3D<sup>39</sup>

End of Ex. 4.4

The angle of the change is

$$\alpha = \tan^{-1} \left( \frac{\frac{b}{3\sqrt{2}} (1 - 2 \sin^2 \theta)}{\frac{\sqrt{2} b \sin \theta}{3}} \right) = \tan^{-1} \left( \frac{(1 - 2 \sin^2 \theta)}{2 \sin \theta} \right) \quad (4.4.l)$$

For small angle  $\theta$  Eq. (4.4.l) becomes

$$\alpha \cong \tan^{-1} \left( \frac{\theta}{2} \right) \quad (4.4.m)$$

The total change is the same as in Eq. (4.3.f)

$$\frac{\Delta r}{b} = \sqrt{\left( \frac{\sqrt{2} \sin \theta}{3} \right)^2 + \left( \frac{2 \sin^2 \theta}{3} \right)^2} \quad (4.4.n)$$

and for small angle  $\theta \ll 1$

$$\frac{\Delta r}{b} = \frac{\sqrt{2} \sin \theta}{3} \cong \frac{\sqrt{2} \theta}{3} \quad (4.4.o)$$

### 4.4 Change of Mass Centroid Due to Addition or Subtraction of Mass in 3D

This topic is extension of the previous topic of two dimensions change of centroid. All bodies are three dimensions thus when no symmetry or extrudations<sup>3</sup> exist the full analysis has to be done. Furthermore, it is interesting to point to the phenomenon none symmetrical body the change and be in a third dimension. This topic to be discussed in stability Chapter.

A centroid of a slob is located in point "o" and additional mass depicted as "a" and the removed/subtracted mass "r" and again the new location of centroid is denoted as "n."

$$x_n = \frac{m_o x_o + m_a x_a - m_r x_r}{m_o - m_a + m_r} \quad (4.10)$$

As before the special case of equal subtracted and added material Eq. (4.10) converted into

$$x_n = \frac{m_o x_o + m (x_a - x_r)}{m_o} \quad (4.11)$$

<sup>3</sup>The word "extrudations" means same meaning it has in blender (software) for example, a flat area that is expanded vertically.

when the density is uniform, Eq. (4.11) can be written

$$x_n = \frac{V_o x_o + V (x_a - x_r)}{V_o} \rightarrow x_n - x_o = \frac{V}{V_o} (x_a - x_r) \quad (4.12)$$

**Example 4.5: Center of Circle inside Circle**

**Level: Complex**

Calculate the change in the centroid of cylinder with length  $d$ , that one wedge is added and one is removed. In Fig. 4.10 describing only one wedge in additional to second side (which is not drawn in the figure.). Calculate the tiding angle,  $\alpha$  from the center of the cylinder.

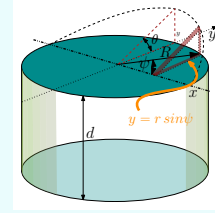


Fig. 4.10 – Mass centroid of cylinder with added wedge and subtracted wedge.

**Solution**

First the centroid of the wedge has to be calculated and Eq. (4.2) can be used as

$$x_i = \frac{\int x_i dV}{\int dV} \quad (4.5.a)$$

The ratio shown in Eq. (4.12) indicates that it is sufficient to ascertain over only half of wedge. The centroid of mass of cylinder wedge is not given in any table in this book and is not readily available other places. Hence, this work will be presented (a bit more than what should be presented in this book.). The volume of the wedge is

$$V = 2 \int_0^r A dx \quad (4.5.b)$$

Notice that instead to carry the integration from  $r$  to  $-r$  it is carried from  $0$  to  $r$  and multiplied by two. The circle equation is  $x^2 + y^2 = r^2$ . The infinitesimal volume made from the triangle (the brown color) with a base of  $\sqrt{r^2 - x^2}$  and the height of  $\sqrt{r^2 - x^2} \tan \theta$  and infinitesimal thickness,  $dx$ , to be integrated as

$$V = 2 \int_0^r \frac{\overbrace{(r^2 - x^2) \tan \theta}^A}{\cancel{2}} dx = \tan \theta \int_0^r (r^2 - x^2) dx \quad (4.5.c)$$

A new variable is defined as  $\xi = x/r$  when  $x = r \rightarrow \rho = 1$ , and equation Eq. (4.5.c) can be transformed

$$V = r^3 \tan \theta \int_0^r \left(1 - \frac{x^2}{r^2}\right) \frac{dx}{r} = r^3 \tan \theta \int_0^1 (1 - \xi^2) d\xi = \frac{2r^3 \tan \theta}{3} \quad (4.5.d)$$



4.4. CHANGE OF MASS CENTROID DUE TO ADDITION OR SUBTRACTION OF MASS IN 3D41

End of Ex. 4.5

The brown triangle with infinitesimal thickness has mass centroid at  $2/3$  of the length of the triangle (from the center or in other words cylinder center). The mass centroid can be calculated utilizing the similar approach but simpler in cylindrical coordinates with the boundaries of zero to  $\pi/2$ . The integration of infinitesimal volume (similar to previous calculations where the base is  $r \cos \psi$ ) can be carried as

$$\int x_i dV = 2 \int_0^{\pi/2} \overbrace{\left(\frac{2}{3} r \cos \psi\right)}^{y_i} \overbrace{\frac{(r \cos \psi)^2 \tan \theta}{2}}^A \overbrace{\cos \psi r d\psi}^{''dx''} \quad (4.5.e)$$

The choice of the coordinate system base on convenience. Eq. (4.5.e) can be written as

$$\int x_i dV = \frac{2 r^4 \tan \theta}{3} \int_0^{\pi/2} (\cos \psi)^4 d\psi \quad (4.5.f)$$

The integral is readily available and can be done by integration by part as

$$\int x_i dV = \frac{2 r^4 \tan \theta}{3} \times \left( \frac{\sin \psi \cos^3 \psi}{4} + \frac{3 \cos \psi \sin \psi}{8} + \frac{3 \psi}{8} \right) \Big|_0^{\pi/2} = \frac{r^4 \pi \tan \theta}{8} \quad (4.5.g)$$

The first and the second terms vanished at the boundaries. Substituting the volumes of both integrals into Eq. (4.5.a) provides

$$x_i = \frac{\frac{r^4 \pi \tan \theta}{8}}{\frac{2 r^3 \tan \theta}{3}} = \frac{3 \pi r}{8} \quad (4.5.h)$$

It is interesting to point out that  $x_i$  is not a function of the angle  $\theta$ . The total change based on Eq. (4.12) is

$$\Delta x = \frac{2 V x_i}{V_0} = \frac{\frac{2 \cancel{\pi} \cancel{\tan \theta} r^2}{8}}{\frac{\cancel{\pi} \cancel{\tan \theta} d}{1}} = \frac{r^2 \tan \theta}{4 d} \quad (4.5.i)$$

The angle of change,  $\alpha$  under assumption of small  $\Delta y$  can be obtained as

$$\alpha \cong \tan^{-1} \frac{\Delta x}{\xi} = \tan^{-1} \left( \frac{r^2 \tan \theta}{4 d \xi} \right) = \tan^{-1} \left( \frac{r^2 \tan \theta}{4 d \xi} \right) \quad (4.5.j)$$

**Example 4.6: Wedge centroid****Level: Intermediate**

In Ex. 4.5  $\Delta y$  was assumed to be zero. Is this assumption correct or/and under what conditions it is correct. Hint: first calculate  $\Delta y$  and then use the results to the estimated results.

**Solution**

under construction

**4.4.1 A Small Change in Angle of Rotation**

This section is dealing with a special topic of change of area due to rotation when the area (volume) is constant topic very is important to stability analysis. The change of the area in Ex. 4.5 dealt with a specific geometry. This procedure can be generalized or even simplified the procedure. The process of calculating the change of the centroid can be converted for small angle.

$$x_i = \frac{\int x dV}{\int dV} = \frac{\int x \overbrace{x \tan \theta}^h dA}{V} = \frac{\tan \theta \int x^2 dA}{V} \quad (4.13)$$

The term in the nominator is called the Moment of Inertia and will be discussed in the following section. The Moment of Inertia symbolized by  $I_{xx}$  and Eq. (4.13) by

$$x_i = \frac{\tan \theta I_{xx}}{V} \quad (4.14)$$

Notice that  $I_{xx}$  is a function of the cross section only and is half of the cross section. Hence for the total moment of inertia double the half (see next section for explanation). The volume of the small wedge is calculated below. The total change is defined in Eq. (4.12)

$$\Delta x = \frac{V}{V_0} \overbrace{\tan \theta I_{xx}}^{2x_1} = \tan \theta \frac{I_{xx}}{V_0} \quad (4.15)$$

It is remarkable that the change location of centroid can be determined from knowing/calculating the moment of inertia of the cross section and by the displaced volume.

The change in centroid in the  $y$  direction and done in a similar fashion when noticing that the  $y$  is actually  $y/2$  (when first  $y$  refers to the  $y$  of the distance and the second refers to its value.). For small angle it can be written that

$$y_i = \frac{\int y dV}{V} = \frac{\int y/2 \overbrace{y dA}^{dV}}{V} = \frac{\int (\tan \theta x)^2 dA}{2V} = \frac{\tan^2 \theta \int x^2 dA}{2V} \quad (4.16)$$

#### 4.4. CHANGE OF MASS CENTROID DUE TO ADDITION OR SUBTRACTION OF MASS IN 3D43

Again as in the  $x$  direction Eq. (4.17) moment of inertia should be recognized and it can be written as

$$y_i = \frac{\tan^2 \theta I_{xx}}{2V} \quad (4.17)$$

Hence,

$$\Delta y = \frac{V}{V_0} \frac{\overbrace{\tan^2 \theta I_{xx}}^{2x_1}}{2V} = \tan^2 \theta \frac{I_{xx}}{2V_0} \quad (4.18)$$

The difference between in the two changes is significant. The amount change in  $y$  smaller by a factor. For practical purposes, the change in  $y$  can be ignored.

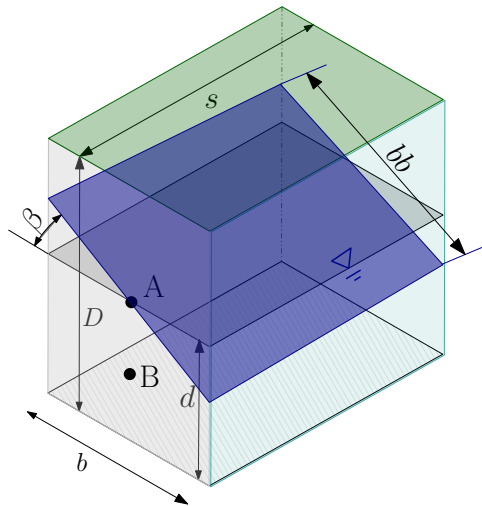


Fig. 4.11 – Rectangular Floating in Various Angles. Notice that blue surface represents all the positions that the body can have up to point that is “fail” on the side.

#### Example 4.7: Centroid of cylinder

Level: Complex

Repeat Ex. 4.3 and Ex. 4.4 with using Eq. (4.15). Use as a base the calculate all change for all the angle  $0 < \theta < 60^\circ$ . Comment on the geometrical limitations. In these two cases the volume is fix and the moment of inertia is varies. The moment of inertia is discussed in the Chapter 5. For the purpose of this exercise assume that moment inertia for rectangular is  $I = ab^3/12$  see the book Appendix table 4.2. Where  $a$  is the length around the rotation occurs and  $b$  is width of the rectangular.

## Solution

As long as the geometry remains the same this analysis can be assumed to be correct. Also note that the  $\beta$  and  $\theta$  are different angle:  $\beta$  is the angle that is as location for which is checked to see if it is stable, while  $\theta$  is the small angle forced on the body to see the body will remain at  $\beta$ . Fig. 4.11 depicts the liquid surface plane (the blue color). The question refers two edge cases, one when  $\beta = 0$  and two when  $\beta = 45^\circ$ . Thus, the solution here will deal with all the angles. So all the properties have to express as a function of angle  $\beta$ . The volume is constant regardless to  $\beta$ . The moments inertia is a strong of  $\beta$ . The distance  $bb$  shown in the figure is

$$bb = \frac{b}{\cos \beta} \quad (4.7.a)$$

Hence, the moment of inertia is

$$I_{xx} = \frac{s \left( \frac{b}{\cos \theta} \right)^3}{12} = \frac{s b^3}{12 \cos^3 \beta} \quad (4.7.b)$$

The volume of the body is

$$V = d b s \quad (4.7.c)$$

The change in buoyancy centroid Eq. (4.15) is

$$\Delta x = \tan \theta \frac{\frac{b^2}{d} \frac{12 \cos^3 \theta}{b}}{d} \longrightarrow \frac{\Delta x}{b} = \frac{b \tan \theta}{d 12 \cos^3 \beta} \quad (4.7.d)$$

The two cases in the question are  $\beta = 0$  and  $\beta = 45^\circ$  can be substituted into Eq. (4.7.d) to get the answers which are plotted on Fig. 4.12. The graph depict Eq. (4.7.d) for the dimensionless parameters, angle of inclination,  $\beta$ , and ratio of the moment of inertia to the volume (below). The results show that as the inclination angle,  $\beta$ , has a dramatic effect. For large angle ( $30^\circ$ – $40^\circ$ ) the change in centroid getting larger and larger. Additionally, larger value of  $b/d$  (width and depth) ratio the effect become more significant. In the case of  $\beta = 0$  and  $\beta = 45^\circ$  the equations respectively are: 0 case;

$$\frac{\Delta x}{b} = \frac{b \tan \theta}{d 12} \quad (4.7.e)$$

The distance  $AB$  is  $d/2$  thus tangent  $\alpha$  is

$$\tan \alpha = \frac{\Delta x}{AB} = \frac{\frac{b^2 \tan \theta}{d 12}}{d/2} \longrightarrow \alpha = \tan^{-1} \left\{ \frac{1}{6} \left( \frac{b}{d} \right)^2 \tan \theta \right\} \quad (4.7.f)$$

$45^\circ$ ;

$$\frac{\Delta x}{b} = \frac{b \tan \theta}{d 12 \frac{1}{2\sqrt{2}}} \longrightarrow \frac{\Delta x}{b} = \frac{\sqrt{2}}{3} \frac{b \tan \theta}{d} \quad (4.7.g)$$

4.4. CHANGE OF MASS CENTROID DUE TO ADDITION OR SUBTRACTION OF MASS IN 3D45

End of Ex. 4.7

In this case  $AB = d/(3\sqrt{2})$ . For the second case, the case angle  $\alpha$  is

$$\alpha = \tan^{-1} \left\{ 2 \left( \frac{b}{d} \right)^2 \tan \theta \right\} \quad (4.7.h)$$

Before the general expression could be written as formula for the centroid has to found.

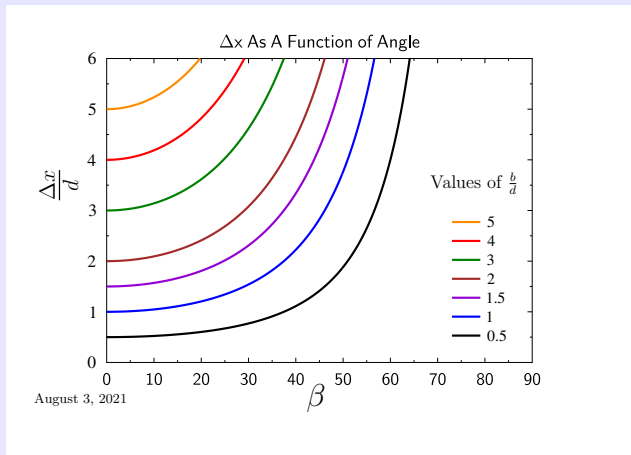


Fig. 4.12 – Change of Buoyancy in The x Direction.

It can be observed from the analysis the change of  $\Delta x$  varies and this will be used in the stability analysis. There several conclusions that can be drawn so far. As long geometry remain the same, larger change on centroid appear for larger angle. When happen when the change of number of angle appear. In other words, if the surface reaches to upper or the lower corner then the analysis has to be modified.

Example 4.8: Ellipse centroid

Level: Intermediate

Repeat Ex. 4.7 for cylinder standing up. Cross section area is ellipse with minor axis of  $D$  while the major radius changes and is function of the geometrical identity as  $r_m = r/\cos\beta$ . Using this information construct a map similar to the above example.

Solution

The moment of inertia of the ellipse (see table 4.2) is

$$I_{xx} = \frac{\pi r \left( \frac{r}{\cos \beta} \right)^3}{4} \longrightarrow I_{xx} = \frac{\pi r^4}{4 \cos^3 \beta} \quad (4.8.a)$$

End of Ex. 4.8

The volume, again, is constant and equable to

$$V = \pi r^2 d \quad (4.8.b)$$

The change in centroid is

$$\Delta x = \tan \theta \frac{\frac{\pi r^2 d}{4 \cos^3 \beta}}{\frac{\pi r^2 d}{4 \cos^3 \beta}} \longrightarrow \frac{\Delta x}{r} = \frac{\tan \theta r}{4 \cos^3 \beta d} \quad (4.8.c)$$

**Remark**

The difference between this Eq. (4.8.c) and Eq. (4.7.d) is the coefficient.

Generally the change in centroid for extruded shape will be in a general form of

$$\frac{\Delta x}{b} = CC \frac{\text{geometrical parameter}}{\text{geometrical parameter}} \frac{\tan \theta}{\cos^3 \beta} \quad (4.19)$$

Where  $CC$  is a coefficient that depend on the geometry. This form change when the shape of the body is radially different. Large ships can be considered as a large box with a round bottom. Thus the analysis provide here in relevant to these bodies close to "boxes." However, for small boat have more complicate shape and it can be numerically analyzed. It also can be noticed that bodies that change shape with  $z$  coordinate like cone the centroid change as well as the center of the surface area (at liquid surface).

#### 4.5 Transformation of Coordinates

Sometimes the information on the centroid is provided in different coordinate system. In this case a demonstration on how to transfer the coordinate is present. A trapezoid centroid are given normally in coordinate system that attached to base as shown below.

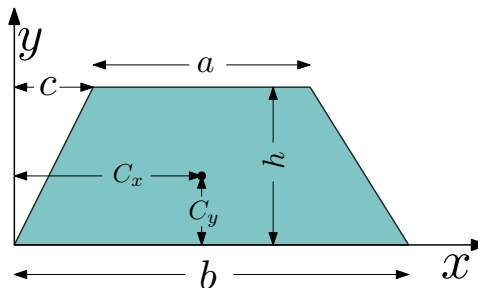


Fig. 4.13 – Standard trapezoid with given data and coordinate system.

For this trapezoid in this specific coordinate system the center is given utilizing the given names in the following:

$$c_x = \frac{a^2 + b^2 + 2ac + bc + ab}{3(a + b)} \tag{4.20a}$$

$$c_y = \frac{h(2a + b)}{3(a + b)} \tag{4.20b}$$

To ascertain the location according to the new specific dimensions. It can be noticed that the following “translation” should be made when the two Parallel lines. Utilizing the equations Eq. (4.20) to the dimensions shown Fig. 4.13 to apply to Fig. 4.14. It reduces the labor in handling with the use of the following definitions are used

$$\Psi = b \tan \beta / 2, \quad h \longrightarrow b \quad a \longrightarrow d - \Psi$$

$$b \longrightarrow d + \Psi \quad c \longrightarrow 2 \Psi.$$

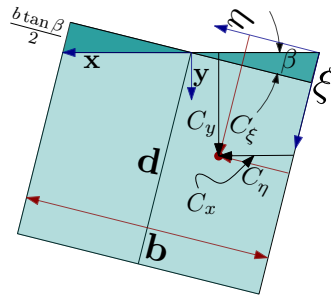


Fig. 4.14 – Trapezoid in a new coordinate system to be transformation from old system.

Several identities can be observed;

Table 4.2 – Trapezoid properties and equations

Trapezoid properties and equations	
Transformation	Identities Units
$\Psi = b/2 \tan \beta$	$a + b = 2d$
$h \longrightarrow b$	$a^2 + b^2 = 2d^2 + 2\Psi^2$
$a \longrightarrow (d - \Psi)$	$ab = d^2 - \Psi^2$
$b \longrightarrow (d + \Psi)$	$2ac = 4d\Psi - 4\Psi^2$
$c \longrightarrow 2\Psi$	$bc = 2d\Psi + 2\Psi^2$
	$2ac + bc = 6d\Psi - 2\Psi^2$

Eq. (4.20) or more specifically Eq. (4.20a) can be evaluated

$$C_\xi = \frac{d}{2} + \Psi - \frac{\Psi^2}{6d} \tag{4.21}$$

or in dimensionless form as

$$\frac{C_\xi}{d} = \frac{1}{2} + \frac{\Psi}{d} - \frac{1}{6} \left( \frac{\Psi}{d} \right)^2 \tag{4.22a}$$

or in terms of  $b$  and  $\tan \beta$  as

$$\frac{C_\xi}{d} = \frac{1}{2} + \frac{1}{2} \frac{b}{d} \tan \beta - \frac{1}{24} \left( \frac{b}{d} \right)^2 \tan^2 \beta \tag{4.22b}$$

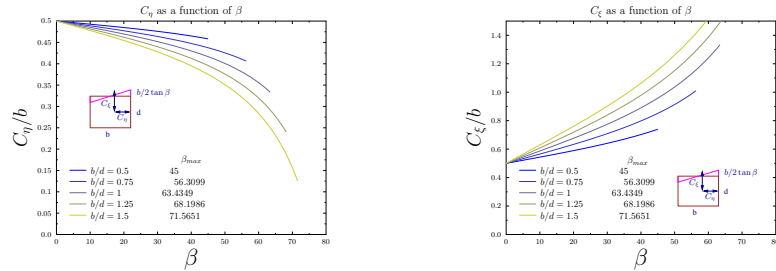
For the  $\eta$  direction

$$C_\eta = \frac{b(2(d - \Psi) + (d + \Psi))}{6d} = \frac{b(3d - \Psi)}{6d} = \frac{b}{2} - \frac{b\Psi}{6d} \tag{4.22c}$$

and dimensionless form as

$$\frac{C_\eta}{d} = \frac{1}{2} \frac{b}{d} - \frac{b\Psi}{6d^2} = \frac{1}{2} \frac{b}{d} - \frac{1}{12} \left( \frac{b}{d} \right)^2 \tan \beta \tag{4.22d}$$

The ratio of  $C_\xi$  is related to  $C_x$  by  $\beta$  through the equation above and the same can be said for the other pair  $(C_y, C_\eta)$ . Assume that  $\beta$  is rotating at a constant angular speed then the centroid is rotating but in different angular velocity. This fact means that the centroid will not be under the point  $G$  or even  $A$ . The results are depicted in Fig. 4.16.



(a) The distance of  $C_\eta$  as a function of  $\beta$ . (b) The distance of  $C_\xi$  as a function of  $\beta$ .

Fig. 4.16 – Distance of the center as a function of  $\beta$ . The distance is measured from the top corner and it is also moving with  $\beta$  as for  $C_\xi$  (it does not effect  $C_\eta$ ).

The corrected distance is depicted in Fig. 4.17.



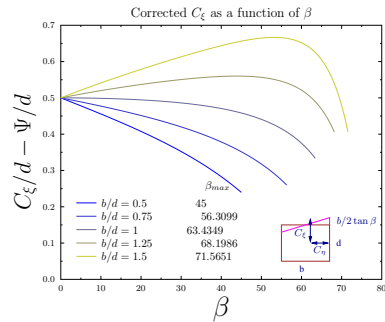


Fig. 4.17 – Corrected  $C_\xi$  as a function of  $\beta$ .

It can be noted that for small value  $b/d$  the distance increases which is very important factor for instability.

### 4.6 Appendix

As a side kick, the integral that was canceled before Eq. (4.15) can be calculated as following

$$\int dV = \int x \tan \theta dA = \tan \theta \int x dA = \tan \theta \bar{x} A \quad (4.23)$$

The value of  $\tan \theta$  is constant in the integration and the value  $\bar{x}$  is the average height of wedge. The value of  $\bar{x}$  is a function of  $\theta$  but not its location and the  $A$  cross area is not function of  $\theta$ . Sometime of values  $\bar{x}$  are tabulated and hence the integration can be readily available.



# 5

## Moment of Inertia

As the Chapter 4 on the centroid was divided, the chapter on moment of inertia is divided into moment of inertia of mass and area. Additionally this chapter another issue that is transformation to different coordinate systems (linear, rotational). Furthermore, the product of inertia will be introduced.

### 5.1 *Moment of Inertia for Mass*

The moment of inertia turns out to be an essential part for the calculations of rotating bodies and stability. Furthermore, it turns out that the moment of inertia has much wider applicability and usefulness than one assumes initially. Moment of inertia of mass is defined as

$$I_{rrm} = \int_V \rho r^2 dV \quad (5.1)$$

If the density is constant then equation (5.1) can be transformed into

$$I_{rrm} = \rho \int_V r^2 dV \quad (5.2)$$

The moment of inertia is independent of the coordinate system used for the calculation, but dependent on the location of axis of rotation relative to the body. Some people define the radius of gyration as an equivalent concepts for the center of mass concept and which means if all the mass were to locate in the one point/distance and to obtain

the same of moment of inertia.

$$r_k = \sqrt{\frac{I_m}{m}} \quad (5.3)$$

The body has a different moment of inertia for every coordinate/axis and they are

$$\begin{aligned} I_{xx} &= \int_V r_x^2 dm = \int_V (y^2 + z^2) dm \\ I_{yy} &= \int_V r_y^2 dm = \int_V (x^2 + z^2) dm \\ I_{zz} &= \int_V r_z^2 dm = \int_V (x^2 + y^2) dm \end{aligned} \quad (5.4)$$

$$\begin{aligned} I_{xx} &= \int_V r_x^2 dm = \int_V (y^2 + z^2) dm \\ I_{yy} &= \int_V r_y^2 dm = \int_V (x^2 + z^2) dm \\ I_{zz} &= \int_V r_z^2 dm = \int_V (x^2 + y^2) dm \end{aligned} \quad (5.5)$$

## 5.2 Moment of Inertia for Area

### 5.2.1 General Discussion

For body with thickness,  $t$  and uniform density the following can be written

$$I_{xxm} = \int_m r^2 dm = \rho t \overbrace{\int_A r^2 dA}^{\text{moment of inertia for area}} \quad (5.6)$$

The moment of inertia about axis is  $x$  can be defined as

$$\boxed{I_{xx} = \int_A r^2 dA = \frac{I_{xxm}}{\rho t}} \quad (5.7)$$

where  $r$  is distance of  $dA$  from the axis  $x$  and  $t$  is the thickness. Any point distance can be calculated from axis  $x$  as

$$r = \sqrt{y^2 + z^2} \quad (5.8)$$

Thus, Eq. (5.7) can be written as

$$I_{xx} = \int_A (y^2 + z^2) dA \quad (5.9)$$

In the same fashion for other two coordinates as

$$I_{yy} = \int_A (x^2 + z^2) dA \quad (5.10)$$

$$I_{zz} = \int_A (x^2 + y^2) dA \quad (5.11)$$

### 5.3 The Parallel Axis Theorem

This section deals with transformation of the moment of from one system to a parallel system. The moment is depended on the coordinate system. The dependency is because moment of inertia changes when the body rotates in different coordinates. The moment of inertial can be calculated for any axis. The knowledge about one axis can help calculating the moment of inertia for a parallel axis.  $I_{xx}$  is denoted as the moment of inertia about axis  $xx$  which is at the center of mass/area. The moment of inertia for axis  $x'$  is

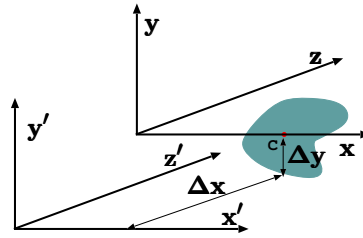


Fig. 5.1 – The schematic that explains the summation of moment of inertia.

$$I_{x'x'} = \int_A r'^2 dA = \int_A (y'^2 + z'^2) dA = \int_A [(y + \Delta y)^2 + (z + \Delta z)^2] dA \quad (5.12)$$

Eq. (5.12) can be expended as

$$I_{x'x'} = \overbrace{\int_A (y^2 + z^2) dA}^{I_{xx}} + \overbrace{2 \int_A (y \Delta y + z \Delta z) dA}^{=0} + \int_A ((\Delta y)^2 + (\Delta z)^2) dA \quad (5.13)$$

The first term in Eq. (5.13) on the right hand side is the moment of inertia about axis  $x$  and the second them is zero. The second therm is zero because it integral of center about center thus is zero. The third term is a new term and can be written as

$$\int_A \overbrace{((\Delta y)^2 + (\Delta z)^2)}^{constant} dA = \overbrace{((\Delta y)^2 + (\Delta z)^2)}^{r^2} \overbrace{\int_A dA}^A = r^2 A \quad (5.14)$$

Hence, the relationship between the moment of inertia at  $xx$  and parallel axis  $x'x'$  is

Parallel Axis Equation

$$I_{x'x'} = I_{xx} + r^2 A \quad (5.15)$$

The moment of inertia of several areas is the sum of moment inertia of each area see Figure 5.2 and therefore,

$$I_{xx} = \sum_{i=1}^n I_{xxi} \quad (5.16)$$

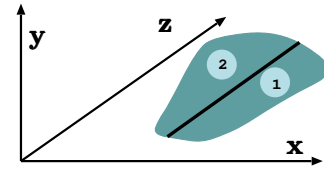


Fig. 5.2 – The schematic to explain the summation of moment of inertia.

If the same areas are similar thus

$$I_{xx} = \sum_{i=1}^n I_{xxi} = n I_{xxi} \quad (5.17)$$

Equation (5.17) is very useful in the calculation of the moment of inertia utilizing the moment of inertia of known bodies. For example, the moment of inertia of half a circle is half of whole circle for axis at the center of circle. The moment of inertia can then move the center of area.

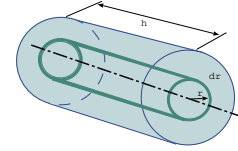


Fig. 5.3 – Cylinder with an element for calculation moment of inertia.

## 5.4 Examples of Moment of Inertia

### Example 5.1: Moment of Cylinder

Level: Easy

Calculate the moment of inertia for the mass of the cylinder about center axis which height of  $h$  and radius,  $r_0$ , as shown in Figure 5.3. The material is with an uniform density and homogeneous.

#### Solution

The element can be calculated using cylindrical coordinate. Here the convenient element is a shell of thickness  $dr$  which shown in Figure 5.3 as

$$I_{rr} = \rho \int_V r^2 dm = \rho \int_0^{r_0} r^2 \overbrace{h 2\pi r dr}^{dV} = \rho h 2\pi \frac{r_0^4}{4} = \frac{1}{2} \rho h \pi r_0^4 = \frac{1}{2} m r_0^2 \quad (5.1.a)$$

The radius of gyration is

$$r_k = \sqrt{\frac{\frac{1}{2} m r_0^2}{m}} = \frac{r_0}{\sqrt{2}} \quad (5.1.b)$$

**Example 5.2: Rectangular moment of Inertia**

**Level: Easy**

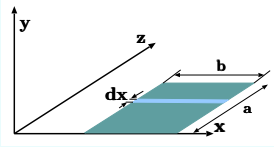


Fig. 5.4 – Description of rectangular in x-y plane for calculation of moment of inertia.

Calculate the moment of inertia of the rectangular shape shown in Figure 5.4 around x coordinate.

**Solution**

The moment of inertia is calculated utilizing equation (5.9) as following

$$I_{xx} = \int_A \left( \overbrace{y^2}^0 + z^2 \right) dA = \int_0^a z^2 \overbrace{bdz}^{dA} = \frac{a^3 b}{3}$$

This value will be used in later examples.

**Example 5.3: Rectangular moment of Inertia**

**Level: Intermediate**

To study the assumption of zero thickness, consider a simple shape to see the effects of this assumption. Calculate the moment of inertia about the center of mass of a square shape with a thickness,  $t$  compare the results to a square shape with zero thickness.

**Solution**

The moment of inertia of transverse slice about  $y'$  (see Fig. 5.5) is

$$dI_{xxm} = \rho \overbrace{dy}^t \overbrace{\frac{I_{xx}}{12}}^{ba^3} \quad (5.3.a)$$

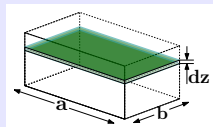


Fig. 5.5 – A square element for the calculations of inertia of two-dimensional to three-dimensional deviations.

End of Ex. 5.3

The transformation into from local axis  $x$  to center axis,  $x'$  can be done as following

$$dI_{x'x'm} = \rho dy \left( \frac{I_{xx}}{12} + \underbrace{z^2}_{r^2} \underbrace{ba}_A \right) \quad (5.3.b)$$

The total moment of inertia can be obtained by integration of equation (5.3.b) to write as

$$I_{xxm} = \rho \int_{-t/2}^{t/2} \left( \frac{ba^3}{12} + z^2 ba \right) dz = \rho t \frac{abt^2 + a^3b}{12} \quad (5.3.c)$$

Comparison with the thin body results in

$$\frac{I_{xx} \rho t}{I_{xxm}} = \frac{ba^3}{t^2 ba + ba^3} = \frac{1}{1 + \frac{t^2}{a^2}} \quad (5.3.d)$$

It can be noticed right away that equation (5.3.d) indicates that ratio approaches one when thickness ratio is approaches zero,  $I_{xxm}(t \rightarrow 0) \rightarrow 1$ . Additionally it can be noticed that the ratio  $a^2/t^2$  is the only contributor to the error<sup>1</sup>. The results are present in Fig. 5.6. I can be noticed that the error becomes significant very fast even for small values of  $t/a$  while the width of the box,  $b$  has no effect on the error.

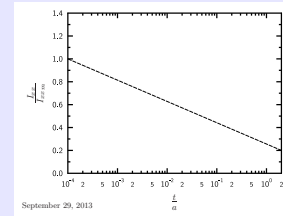


Fig. 5.6 – The ratio of the moment of inertia of two-dimensional to three-dimensional.

**Example 5.4: Moment of Inertia in z direction**

Level: Intermediate

Calculate the rectangular moment of Inertia for the rotation trough center in  $zz$  axis (axis of rotation is out of the page). Hint, construct a small element and build longer build out of the small one. Using this method calculate the entire rectangular. Calculate the rectangular moment of Inertia for the rotation trough

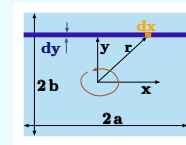


Fig. 5.7 – Rectangular Moment of inertia in the z direction.

**Solution**

<sup>1</sup>This ratio is a dimensionless number that commonly has no special name. This author suggests to call this ratio as the B number.



End of Ex. 5.4

The moment of inertia for a long element with a distance  $y$  shown in Figure 5.7 is

$$dI_{zz}|_{dy} = \int_{-a}^a \overbrace{(y^2 + x^2)}^{r^2} dy dx = \frac{2(3ay^2 + a^3)}{3} dy \quad (5.4.a)$$

The second integration (no need to use (5.15), why?) is

$$I_{zz} = \int_{-b}^b \frac{2(3ay^2 + a^3)}{3} dy \quad (5.4.b)$$

Results in

$$I_{zz} = \frac{a(2ab^3 + 2a^3b)}{3} = \overbrace{\frac{4ab}{A}}^{\frac{4ab}{A}} \left( \frac{(2a)^2 + (2b)^2}{12} \right) \quad (5.4.c)$$

Or

### Example 5.5: Parabola Moment of Inertia

Level: Intermediate

Calculate the center of area and moment of inertia for the parabola,  $y = \alpha x^2$ , depicted in Figure 5.8. Hint, calculate the area first. Use this area to calculate moment of inertia. There are several ways to approach the calculation (different infinitesimal area).

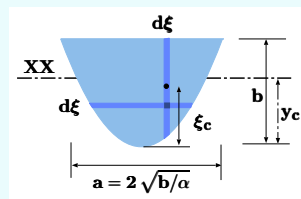


Fig. 5.8 – Parabola for calculations of moment of inertia and the area centroid.

### Solution

For  $y = b$  the value of  $x = \sqrt{b/\alpha}$ . First the area inside the parabola calculated as

$$A = 2 \int_0^{\sqrt{b/\alpha}} \overbrace{(b - \alpha\xi^2)}^{dA/2} d\xi = \frac{2(3\alpha - 1)}{3} \left( \frac{b}{\alpha} \right)^{\frac{3}{2}}$$

The center of area can be calculated utilizing equation (4.5). The center of every element is at,  $\left( \alpha\xi^2 + \frac{b - \alpha\xi^2}{2} \right)$  the element area is used before and therefore

$$x_c = \frac{1}{A} \int_0^{\sqrt{b/\alpha}} \overbrace{\left( \alpha\xi^2 + \frac{b - \alpha\xi^2}{2} \right)}^{x_c} \overbrace{(b - \alpha\xi^2)}^{dA} d\xi = \frac{3\alpha b}{15\alpha - 5} \quad (5.5.a)$$

The moment of inertia of the area about the center can be found using in equation (5.5.a) can be done in two steps first calculate the moment of inertia in this coordinate system and then move the coordinate system to center. Utilizing equation (5.9) and

End of Ex. 5.5

doing the integration from 0 to maximum  $y$  provides

$$I_{x'x'} = 4 \int_0^b \xi^2 \sqrt{\frac{\xi}{\alpha}} d\xi = \frac{2b^{7/2}}{7\sqrt{\alpha}}$$

Utilizing equation (5.15)

$$I_{xx} = I_{x'x'} - A \Delta x^2 = \frac{4b^{7/2}}{7\sqrt{\alpha}} - \frac{3\alpha - 1}{3} \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{(\Delta x = x_c)^2}{\left(\frac{3\alpha b}{15\alpha - 5}\right)^2}$$

or after working the details results in

$$I_{xx} = \frac{\sqrt{b} (20b^3 - 14b^2)}{35\sqrt{\alpha}}$$

**Example 5.6: Triangle Moment of Inertia**

Level: Complex

Calculate the moment of inertia of strait angle triangle about its  $y$  axis as shown in the Figure on the right. Assume that base is  $a$  and the height is  $h$ . What is the moment when a symmetrical triangle is attached on left? What is the moment when a symmetrical triangle is attached on bottom? What is the moment inertia when  $a \rightarrow 0$ ? What is the moment inertia when  $h \rightarrow 0$ ?

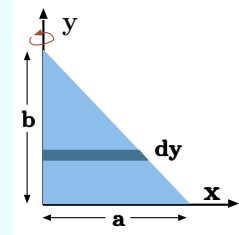


Fig. 5.9 – Triangle for example 5.6.

**Solution**

The right wedge line equation can be calculated as

$$\frac{y}{h} = \left(1 - \frac{x}{a}\right)$$

or

$$\frac{x}{a} = \left(1 - \frac{y}{h}\right)$$

Now using the moment of inertia of rectangle on the side ( $y$ ) coordinate (see example 5.2)

$$\int_0^h \frac{a \left(1 - \frac{y}{h}\right)^3 dy}{3} = \frac{a^3 h}{4}$$

For two triangles attached to each other the moment of inertia will be sum as  $\frac{a^3 h}{2}$ .  
The rest is under construction.

## 5.5 Product of Inertia

In addition to the moment of inertia, the product of inertia is commonly used. Here only the product of the area is defined and discussed. The product of inertia defined as

Moment of Inertia

$$I_{x_i x_j} = \int_A x_i x_j dA \quad (5.18)$$

For example, the product of inertia for  $x$  and  $y$  axis is

$$I_{xy} = \int_A x y dA \quad (5.19)$$

Product of inertia can be positive or negative value as oppose the moment of inertia. The calculation of the product of inertia isn't different much for the calculation of the moment of inertia. The units of the product of inertia are the same as for moment of inertia.

### 5.5.1 Transfer of Axis Theorem

Same as for moment of inertia there is also similar theorem.

$$I_{x' y'} = \int_A x' y' dA = \int_A (x + \Delta x)(y + \Delta y) dA \quad (5.20)$$

expanding equation (5.20) results in

$$I_{x' y'} = \underbrace{\int_A x y dA}_{I_{xy}} + \underbrace{\Delta y \int_A x dA}_{\overset{0}{\int_A x dA}} + \underbrace{\Delta x \int_A y dA}_{\overset{0}{\int_A y dA}} + \underbrace{\int_A \Delta x \Delta y dA}_{\Delta x \Delta y A} \quad (5.21)$$

The final form is

$$I_{x' y'} = I_{xy} + \Delta x \Delta y A \quad (5.22)$$

There are several relationships should be mentioned

$$I_{xy} = I_{yx} \quad (5.23)$$

Symmetrical area has zero product of inertia because integration of odd function (asymmetrical function) left part cancel the right part.

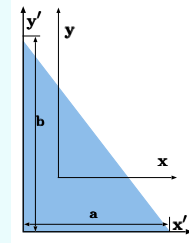
**Example 5.7: Triangle Moment of Inertia****Level: Intermediate**

Fig. 5.10 – Product of inertia for triangle.

Calculate the product of inertia of straight wedge triangle. Assume that body is two dimensional.

**Solution**

The equation of the line is

$$y = \frac{a}{b}x + a$$

The product of inertia at the center is zero. The total product of inertia is

$$I_{x'y'} = 0 + \underbrace{\frac{\Delta x}{a}}_{\frac{a}{3}} \underbrace{\frac{\Delta y}{b}}_{\frac{b}{3}} \underbrace{\left(\frac{A}{2}\right)}_{\left(\frac{ab}{2}\right)} = \frac{a^2 b^2}{18}$$

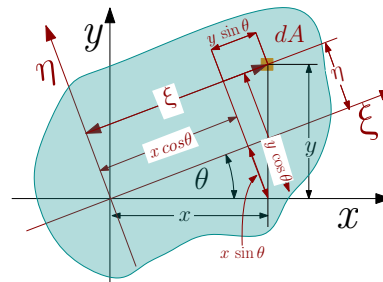
**5.6 Principal Axes of Inertia**

Fig. 5.11 – Rotated coordinate system to explain rotation of axes.

The element area shown in Fig. 5.11 can be measured in two coordinate systems  $x$ - $y$  and  $\xi$ - $\eta$ . The transformation of from one system to another can be expressed by the geometrical distance shown in the figure.

$$\xi = x \cos \theta + y \sin \theta \quad (5.24)$$

and

$$\xi = y \cos \theta - x \sin \theta \quad (5.25)$$

Using these identities and writing by collecting all the right terms yields the following relationship

$$I_{\xi\xi} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\theta - I_{xy} \sin 2\theta \quad (5.26a)$$

$$I_{\eta\eta} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\theta + I_{xy} \sin 2\theta \quad (5.26b)$$

$$I_{\xi\eta} = \frac{I_{xx} - I_{yy}}{2} \sin 2\theta + I_{xy} \cos 2\theta \quad (5.26c)$$

This Eq. (5.26) can be manipulated to show

$$I_{\xi\xi} + I_{\eta\eta} = I_{xx} + I_{yy} \quad (5.27)$$

By differentiating the Eq. (5.26a) with respect to  $\theta$  and setting the result to zero to obtain

$$\frac{dI_{\xi\xi}}{d\theta} = -2 \left( \frac{I_{xx} - I_{yy}}{2} \right) \sin 2\theta - 2 I_{xy} \cos 2\theta \quad (5.28)$$

Thus, the maximum or minimum is at

$$\tan 2\theta_{principal} = \frac{2 I_{xy}}{I_{yy} - I_{xx}} \quad (5.29)$$

Using these results it can be shown that the maximum/minimum moment of inertia is

$$I_{\frac{max}{min}} = \frac{I_{xx} + I_{yy}}{2} \pm \sqrt{\left( \frac{I_{xx} - I_{yy}}{2} \right)^2 + I_{xy}^2} \quad (5.30)$$

The topic of Mohrs Circle is not discussed here because it is out the possible scope of this book. Only for completion presentation of 3D tensor of the moment of inertia is presented.

The inertia matrix or inertia tensor is

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \quad (5.31)$$

In linear algebra it was shown that for some angle equation (5.31) can be transform into

$$\begin{bmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{bmatrix} \quad (5.32)$$

System which creates equation (5.32) referred as principle system.



# 6

## Pressure

### 6.1 Introduction

The concept of pressure is very important in stability. As opposed to regular treatment in Fluid Mechanics books or class, it will be treated as a scalar and care will be given to minimize the actual complex mathematical details. It is assumed that some of the readers of this book lack the mathematical background, hence the complexity will be abstracted.

In simple terms, pressure is the ratio of force and the area such as

$$P = \frac{F}{A} \quad (6.1)$$

The force,  $F$  is a vector as well the area,  $A$ . So, division of vector by vector creates a complex creation with nine (9) terms. However, here only one term is used since the pressure is equal in all three directions and there no other terms (at no flow situations). Here, the pressure is assumed to be acting force magnitude divided by the area in the same direction. When the "bureaucratic" formality is cleared, a simple pressure explanation can be presented. A cylinder of liquid shown in the Fig. 6.1 the top is at liquid surface and is weight

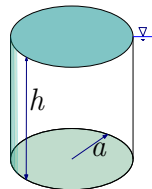


Fig. 6.1 – Cylinder to explain the pressure concept.

$$W = \underbrace{\pi a^2}_A \overbrace{h}^V \rho g \quad (6.2)$$

The weight is the force that acting on the surface at the bottom of the cylinder. The pressure at the surface according to the definition 6.1 is

$$\frac{\cancel{\pi a^2} h \rho g}{\cancel{\pi a^2}} = \rho g h \quad (6.3)$$

The pressure in stationary liquid is simply depends on the density and the height (and the gravity). The pressure is not depends on the area.

This concept that pressure depends solely on the depth of liquid is extended to the fact that there is no need for it be in a continuous line. That is the pressure constant pressure lines that goes where the liquid is connected. For example, in Fig. 6.2 depicts constant pressure line across solid walls.

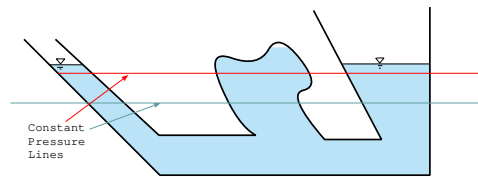


Fig. 6.2 – Constant pressure lines across solid boundaries. It can be noticed that free surface are equal while the close could be higher.

In fact the this idea used in may places as way to measure two locations to be in the same height.

## 6.2 Fluid Forces on Surfaces

The forces that fluids (at static conditions) extracts on surfaces are very important for engineering purposes. This section deals with these calculations which are divided into two categories, straight surfaces and curved surfaces. These issues especially applied to floating bodies and ships.

### 6.2.1 Fluid Forces on Straight Surfaces

A motivation is needed before going through the routine of derivations. Initially, a simple case will be examined. Later, how the calculations can be simplified will be shown.

#### Example 6.1: Pressure on Door

Level: Intermediate

A straight vertical ship door is exposed to salt water at density of  $\rho_{ws}$ . The width of door is 1[m] and its length is 2[m] and door hinges are located at the surface of the water. The calculate the total force acted on the door and the moment the water acts on the door. If water was acting both sides of the door the net force and momentum is zero. Why?



## Solution

The pressure that acting on door varied from zero at the top (since it is at the water surface) and maximum pressure at the bottom. To avoid using the mathematics the two values can be taken and averaged and multiply to get the force. The pressure at the bottom is  $\rho g h$  and the height is given the  $h$  and thus average pressure

$$P_{avg} = \frac{\rho g h}{2} \quad (6.1.a)$$

The total force will be

$$F = \overbrace{\frac{\rho g h}{2}}^P A = 1000 \times 9.8 \times 2/2 \times \overbrace{2 \times 1}^A \sim 19,600[N] \quad (6.1.b)$$

The force is incredible almost 20 tons. According to same logic the moment (assuming that this concept is introduced somewhere else before) will be at the center and value will be  $20 \text{ ton} \times m$ . At this part a bit must be inserted and check against these assumptions. It is regular procedure to take an very small element in which the pressure is uniform and add these elements. The element chosen is an element  $dh$  and the width of the door. The pressure in the element ( $a dh$ ) is uniform since the element depth is uniform. The element force is  $dF = \rho g h a dh$  and total force is

$$F = \int_0^h \rho g h a dh = \rho g a \frac{h^2}{2} \quad (6.1.c)$$

which identical to what was computed using the average pressure. Note the integral of  $h$  is  $h^2/2$ . To find the point of equivalent force (that force substitute pressure) obtained by

$$h_{effective} = \frac{M}{F} = \frac{\int_0^h h \overbrace{\rho g h a dh}^{dF}}{0.5 \rho g a h^2} = \frac{\int_0^h h^2 dh}{0.5 h} = \frac{h^3/3}{0.5 h} = \frac{2h}{3} \quad (6.1.d)$$

The result is the center of triangle which is the shape of the pressure. So while the pressure assumption was correct the center of pressure is at center of the triangle.

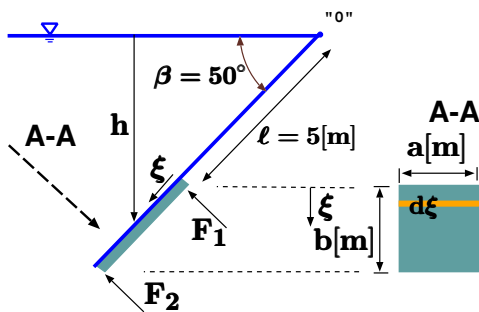


Fig. 6.3 – Rectangular area under pressure.

**Example 6.2: Pressure on Rectangular****Level: Intermediate**

Consider a rectangular shape gate as shown in Fig. 6.3. Calculate the minimum forces,  $F_1$  and  $F_2$  to maintain the gate in position. Assuming that the atmospheric pressure can be ignored.

**Solution**

The forces can be calculated by looking at the moment around point "O." The element of moment is  $a d\xi$  for the width of the gate and is

$$dM = \overbrace{P a d\xi}^{dF} (\ell + \xi) \quad (6.2.a)$$

The pressure,  $P$  can be expressed as a function  $\xi$  as the following

$$P = g \rho (\ell + \xi) \sin \beta$$

The liquid total moment on the gate is

$$M = \int_0^b g \rho (\ell + \xi) \sin \beta a d\xi (\ell + \xi)$$

The integral can be simplified as

$$M = g a \rho \sin \beta \int_0^b (\ell + \xi)^2 d\xi \quad (6.4)$$

The solution of the above integral is

$$M = g \rho a \sin \beta \left( \frac{3 b \ell^2 + 3 b^2 \ell + b^3}{3} \right)$$

This value provides the moment that  $F_1$  and  $F_2$  should extract. Additional equation is needed. It is the total force, which is

$$F_{total} = \int_0^b g \rho (\ell + \xi) \sin \beta a d\xi$$

The total force integration provides

$$F_{total} = g \rho a \sin \beta \int_0^b (\ell + \xi) d\xi = g \rho a \sin \beta \left( \frac{2 b \ell + b^2}{2} \right)$$

The forces on the gate have to provide

$$F_1 + F_2 = g \rho a \sin \beta \left( \frac{2 b \ell + b^2}{2} \right)$$

Additionally, the moment of forces around point "O" is

$$F_1 \ell + F_2 (\ell + b) = g \rho a \sin \beta \left( \frac{3 b \ell^2 + 3 b^2 \ell + b^3}{3} \right)$$

End of Ex. 6.2

The solution of these equations is

$$F_1 = \frac{(3\ell + b) a b g \rho \sin \beta}{6}$$

$$F_2 = \frac{(3\ell + 2b) a b g \rho \sin \beta}{6}$$

The above calculations are time consuming and engineers always try to make life simpler. Looking at the above calculations, it can be observed that there is a moment of area in Eq. (6.4) and also a center of area. These concepts have been introduced in Chapter 5. Several represented areas for which moment of inertia and center of area have been tabulated in Chapter 5. These tabulated values can be used to solve this kind of problems.

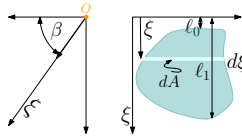


Fig. 6.4 – Schematic of submerged area to explain the center forces and moments.

### 6.2.1.1 Symmetrical Shapes

Consider the two-dimensional symmetrical area that are under pressure as shown in Figure 6.4. The symmetry is around any axes parallel to axis  $x$ . The total force and moment that the liquid extracting on the area need to be calculated. First, the force is

$$F = \int_A P dA = \int (P_{atmos} + \rho g h) dA = A P_{atmos} + \rho g \int_{\ell_0}^{\ell_1} \overbrace{(\xi + \ell_0) \sin \beta}^{h(\xi)} dA \quad (6.5)$$

In this case, the atmospheric pressure can include any additional liquid layer above layer “touching” area. The “atmospheric” pressure can be set to zero.

The boundaries of the integral of equation (6.5) refer to starting point and ending points not to the start area and end area. The integral in equation (6.5) can be further developed as

$$F_{total} = A P_{atmos} + \rho g \sin \beta \left( \ell_0 A + \overbrace{\int_{\ell_0}^{\ell_1} \xi dA}^{x_c A} \right) \quad (6.6)$$

In a final form as

$$F_{total} = A [P_{atmos} + \rho g \sin \beta (\ell_0 + x_c)] \quad (6.7)$$

The moment of the liquid on the area around point "O" is

$$M_y = \int_{\xi_0}^{\xi_1} P(\xi) \xi dA \quad (6.8)$$

$$M_y = \int_{\xi_0}^{\xi_1} (P_{atmos} + g \rho \overbrace{h(\xi)}^{\xi \sin \beta}) \xi dA \quad (6.9)$$

Or separating the parts as

$$M_y = P_{atmos} \overbrace{\int_{\xi_0}^{\xi_1} \xi dA}^{x_c A} + g \rho \sin \beta \overbrace{\int_{\xi_0}^{\xi_1} \xi^2 dA}^{I_{x'x'}} \quad (6.10)$$

The moment of inertia,  $I_{x'x'}$ , is about the axis through point "O" into the page. Equation (6.10) can be written in more compact form as

Total Moment in Inclined Surface

$$M_y = P_{atmos} x_c A + g \rho \sin \beta I_{x'x'} \quad (6.11)$$

Example 6.2 can be generalized to solve any two forces needed to balance the area/gate. Consider the general symmetrical body shown in Fig. 6.5 which has two forces that balance the body. Equations (6.7) and (6.11) can be combined the moment and force acting on the general area. If the "atmospheric pressure" can be zero or include additional layer of liquid. The forces balance reads

$$F_1 + F_2 = A [P_{atmos} + \rho g \sin \beta (\ell_0 + x_c)] \quad (6.12)$$

and moments balance reads

$$F_1 a + F_2 b = P_{atmos} x_c A + g \rho \sin \beta I_{x'x'} \quad (6.13)$$

The solution of these equations is

$$F_1 = \frac{\left[ \left( \rho \sin \beta - \frac{P_{atmos}}{g b} \right) x_c + \ell_0 \rho \sin \beta + \frac{P_{atmos}}{g} \right] b A - I_{x'x'} \rho \sin \beta}{g (b - a)} \quad (6.14)$$

and

$$F_2 = \frac{I_{x'x'} \rho \sin \beta - \left[ \left( \rho \sin \beta - \frac{P_{atmos}}{g a} \right) x_c + \ell_0 \rho \sin \beta + \frac{P_{atmos}}{g} \right] a A}{g (b - a)} \quad (6.15)$$

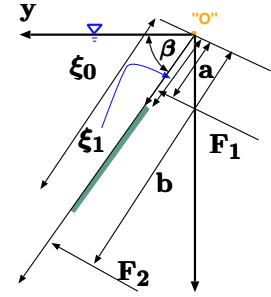


Fig. 6.5 – The general forces acting on submerged area.

In the solution, the forces can be negative or positive, and the distance  $a$  or  $b$  can be positive or negative. Additionally, the atmospheric pressure can contain either an additional liquid layer above the “touching” area or even atmospheric pressure simply can be set up to zero. In symmetrical area only two forces are required since the moment is one dimensional. However, in non-symmetrical area there are two different moments and therefore three forces are required. Thus, additional equation is required. This equation is for the additional moment around the  $x$  axis (see for explanation in Figure 6.6). The moment around the  $y$  axis is given by equation (6.11) and the total force is given by (6.7). The moment around the  $x$  axis (which was arbitrary chosen) should be

$$M_x = \int_A y P dA \quad (6.16)$$

Substituting the components for the pressure transforms equation (6.16) into

$$M_x = \int_A y (P_{atmos} + \rho g \xi \sin \beta) dA \quad (6.17)$$

The integral in equation (6.16) can be written as

$$M_x = P_{atmos} \overbrace{\int_A y dA}^{Ay_c} + \rho g \sin \beta \overbrace{\int_A \xi y dA}^{I_{x'y'}} \quad (6.18)$$

The compact form can be written as

$$M_x = P_{atmos} A y_c + \rho g \sin \beta I_{x'y'} \quad (6.19)$$

The product of inertia was presented in Chapter 5. These equations (6.7), (6.11) and (6.19) provide the base for solving any problem for straight area under pressure with uniform density. There are many combinations of problems (e.g. two forces and moment) but no general solution is provided. Example to illustrate the use of these equations is provided.

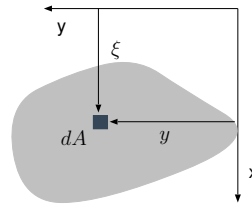


Fig. 6.6 – The general forces acting on non symmetrical straight area.

### Example 6.3: Pressure on Non Symmetrical

Level: Intermediate

Calculate the forces which required to balance the triangular shape shown in the Figure 6.7.

## Solution

The three equations that needs to be solved are

$$F_1 + F_2 + F_3 = F_{total} \quad (6.3.a)$$

The moment around  $x$  axis is

$$F_1 b = M_y \quad (6.3.b)$$

The moment around  $y$  axis is

$$F_1 \ell_1 + F_2 (a + \ell_0) + F_3 \ell_0 = M_x \quad (6.3.c)$$

The right hand side of these equations are given before in equations (6.7), (6.11) and (6.19).

The moment of inertia of the triangle around  $x$  is made of two triangles (as shown in the Figure (6.7) for triangle 1 and 2). Triangle 1 can be calculated as the moment of inertia around its center which is  $\ell_0 + 2 * (\ell_1 - \ell_0) / 3$ . The height of triangle 1 is  $(\ell_1 - \ell_0)$  and its width  $b$  and thus, moment of inertia about its center is  $I_{xx1} = b(\ell_1 - \ell_0)^3 / 36$ . The moment of inertia for triangle 1 about  $y$  is

$$I_{xx1} = \frac{b(\ell_1 - \ell_0)^3}{36} + \overbrace{\frac{b(\ell_1 - \ell_0)}{3}}^{A_1} \overbrace{\left( \ell_0 + \frac{2(\ell_1 - \ell_0)}{3} \right)^2}^{\Delta x_1^2}$$

The height of the triangle 2 is  $a - (\ell_1 - \ell_0)$  and its width  $b$  and thus, the moment of inertia about its center is

$$I_{xx2} = \frac{b[a - (\ell_1 - \ell_0)]^3}{36} + \overbrace{\frac{b[a - (\ell_1 - \ell_0)]}{3}}^{A_2} \overbrace{\left( \ell_1 + \frac{[a - (\ell_1 - \ell_0)]}{3} \right)^2}^{\Delta x_2^2}$$

and the total moment of inertia

$$I_{xx} = I_{xx1} + I_{xx2}$$

The product of inertia of the triangle can be obtain by integration. It can be noticed that upper line of the triangle is  $y = \frac{(\ell_1 - \ell_0)x}{b} + \ell_0$ . The lower line of the triangle is  $y = \frac{(\ell_1 - \ell_0 - a)x}{b} + \ell_0 + a$ .

$$I_{xy} = \int_0^b \left[ \int_{\frac{(\ell_1 - \ell_0)x}{b} + \ell_0}^{\frac{(\ell_1 - \ell_0 - a)x}{b} + \ell_0 + a} x y dx \right] dy = \frac{2 a b^2 \ell_1 + 2 a b^2 \ell_0 + a^2 b^2}{24}$$

The solution of this set equations is

$$F_1 = \left[ \frac{ab}{3} \right] \frac{(g(6\ell_1 + 3a) + 6g\ell_0) \rho \sin \beta + 8 P_{atmos}}{24},$$

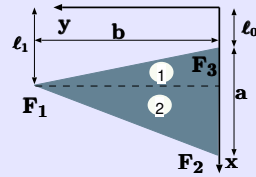


Fig. 6.7 – The general forces acting on a non symmetrical straight area.

End of Ex. 6.3

$$\frac{F_2}{\left[\frac{ab}{3}\right]} = - \frac{\left( (3\ell_1 - 14a) - \ell_0 \left( \frac{12\ell_1}{a} - 27 \right) + \frac{12\ell_0^2}{a} \right) g \rho \sin \beta}{72} - \frac{\left( \left( \frac{24\ell_1}{a} - 24 \right) + \frac{48\ell_0}{a} \right) P_{atmos}}{72}, \quad (6.3.d)$$

$$\frac{F_3}{\left[\frac{ab}{3}\right]} = \frac{\left( \left( a - \frac{15\ell_1}{a} \right) + \ell_0 \left( 27 - \frac{12\ell_1}{a} \right) + \frac{12\ell_0^2}{a} \right) g \rho \sin \beta}{72} + \frac{\left( \left( \frac{24\ell_1}{a} + 24 \right) + \frac{48\ell_0}{a} \right) P_{atmos}}{72} \quad (6.3.e)$$

### 6.2.2 Pressure Center

In the literature, pressure centers are commonly defined. These definitions are mathematical in nature and has physical meaning of equivalent force that will act through this center. The definition is derived or obtained from equation (6.11) and equation (6.19). The pressure center is the distance that will create the moment with the hydrostatic force on point "O." Thus, the pressure center in the  $x$  direction is

$$x_p = \frac{1}{F} \int_A x P dA \quad (6.20)$$

In the same way, the pressure center in the  $y$  direction is defined as

$$y_p = \frac{1}{F} \int_A y P dA \quad (6.21)$$

To show relationship between the pressure center and the other properties, it can be found by setting the atmospheric pressure and  $\ell_0$  to zero as following

$$x_p = \frac{g \rho \sin \beta I_{x'x'}}{A \rho g \sin \beta x_c} \quad (6.22)$$

Expanding  $I_{x'x'}$  according to equation (5.12) results in

$$x_p = \frac{I_{xx}}{x_c A} + x_c \quad (6.23)$$

and in the same fashion in  $y$  direction

$$y_p = \frac{I_{yy}}{y_c A} + y_c \quad (6.24)$$

It has to emphasis that these definitions are useful only for case where the atmospheric pressure can be neglected or canceled and where  $\ell_0$  is zero. Thus, these limitations diminish the usefulness of pressure center definitions. In fact, the reader can find that direct calculations can sometimes simplify the problem.

### 6.3 Pressure on Floating Body

While the fundamentals of the pressure where discussed before it worth while to review with a typical body like a block. The pressure around the block has a shape of triangle. It can be noticed that pressure on the front is identical to the pressure at the back. Hence it cancel each other. The same can be said for the right side and the left side. Similar argument is made for diagonal surfaces. For diagonal surface the net force in any direction is just the "shadow" in that direction times the pressure. The pressure is the same in the same depth and hence it cancel. The only side that pressure is not cancel it the pressure on the bottom. The pressure is total (force =  $PA$ ) in the weight of the ship or the body. When the body is total immersed in the liquid, the force at the bottom is reduced by the force at the top. The net force in the vertical direction, at the stationary conditions, is the weight of the body.

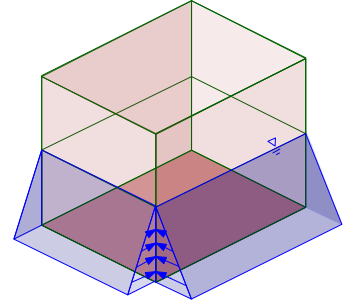


Fig. 6.8 – Pressure on a floating block.

### 6.4 Summary

The main points that this chapter was indent to illustrate are the following. Pressure lines are constant depth form the surface. The pressure acting surfaces from both sides has zero force and zero moment because there is a counter force and momentum. On straight surfaces the forces are at  $2/3$  of the distance.



# 7

## Buoyancy

### 7.1 Introduction

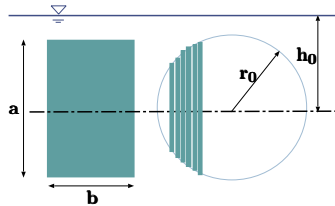


Fig. 7.1 – Schematic of Immersed Cylinder.

One of the oldest known scientific research on fluid mechanics relates to buoyancy due to question of money was carried by Archimedes. Archimedes principle is related to question of density and volume. While Archimedes did not know much about integrals, he was able to capture the essence. Here, because this material is presented in a different era, more advance mathematics will be used. While the question of the stability was not scientifically examined in the past, the floating vessels structure (more than 300 years ago) show some understanding<sup>1</sup>.

The total net forces the liquid and gravity exact on a body are considered as a buoyancy issue while the moment these force considered as a stability issue. The buoyancy issue was solved by Archimedes and for all practical purpose is really solved

<sup>1</sup>This topic was the author's high school class name (ship stability). It was taught by people like these, 300 years ago and more, ship builders who knew how to calculate  $GM$  but weren't aware of scientific principles behind it. If the reader wonders why such a class is taught in a high school, perhaps the name can explain it: Sea Officers High School ( or Acco (sometimes spell Akko) Nautical College)

issue. Furthermore, as a derivative issue, the stability in the perpendicular direction to the liquid surface is a solved problem which did not cause any real question (like oscillating of body is solved problem). While there are recent papers which deal the issue but they do solve any issue in this respect. However, the rotation stability is issue that continue to be evolved even after this work. There three approaches that deal with issue which are in historical order are Metacenter, Potential, and Direct Examination<sup>2</sup>. These stability issues will discussed in the next several chapters.

To understand this issue, consider a cubical and a cylindrical body that is immersed in liquid and center in a depth of,  $h_0$  as shown in Fig. 7.1. The force which hold the cylinder at the place made from the integration of the pressure around the surface of the bodies. The forces on square geometry body are made only of vertical forces because the two sides cancel each other. However, on the vertical direction, the pressure on the two surfaces are different. On the upper surface the pressure is  $\rho g (h_0 - a/2)$ . On the lower surface the pressure is  $\rho g (h_0 + a/2)$ .

The force due to the liquid pressure per unit depth (into the page) is

$$F = \rho g ((h_0 - a/2) - (h_0 + a/2)) \ell b = -\rho g a b \ell = -\rho g V \quad (7.1)$$

In this case the  $\ell$  represents a depth (into the page). Rearranging equation (7.1) to be

$$\frac{F}{V} = \rho g \longrightarrow F = V \rho g \quad (7.2)$$

The force acting on the immersed body is equal to the weight of the displaced liquid. This analysis can be generalized by noticing two things. All the horizontal forces are canceled. Any body that has a projected area that has two sides, those will cancel each other in the perpendicular to surface direction. Another way to look at this point is by approximation. For any two rectangle bodies, the horizontal forces are canceling each other. Thus, even these bodies are in contact with each other, the imaginary pressure make it so that they cancel each other.

On the other hand, any shape is made of many small rectangles. The force on every rectangular shape is made of its weight of the volume. Thus, the total force is made of the sum of all the small rectangles which is the weight of the sum of all volume.

In illustration of this concept, consider the cylindrical shape in Figure 7.1. The force per area (see Figure 7.2) is

$$dF = \overbrace{\rho g (h_0 - r \sin \theta)}^P \overbrace{\sin \theta r d\theta}^{dA_{vertical}} \quad (7.3)$$

<sup>2</sup>The first method was developed 300 years ago, the potential was developed about 30 years ago and Direct Examination is present here for the first time.

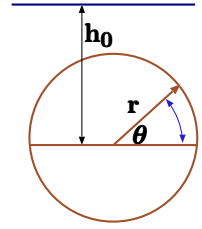


Fig. 7.2 – The floating forces on Immersed Cylinder.

The total force will be the integral of the equation (7.3)

$$F = \int_0^{2\pi} \rho g (h_0 - r \sin \theta) r d\theta \sin \theta \quad (7.4)$$

Rearranging equation (7.3) transforms it to

$$F = r g \rho \int_0^{2\pi} (h_0 - r \sin \theta) \sin \theta d\theta \quad (7.5)$$

The solution of equation (7.5) is

$$F = -\pi r^2 \rho g \quad (7.6)$$

The negative sign indicate that the force acting upwards. While the horizontal force is

$$F_v = \int_0^{2\pi} (h_0 - r \sin \theta) \cos \theta d\theta = 0 \quad (7.7)$$

## 7.2 Examples

### Example 7.1: Floating Log

Level: Advance

To what depth will a long log with radius,  $r$ , a length,  $\ell$  and density,  $\rho_s$  in liquid with density,  $\rho_\ell$ . Assume that  $\rho_\ell > \rho_s$ . You can provide the angle or the depth as the solution.

#### Solution

This example actually deals with two dimensional problem. Archimedes formula reads

$$V \rho_s = V_0 \rho_\ell \quad (7.1.a)$$

The equations for both volumes linearly depended on the length  $s$  as

$$A \rho_s = A_0 \rho_\ell \quad (7.1.b)$$

where  $A_0$  is the area of the segment. The area of segment as a function of  $\theta$  is

$$A_0 = \frac{r^2 (2\theta - \sin(2\theta))}{2} \quad (7.1.c)$$

where  $\theta$  is the angle measured from the bottom to the contact point between the liquid surface and the circle and the center is at the center of the circle. Substituting Eq. (7.1.c) into Eq. (7.1.b) results in

$$\pi r^2 \rho_s = \left( \frac{r^2 (2\theta - \sin(2\theta))}{2} \right) \rho_\ell \quad (7.1.d)$$

End of Ex. 7.1

which can be somewhat simplified as

$$\frac{2\pi\rho_s}{\rho_\ell} = (2\theta - \sin(2\theta)) \quad (7.1.e)$$

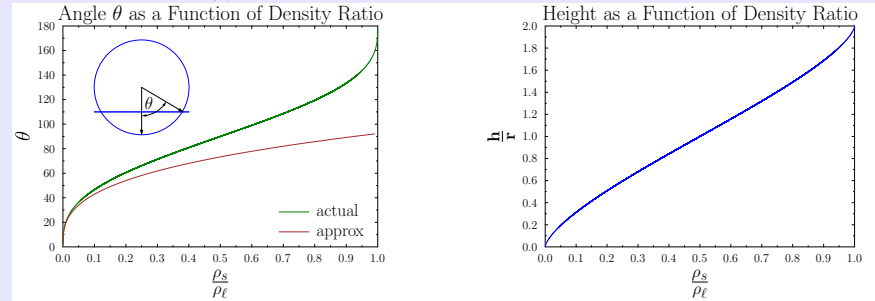
From practical point of view, this example is solved. Nevertheless, for the fun of the math the  $\sin\theta$  can be expanded in Taylor series as

$$\frac{2\pi\rho_s}{\rho_\ell} = \frac{(2\theta)^3}{3!} - \frac{(2\theta)^5}{5!} + \frac{(2\theta)^7}{7!} + \dots \quad (7.1.f)$$

And for small  $\theta$  (or small ratio of the density) Eq. (7.1.f) is reduced into

$$\theta \sim \sqrt[3]{\frac{4\pi\rho_s}{3\rho_\ell}} \quad \frac{\rho_s}{\rho_\ell} \lesssim 0.4 \quad (7.1.g)$$

Which means that  $\frac{\rho_s}{\rho_\ell} < 0.0002$  for this approximation to be valid.



(a) The angle as a function of the density ratio

(b) The angle as a function of the density ratio

Fig. 7.4 – The effect of the density ratio on the buoyancy. Note the angle is from the bottom to the contact of the liquid with cylinder.

Typical examples to explain the buoyancy are of the vessel with thin walls put upside down into liquid. The second example of the speed of the floating bodies. Since there are no better examples, these examples are a must.

**Example 7.2: Reverse Container****Level: Advance**

A cylindrical body, shown in Figure 7.5, is floating in liquid with density,  $\rho_l$ . The body was inserted into liquid in a such a way that the air had remained in it. Express the maximum wall thickness,  $t$ , as a function of the density of the wall,  $\rho_s$  liquid density,  $\rho_l$  and the surroundings air temperature,  $T_1$  for the body to float. In the case where thickness is half the maximum, calculate the pressure inside the container. The container diameter is  $w$ . Assume that the wall thickness is small compared with the other dimensions ( $t \ll w$  and  $t \ll h$ ).

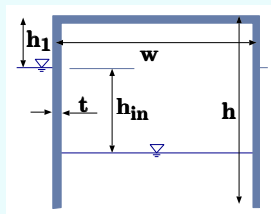


Fig. 7.5 – Schematic of a thin wall floating body.

**Solution**

The air mass in the container is

$$m_{air} = \underbrace{\pi w^2 h}_{V} \overbrace{\frac{P_{atmos}}{RT}}^{\rho_{air}}$$

The mass of the container is

$$m_{container} = \left( \underbrace{\pi w^2 + 2\pi w h}_A \right) t \rho_s$$

The liquid amount enters into the cavity is such that the air pressure in the cavity equals to the pressure at the interface (in the cavity). Note that for the maximum thickness, the height,  $h_1$  has to be zero. Thus, the pressure at the interface can be written as

$$P_{in} = \rho_l g h_{in}$$

On the other hand, the pressure at the interface from the air point of view (ideal gas model) should be

$$P_{in} = \frac{m_{air} R T_1}{\underbrace{h_{in} \pi w^2}_V}$$

Since the air mass didn't change and it is known, it can be inserted into the above equation.

$$\rho_l g h_{in} + P_{atmos} = P_{in} = \frac{(\pi w^2 h) \overbrace{\frac{P_{atmos}}{R T_1}}^{\rho}}{h_{in} \pi w^2} R T_1$$

The last equation can be simplified into

$$\rho_l g h_{in} + P_{atmos} = \frac{h P_{atmos}}{h_{in}}$$

End of Ex. 7.2

And the solution for  $h_{in}$  is

$$h_{in} = -\frac{P_{atmos} + \sqrt{4ghP_{atmos}\rho_l + P_{atmos}^2}}{2g\rho_l}$$

and

$$h_{in} = \frac{\sqrt{4ghP_{atmos}\rho_l + P_{atmos}^2} - P_{atmos}}{2g\rho_l}$$

The solution must be positive, so that the last solution is the only physical solution.

— — — — — Advance material can be skipped — — — — —

### Example 7.3: Reverse Container

Level: Hard

Calculate the minimum density an infinitely long equilateral triangle Triangle shape (three equal sides) has to be so that the sharp end is in the water. Assume that the body is stable (it is not for most).

#### Solution

The solution demonstrates that when  $h \rightarrow 0$  then  $h_{in} \rightarrow 0$ . When the gravity approaches zero (macro gravity) then

$$h_{in} = \frac{P_{atmos}}{\rho_l g} + h - \frac{h^2 \rho_l g}{P_{atmos}} + \frac{2h^3 \rho_l^2 g^2}{P_{atmos}^2} - \frac{5h^4 \rho_l^3 g^3}{P_{atmos}^3} + \dots$$

This “strange” result shows that bodies don’t float in the normal sense. When the floating is under vacuum condition, the following height can be expanded into

$$h_{in} = \sqrt{\frac{hP_{atmos}}{g\rho_l}} + \frac{P_{atmos}}{2g\rho_l} + \dots$$

which shows that the large quantity of liquid enters into the container as it is expected. Archimedes theorem states that the force balance is at displaced weight liquid (of the same volume) should be the same as the container, the air. Thus,

$$\underbrace{\pi w^2 (h - h_{in}) g}_{\text{net displaced water}} = \underbrace{(\pi w^2 + 2\pi w h) t \rho_s g}_{\text{container}} + \underbrace{\pi w^2 h \left(\frac{P_{atmos}}{RT_1}\right) g}_{\text{air}} \quad (7.3.a)$$

If air mass is neglected the maximum thickness is

$$t_{max} = \frac{2ghw\rho_l + P_{atmos}w - w\sqrt{4ghP_{atmos}\rho_l + P_{atmos}^2}}{(2gw + 4gh)\rho_l\rho_s} \quad (7.3.b)$$

The condition to have a physical value for the maximum thickness is

$$2gh\rho_l + P_{atmos} \geq \sqrt{4ghP_{atmos}\rho_l + P_{atmos}^2} \quad (7.3.c)$$

## End of Ex. 7.3

The full solution is

$$t_{max} = \frac{(w R \sqrt{4 g h P_{atmos} \rho_l + P_{atmos}^2} - 2 g h w R \rho_l - P_{atmos} w R) T_1 + 2 g h P_{atmos} w \rho_l}{(2 g w + 4 g h) R \rho_l \rho_s T_1} \quad (7.3.d)$$

In this analysis the air temperature in the container immediately after insertion in the liquid has different value from the final temperature. It is reasonable as the first approximation to assume that the process is adiabatic and isentropic. Thus, the temperature in the cavity immediately after the insertion is

$$\frac{T_i}{T_f} = \left( \frac{P_i}{P_f} \right) \quad (7.3.e)$$

The final temperature and pressure were calculated previously. The equation of state is

$$P_i = \frac{m_{air} R T_i}{V_i} \quad (7.3.f)$$

The new unknown must provide additional equation which is

$$V_i = \pi w^2 h_i \quad (7.3.g)$$

### Thickness Below The Maximum

For the half thickness  $t = \frac{t_{max}}{2}$  the general solution for any given thickness below maximum is presented. The thickness is known, but the liquid displacement is still unknown. The pressure at the interface (after long time) is

$$\rho_l g h_{in} + P_{atmos} = \frac{\pi w^2 h \frac{P_{atmos}}{R T_1} R T_1}{(h_{in} + h_1) \pi w^2}$$

which can be simplified to

$$\rho_l g h_{in} + P_{atmos} = \frac{h P_{atmos}}{h_{in} + h_1}$$

The second equation is Archimedes' equation, which is

$$\pi w^2 (h - h_{in} - h_1) = (\pi w^2 + 2 \pi w h) t \rho_s g + \pi w^2 h \left( \frac{P_{atmos}}{R T_1} \right) g$$

— — — — — End Advance material — — — — —

### Example 7.4: Reverse Container

Level: Intermediate

A body is pushed into the liquid to a distance,  $h_0$  and left at rest. Calculate acceleration and time for a body to reach the surface. The body's density is  $\alpha \rho_l$ , where  $\alpha$  is ratio between the body density to the liquid density and ( $0 < \alpha < 1$ ). Is the body volume important?

## Solution

The net force is

$$F = \overbrace{V g \rho_l}^{\text{liquid weight}} - \overbrace{V g \alpha \rho_l}^{\text{body weight}} = V g \rho_l (1 - \alpha)$$

But on the other side the internal force is

$$F = m a = \overbrace{V \alpha \rho_l}^m a$$

Thus, the acceleration is

$$a = g \left( \frac{1 - \alpha}{\alpha} \right)$$

If the object is left at rest (no movement) thus time will be ( $h = 1/2 a t^2$ )

$$t = \sqrt{\frac{2 h \alpha}{g(1 - \alpha)}}$$

If the object is very light ( $\alpha \rightarrow 0$ ) then

$$t_{min} = \sqrt{\frac{2 h \alpha}{g}} + \frac{\sqrt{2 g h} \alpha^{\frac{3}{2}}}{2 g} + \frac{3 \sqrt{2 g h} \alpha^{\frac{5}{2}}}{8 g} + \frac{5 \sqrt{2 g h} \alpha^{\frac{7}{2}}}{16 g} + \dots$$

From the above equation, it can be observed that only the density ratio is important. This idea can lead to experiment in “large gravity” because the acceleration can be magnified and it is much more than the reverse of free falling.

## Example 7.5: Half Sphere

Level: Intermediate

In some situations, it is desired to find equivalent of force of a certain shape to be replaced by another force of a “standard” shape. Consider the force that acts on a half sphere. Find equivalent cylinder that has the same diameter that has the same force.

## Solution

The force act on the half sphere can be found by integrating the forces around the sphere. The element force is

$$dF = (\rho_L - \rho_S) g \overbrace{r \cos \phi}^h \overbrace{\cos \theta \cos \theta}^{dA_x} \overbrace{\cos \phi r^2 d\theta d\phi}^{dA}$$



End of Ex. 7.5

The total force is then

$$F_x = \int_0^\pi \int_0^\pi (\rho_L - \rho_S) g \cos^2 \phi \cos^2 \theta r^3 d\theta d\phi$$

The result of the integration the force on sphere is

$$F_s = \frac{\pi^2 (\rho_L - \rho_S) r^3}{4}$$

The force on equivalent cylinder is

$$F_c = \pi r^2 (\rho_L - \rho_S) h$$

These forces have to be equivalent and thus

$$\frac{\pi^2 (\rho_L - \rho_S) r^3}{4} = \pi r^2 (\rho_L - \rho_S) h$$

Thus, the height is

$$\frac{h}{r} = \frac{\pi}{4}$$

### Example 7.6: Two Liquid Layers

Level: Intermediate

In the introduction to this section, it was assumed that above liquid is a gas with inconsequential density. Suppose that the above layer is another liquid which has a bit lighter density. Body with density between the two liquids,  $\rho_l < \rho_s < \rho_h$  is floating between the two liquids. Develop the relationship between the densities of liquids and solid and the location of the solid cubical. There are situations where density is a function of the depth. What will be the location of solid body if the liquid density varied parabolically.

### Solution

In the discussion to this section, it was shown that the net force is the body volume times the density of the liquid. In the same vein, the body can be separated into two: one in first liquid and one in the second liquid. In this case there are two different liquid densities. The net force down is the weight of the body  $\rho_c h A$ . Where  $h$  is the height of the body and  $A$  is its cross section. This force is balance according to above explanation by the two liquid as

$$\rho_c h A = A h (\alpha \rho_l + (1 - \alpha) \rho_h)$$

Where  $\alpha$  is the fraction that is in low liquid. After rearrangement it became

$$\alpha = \frac{\rho_c - \rho_h}{\rho_l - \rho_h}$$

the second part deals with the case where the density varied parabolically. The density

End of Ex. 7.6

as a function of  $x$  coordinate along  $h$  starting at point  $\rho_h$  is

$$\rho(x) = \rho_h - \left(\frac{x}{h}\right)^2 (\rho_h - \rho_l)$$

Thus the equilibration will be achieved,  $A$  is canceled on both sides, when

$$\rho_c h = \int_{x_1}^{x_1+h} \left[ \rho_h - \left(\frac{x}{h}\right)^2 (\rho_h - \rho_l) \right] dx$$

After the integration the equation transferred into

$$\rho_c h = \frac{(3\rho_l - 3\rho_h) x_1^2 + (3h\rho_l - 3h\rho_h) x_1 + h^2\rho_l + 2h^2\rho_h}{3h}$$

And the location where the lower point of the body (the physical),  $x_1$ , will be at

$$X_1 = \frac{\sqrt{3} \sqrt{3h^2\rho_l^2 + (4\rho_c - 6h^2\rho_h)\rho_l + 3h^2\rho_h^2 - 12\rho_c\rho_h + 3h\rho_l - 3h\rho_h}}{6\rho_h - 2\rho_l}$$

For linear relationship the following results can be obtained.

$$x_1 = \frac{h\rho_l + h\rho_h - 6\rho_c}{2\rho_l - 2\rho_h}$$

In many cases in reality the variations occur in small zone compare to the size of the body. Thus, the calculations can be carried out under the assumption of sharp change. However, if the body is smaller compare to the zone of variation, they have to accounted for.

### Example 7.7: Sphere

Level: Advance

A hollow sphere is made of steel ( $\rho_s/\rho_w \cong 7.8$ ) with a  $t$  wall thickness. What is the thickness if the sphere is neutrally buoyant? Assume that the radius of the sphere is  $R$ . For the thickness below this critical value, develop an equation for the depth of the sphere.

#### Solution

The weight of displaced water has to be equal to the weight of the sphere

$$\rho_s g \frac{4\pi R^3}{3} = \rho_w g \left( \frac{4\pi R^3}{3} - \frac{4\pi (R-t)^3}{3} \right) \quad (7.7.a)$$

after simplification equation (7.7.a) becomes

$$\frac{\rho_s R^3}{\rho_w} = 3tR^2 - 3t^2R + t^3 \quad (7.7.b)$$

Equation (7.7.b) is third order polynomial equation which it's solution (see the ap-

End of Ex. 7.7

pendix) is

$$\begin{aligned}
 t_1 &= \left(-\frac{\sqrt{3}i}{2} - \frac{1}{2}\right) \left(\frac{\rho_s}{\rho_w} R^3 - R^3\right)^{\frac{1}{3}} + R \\
 t_2 &= \left(\frac{\sqrt{3}i}{2} - \frac{1}{2}\right) \left(\frac{\rho_s}{\rho_w} R^3 - R^3\right)^{\frac{1}{3}} + R \\
 t_3 &= R \left(\sqrt[3]{\frac{\rho_s}{\rho_w} - 1} + 1\right)
 \end{aligned} \tag{7.7.c}$$

The first two solutions are imaginary thus not valid for the physical world. The last solution is the solution that was needed. The depth that sphere will be located depends on the ratio of  $t/R$  which similar analysis to the above. For a given ratio of  $t/R$ , the weight displaced by the sphere has to be same as the sphere weight. The volume of a sphere cap (segment) is given by

$$V_{cap} = \frac{\pi h^2 (3R - h)}{3} \tag{7.7.d}$$

Spherical volume Where  $h$  is the sphere height above the water. The volume in the water is

$$V_{water} = \frac{4\pi R^3}{3} - \frac{\pi h^2 (3R - h)}{3} = \frac{4\pi (R^3 - 3Rh^2 + h^3)}{3} \tag{7.7.e}$$

When  $V_{water}$  denotes the volume of the sphere in the water. Thus the Archimedes law is

$$\frac{\rho_w 4\pi (R^3 - 3Rh^2 + h^3)}{3} = \frac{\rho_s 4\pi (3tR^2 - 3t^2R + t^3)}{3} \tag{7.7.f}$$

or

$$(R^3 - 3Rh^2 + h^3) = \frac{\rho_w}{\rho_s} (3tR^2 - 3t^2R + t^3) \tag{7.7.g}$$

The solution of (7.7.g) is

$$\begin{aligned}
 h &= \left( \frac{\sqrt{-fR(4R^3 - fR)}}{2} - \frac{fR - 2R^3}{2} \right)^{\frac{1}{3}} \\
 &\quad + \frac{R^2}{\left( \frac{\sqrt{-fR(4R^3 - fR)}}{2} - \frac{fR - 2R^3}{2} \right)^{\frac{1}{3}}}
 \end{aligned} \tag{7.7.h}$$

Where  $-fR = R^3 - \frac{\rho_w}{\rho_s} (3tR^2 - 3t^2R + t^3)$  There are two more solutions which contains the imaginary component. These solutions are rejected.

**Example 7.8: Variable Weight****Level: Easy**

One of the common questions in buoyancy is the weight with variable cross section and fix load. For example, a wood wedge of wood with a fix weight/load. The general question is at what the depth of the object (i.e. wedge) will be located. For simplicity, assume that the body is of a solid material.

**Solution**

It is assumed that the volume can be written as a function of the depth. As it was shown in the previous example, the relationship between the depth and the displaced liquid volume of the sphere. Here it is assumed that this relationship can be written as

$$V_w = f(d, \text{other geometrical parameters}) \quad (7.8.a)$$

The Archimedes balance on the body is

$$\rho_\ell V_a = \rho_w V_w \quad (7.8.b)$$

$$d = f^{-1} \frac{\rho_\ell V_a}{\rho_w} \quad (7.8.c)$$

**Example 7.9: Wood Cone****Level: Easy**

In example 7.8 a general solution was provided. Find the reverse function,  $f^{-1}$  for cone with  $30^\circ$  when the tip is in the bottom.

**Solution**

First the function has to built for  $d$  (depth).

$$V_w = \frac{\pi d \left( \frac{d}{\sqrt{3}} \right)^2}{3} = \frac{\pi d^3}{9} \quad (7.9.a)$$

Thus, the depth is

$$d = \sqrt[3]{\frac{9\pi\rho_w}{\rho_\ell V_a}} \quad (7.9.b)$$

**Example 7.10: Isosceles Triangle****Level: Intermediate**

Extruded isosceles triangle is submerged in liquid and the net force of this triangle is  $F$ . Outside the liquid the weight of the triangle is  $W$ . Calculate the ratio of solid density and liquid density. What is the volume of the wood? Assume the liquid density is known.

**Solution**

**End of Ex. 7.10**

$$W = \rho_s V g \quad (7.10.a)$$

where  $\rho_s$  density of the wood,  $V$  is the volume of the wood. In the above equation there are two unknowns and another equation is needed. In the liquid net force on submerged triangle is

$$F = W - \rho_\ell V g = \rho_s V g - \rho_\ell V g = V g (\rho_s - \rho_\ell) \quad (7.10.b)$$

from Eq. (7.10.a) the volume can be expressed and substitute in Eq. (7.10.b)

$$F = \frac{W}{\rho_s g} g (\rho_s - \rho_\ell) \quad (7.10.c)$$

and ultimately the solid density is

$$\frac{\rho_s}{\rho_\ell} = \frac{W}{W - F} \quad (7.10.d)$$

This found density ratio can be used to calculate the volume

$$V = \frac{W}{\rho_s g} \quad (7.10.e)$$

### 7.3 Applications of Buoyancy

**Example 7.11: Hydrometer****Level: Intermediate**

Instrument used to measure the density liquid is called a hydrometer. One possible design is a big heavy spherical ball attached to thin glass with a pipe of diameter,  $D$ . The spherical ball is designed to weighed down so the rig will float straight up. The hydrometer is placed in two liquids with different densities. Assume that  $\rho_1 > \rho_2$ . What is the difference in height above surface for these two situations. Assume the air has no weight.

**Solution**

Again this question deals with Archimedes' law. In fact this device has a long history and it was "invented" numerous times. This question deals with the calibration of this device. The volume of the spherical attachment (acting like a ballast) denoted as  $V_0$ . The pipe has lower density than the liquid under the examination. The total mass of the hydrometer is fix and denoted as  $m_0$  and total weight is  $m_0 g$ . According to Archimedes law it should be placed by the buoyancy force.

$$m_0 g = \rho_1 \left( V_0 + l_1 \overbrace{\pi D^2/4}^A \right) \quad (7.11.a)$$

Where  $l_1$  is the length of the pipe in the case 1.

**End of Ex. 7.11**

In the case 2 the same equation can be written

$$m_0 g = \rho_2 (V_0 + l_2 \pi D^2/4) \quad (7.11.b)$$

The length can be expressed from Eq. (7.11.a) as

$$l_1 = \frac{\frac{m_0 g}{\rho_1} - V_0}{\pi D^2/4} \quad (7.11.c)$$

and the difference in the height is

$$\Delta h = \frac{\frac{m_0 g}{\rho_2} - V_0}{\pi D^2/4} - \frac{\frac{m_0 g}{\rho_1} - V_0}{\pi D^2/4} \quad (7.11.d)$$

**Example 7.12: Dimensionless Hydrometer****Level: Intermediate**

Use the solution of Ex. 7.11 to convert it to a dimensional form.

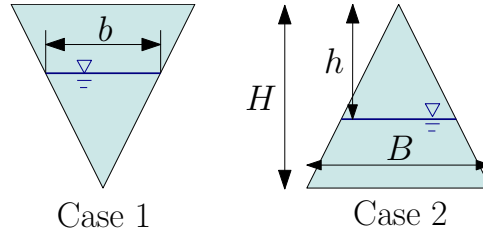


Fig. 7.6 – caption.

**Example 7.13: Extruded Isosceles Triangle****Level: Intermediate**

Extruded wooden isosceles triangle can float in liquid in many positions. For this excises consider two positions triangle floats on the head or the bottom. There is difference how much height triangle will be above the liquid. Draw the ratio of these two heights as a function of the density of the wood to the liquid. That is the density of the wood obeys  $0 < \frac{\rho_s}{\rho_\ell} < 1$ . Show that the length of the triangle is irrelevant for this calculations. This question is a preparation of the stability as this height above the liquid.

**Solution**

While the question deals with geometry and it effects on physics. Archimedes' law dictates that the area in both cases must be same. The relationship between the densities ratio and the area has to be established. In Case 1 the Archimedes' law is

$$\rho_s \frac{B H}{2} \cancel{D L} = \rho_\ell \frac{h_1 b_1}{2} \cancel{D L} \quad (7.13.a)$$

End of Ex. 7.13

where index 1 and 2 refer to the case 1 and case 2 respectively. The geometry identity relates the different sides of the triangle for both cases as

$$\frac{H}{B} = \frac{h}{b} \longrightarrow b = \frac{B h}{H} \quad (7.13.b)$$

Eq. (7.13.a) utilizing Eq. (7.13.b) as

$$\rho_s B H = \rho_\ell h_1^2 \frac{B}{H} \longrightarrow \frac{\rho_s}{\rho_\ell} = \left( \frac{h_1}{H} \right)^2 \quad (7.13.c)$$

For the case 2 the Archimedes's law is (now without the length and the gravity)

$$\rho_s \frac{B H}{2} = \rho_\ell (H - h_2) \frac{b_2 + B}{2} \quad (7.13.d)$$

Substituting Eq. (7.13.b) into Eq. (7.13.d) yields

$$\rho_s B H = \rho_\ell (H - h_2) \left( \frac{B h_2}{H} + B \right) \quad (7.13.e)$$

Eq. (7.13.f) can be simplified as

$$\frac{\rho_s}{\rho_\ell} = \left( 1 - \frac{h_2}{H} \right) \left( \frac{h_2}{H} + 1 \right) \longrightarrow 1 - \left( \frac{h_2}{H} \right)^2 \quad (7.13.f)$$

It common to denote two new parameters as  $\bar{\rho} = \rho_s/\rho_\ell$  and  $\bar{h} = h/H$ . Hence Eq. (7.13.f) can be written as

$$\bar{\rho} = 1 - \bar{h}^2 \quad (7.13.g)$$

or case 1 as

$$\bar{\rho} = \bar{h}^2 \quad (7.13.h)$$

Eqs. (7.13.h) and (7.13.g) are drawn Fig. 7.7.

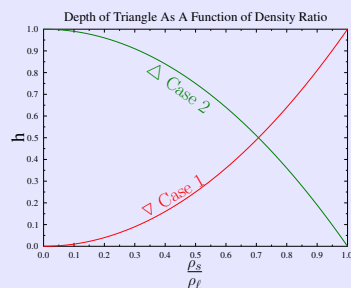


Fig. 7.7 – The size of the tip height above the liquid. The red line represents the case

The density ratio is approaching Zero the most the triangle is out the liquid. Notice that for case 1 (upside and color red) the zero means that  $h$  most the triangle is out of the liquid. While in case 2 the situation is the opposite. Notice that  $h \rightarrow 1$  most of the triangle is out of the liquid. For both cases, the function is a parabola.

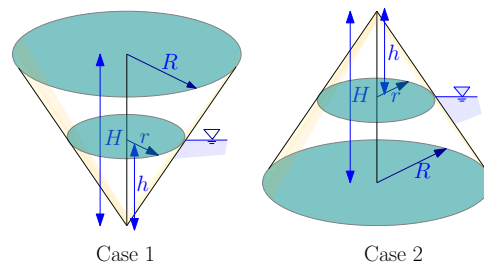


Fig. 7.8 – Floating cone height in liquid as a function of density ratio.

### Example 7.14: Symmetrical Cone

Level: Intermediate

As in Ex. 7.13 repeat the calculation for a symmetrical cone shown in Fig. 7.8.

#### Solution

The point of this example is not merely to repeat Ex. 7.13 but rather to show the idea of symmetry of calculations. That is, the solution in the more convenient (at least for this author) will be used to obtain the solution in the second case.

The cone has several properties that are well documented which include

Table 7.1 – Basic Cone properties

Cone Properties		
Name	Equation(s)	Units
<b>V</b>	$\frac{1}{3} \pi R^2 H$	none
<b>Centroid</b>	$H/4$	none



## End of Ex. 7.14

Archimedes' laws written in this case as

$$\rho_s \frac{\pi R^2 H}{3} = \rho_\ell \frac{\pi r^2 h}{3} \longrightarrow \rho_s R^2 H = \rho_\ell r^2 h \quad (7.14.a)$$

Geometrical relationship for the cone are

$$\frac{r}{R} = \frac{h}{H} \longrightarrow r = \frac{R h}{H} \quad (7.14.b)$$

Combine Eqs. (7.14.a) and (7.14.b) yields

$$\rho_s R^2 H = \rho_\ell \left( \frac{R h}{H} \right)^2 h \longrightarrow \frac{\rho_s}{\rho_\ell} = \left( \frac{h}{H} \right)^3 \quad (7.14.c)$$

Now Eq. (7.14.c) and from symmetry shown in Ex. 7.13 that case 2 the relationship is

$$\rho_s R^2 H = \rho_\ell \frac{\rho_s}{\rho_\ell} = 1 - \left( \frac{h}{H} \right)^3 \quad (7.14.d)$$

Additional argument or rational is that ratio should be same because the volume will be from both sides.

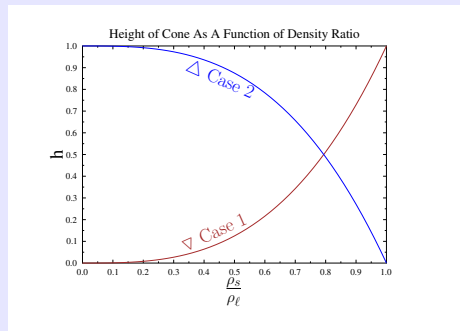


Fig. 7.9 – Cone height for various densities ratio of solid to liquid. Because the most the volume is the the base the claiming is slow and become rapid at ratio of 0.6

## 7.4 Practical questions

This section indent to deal with practical problems that may or may not creates physical questions.

### Example 7.15: Ever Given Ship

Level: Intermediate

The ship Ever Given blocked the Suez Canal in on 23 March 2021 because navigation that might or might relate to stability. With this introduction this ship length dimensions became famous. Assume that Ever Given is a rectangular

**End of Ex. 7.15**

(it is not far away from the shape of the ship). For the following shape calculate the buoyancy centroid. When the ship is empty draft is  $14.5[m]$  and when the ship is the fully loaded the draft is  $16[mm]$ . The ship's width is  $59[m]$  and ship length is about  $399[m]$ . Assuming that the ship is made from uniform material what the distance  $GB$ ?

### Solution

The width and the length of the ship do effect the buoyancy centroid. The centroid is half of the depth. In this case, the value is  $7.25[m]$  when empty and  $8[m]$  when the ship is full.

# 8

## Direct Examination

### 8.1 Introduction

Simplistically, the stability (of floating body) is divided into three categories. When moments/forces are such that they returned the immersed body to its original position state is referred to as the stable body and vice versa. The third state is when the couple forces do have zero moment, it is referred to as the neutral stable. An example of such situation is a rounded body, like a marble, on flat surface<sup>1</sup>

Floating **uniform** density bodies are, as it can be observed, are inherently “unstable” because the gravity centroid is always above buoyancy centroid (case c in Fig. 8.1). Only at extreme cases where liquid density is almost equal to the density of solid body it will be neutral stability. Bodies with none uniform densities can be both situations, in stable and unstable. The bodies with none uniform density can be arranged so that the mass centroid in lower position and the buoyancy centroid. This fact can be illustrated by Fig. 8.4 where the mass/gravity centroid can be at lower point than the buoyant center. The discussion here will be focused on uniform bodies as they provide more complicated situations. The none uniformed bodies are like uniform bodies but with a movable gravity centroid. To understand the unstable zone consider Fig. 8.1 which shows a body made of a hollow balloon and a heavy sphere connected by a thin and

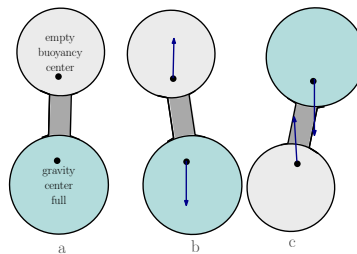


Fig. 8.1 – Schematic of floating bodies.

<sup>1</sup>It happen that the same for ball (spherical) and for the same reasons.

light rod in three different configurations. The left one (a) shows the sphere just under the balloon in middle (b) there is a slight deviation from the previous case. Case 3 depicts (right side) almost opposite to case (a). This arrangement has mass centroid close to the middle of the sphere. The buoyant centroid is below the middle of the balloon. If this arrangement is inserted into liquid and will be floating, the balloon will be on the top and sphere on the bottom Fig. 8.1a. Tilting the body with a small angle from its resting position creates a shift in the forces direction to return original state (examine Fig. 8.1a). These forces create a moment which wants to return the body to the resting (original) position. When the body is at the position shown in Fig. 8.1c, the body is unstable and any tilt from the original position creates moment that will further continue to move the body from its original position. This analysis doesn't violate the second law of thermodynamics because it takes energy to move the body to the unstable situation.

When a solid object is placed on a straight horizontal surface as long as the gravity centroid is placed above the contact surface between the body and the surface, it is stable (see Fig. 8.2). The body can be stable in other situations but they equivalent to the above statement. Yet, when a

body (solid) is placed on top of liquid, the conditions of stability are more complicated. It is not longer sufficient for the gravity centroid to be about the contact area (the same area as in the solid). Furthermore, in one liquid the body can be stable while in another liquid the body is unstable. Even the number of stable points and their locations are different between the liquid and the solid.

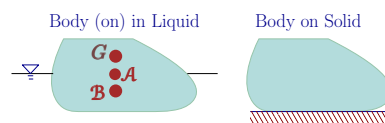


Fig. 8.2 – Center of mass arbitrary floating body on solid surface and in liquid.

## 8.2 Centroid of Floating Body or Buoyancy Centroid

To carry this analysis a new concept has to introduce, the centroid of displaced liquid or Buoyancy Centroid which is denoted “**B**.” The pressure center discussed in Section 6.2.2 in this section expanded to deals with the equivalent force that acting on the floating bodies. To illustrate this point consider an arbitrary shape floats on liquid shown in Fig. 8.2. It was shown, in this book, that the force acting on floating body must be only in the vertical direction. Furthermore, the liquid pressure must be balanced the displaced liquid. The equivalent force of the pressure acting on the body in equilibrium can be obtained from calculating the (mass) centroid of the displaced liquid. Note that the above statement is correct for arbitrary density (for example, if the density,  $\rho = f(h)$ ). If the body is not in equilibrium with the floating force, it does not act at the mass centroid. The location and direction of the force is some distance from the mass centroid yet in the vertical direction under stationary conditions.

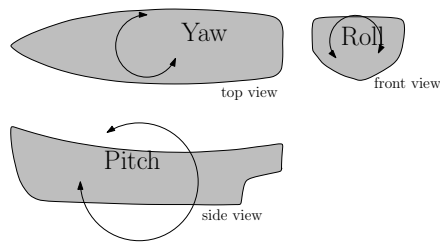
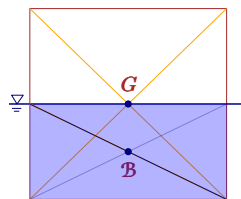


Fig. 8.3 – Typical rotation of ship/floating body.

### 8.3 Introduction to Direct Examination Method

A cubic (for example made of pine) is inserted into liquid. In this specific case, half the block floats above liquid line. The cubic mass (weight) is in the middle of the cubic (assuming uniform density). However the buoyancy center is the middle of the volume under the water (see Fig. 8.4). This situation is similar to Fig. 8.1c. However, any experiment of this cubic shows that the cubic is stable only under special conditions.

Fig. 8.4 – Schematic of Cubic showing the body centroid ( $G$ ) and buoyancy centroid ( $B$ ).

Small amount of tilting of the cubic results in immediate returning away from the original position. For example, under the conditions where the solid density is half of the liquid, the distance between  $G$  and  $B$  (also  $AB$ ) is exactly quarter of the side ( $b/4$ ) as it can be observed from the drawing. The location of gravity centroid is fixed at  $b/4$  and centroid of the immersed part is  $b/4$  and hence  $GB = b/2 - b/4 = b/4$ . The buoyancy force is the weight of the cubic. When the buoyancy centroid is exactly under the mass/gravity centroid (of the cubic), it can be in equilibrium. What happens when the buoyancy force and gravity force are slightly deviate from the equilibrium? This question is the core of stability analysis.

The stability can be answered by looking in what direction the moment created. If the moment tries to return it to “original” and tries to keep the two forces in the same line then the situation is stable. This topic is explained in several stages. In Ex. 4.3 examined the change angle  $\alpha$ , to change of imposed angle  $\theta$ . Fig. 8.5 describes the new location of the inclination of the body by purple line. When the centroid point appears right to the purple line the body is unstable and conversely the body is stable (to be on the left hand side of the purple line  $\alpha \geq \theta$ ). In Ex. 4.7 it was shown that

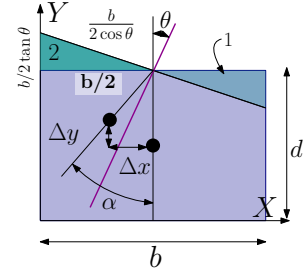


Fig. 8.5 – The Change Of Angle Due Tilt-ing.

$$\alpha = \tan^{-1} \left( \frac{\tan \theta}{6} \left( \frac{b}{d} \right)^2 \right) \cong \tan^{-1} \left( \frac{\theta}{6} \left( \frac{b}{d} \right)^2 \right) \tag{8.1}$$

In this case  $b/d = 2$  and hence Eq. (8.1) reads  $4/3 > 1$  and thus  $\alpha > \theta$  because  $\theta$  is a monotonic linear increase function. Thus, this situation is unstable and the cubic will tilted away.

The prerequisite condition for stability is that the gravity centroid and the buoyancy centroid must lay on the same vertical line. When this condition is not met the situation is unstable. For cylindrical shape, (z-coordinate which is perpendicular to liquid surface)<sup>2</sup> this condition is fulfilled at all angles. Hence the cylindrical shape fulfills the prerequisite stability requirement. However, for all other shapes this condition is fulfilled only under specific angles. For example, in a rectangular (square) extruded shape this requirement appear in two angles: one vertically up and one at 45°. Eq. (4.22)

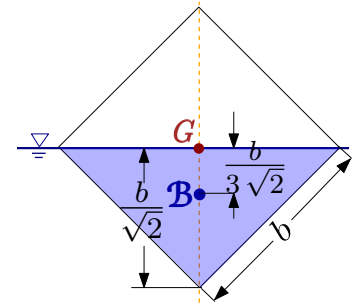


Fig. 8.6 – Cubic on the side (45°) stability analysis.

describes the change of buoyancy center as a function of the rotation<sup>3</sup>. The change in buoyancy centroid does not follow the change in the gravity rotation for the rectangular. In other words the two centroids are not on the same vertical line. Thus, for practical purposes there are only two angles that this condition occur and thus can be stable.

<sup>2</sup>Note what is the definition of surface direction.

<sup>3</sup>in that case it was denoted  $\beta$  to differentiate from the small rotation  $\theta$ .

The first location was role out and now the second angle is under the investigation.<sup>4</sup>

When the cubic is floating at 45° degree the mass centroid remains in the same location. Yet, the immersed volume (area) is now an isosceles triangle with two 45 degree angles. Note the gravity centroid and buoyancy centroid are on the same vertical line. To keep the same volume (area) the base is at the corner of the cubic shown in Fig. 8.6. Under the same procedure shown in Fig. 4.8 the relationship between the  $\alpha$  is given as by Eq. (4.7.h). For the sake of explanation, the equation is copied to here

$$\alpha = \tan^{-1} \left\{ 2 \left( \frac{b}{d} \right)^2 \tan \theta \right\} \quad (8.2)$$

Here, evidently,  $1 \leq 2(b/d)^2$  and  $b/d = 2$  and thus it allays larger than one. That means that the moment will return to the body back and therefore the body is stable in this situation. It has to emphasis that it referred to specific density (in other words that body density is half of the liquid).

From geometrical consideration (see Table 4.2 page 47) center is 1/3 of the height. The height is  $b/\sqrt{2}$  and hence the  $\mathbf{AB}$  (more importantly  $\mathbf{GB}$ ) is  $b/3\sqrt{2}$ . It can be noticed that in this case the value  $\mathbf{GB}$  is smaller than the  $\mathbf{GB}$  distance in the upright situation, that is  $b/4 > b/3\sqrt{2}$ . The value of  $\mathbf{GB}$  (or  $\mathbf{AB}$ ) in the upright is about 0.01429774  $b$  larger than the tilted case but any other configuration. Yet for both cases the forces are identical (why? because body has the same mass yet the moment is smaller due to a small leverage.). This point is actually the base for the energy method.

In this case, all the situations are “unstable” (the term unstable is used because  $\mathbf{G}$  is above  $\mathbf{B}$  and therefor forces are pointing to each other) yet the case with the 45 degree is the least “unstable” (shown in Fig. 8.1c) because when turned the moment turns body to the original state. Hence, the (45 degree) location is the most stable. Also note body has the smallest moment (the force is the same). This topic is related to curve of dynamical stability and Moseley’s formula (for stability not rays). Yet, this topic will not be covered in this book.

In other geometries and/or other densities of liquid and floating body, this kind of analysis has to done to determine the least “unstable” situation. This analysis can be done in a conventional way which will presented first and in new innovative approach. The conventional method introduces a new geometrical location which used to describe the stability while this location is physical it requires calculations and it is not “visible.” While the conventional approach is used by many, now this undersign recommends to utilize the new direct examination method. Additional advance of the new method is simple explain and demonstration of the concepts. For instance, the diagrams build with the help of this methods shows the transition from the stability to unstable because change of density ratio. This method also shows the important parameters and their combination. Thus, the  $\mathbf{GM}$  should not appear but the the ratio of  $\mathbf{GM}/D$  should be used.

<sup>4</sup>From thermodynamical point of view there must be at least one point in which body must be stable. Other wise the second law will be violated and the body because perpetual motion machine. There is not official proof for such point that this author is aware of. Nevertheless, this proof should be build geometrical consideration in which the are based on the potential energy analysis.

The potential method is simpler and practical but requires some theoretical understanding and abstraction of the physics.

### 8.4 The Direct Examination

The introduction provided an example for 2-D examination of a square body under two situations. First, upright square was examined and found to be unstable (the specific case). Second,  $45^\circ$  case which was found to be stable. In this section the general principles of the direct examination are described. As it was shown, the critical point for the stability requirement is  $\alpha = \theta$  which determines where are boundaries. Hence, the position under investigation is a given small tilting angle,  $\theta$ , results in angle  $\alpha$  representing the change in buoyancy centroid. If  $\alpha > \theta$  then the body under the investigation is stable and vice versa. The examples provided earlier are in three dimensions that could be represented in 2-D i.e. a long square. In reality, the body is not completely symmetrical. Thus, even a body is given a small rotation in one direction the buoyancy centroid can move in 3-D. However, in most cases, the change in the other directions is “minor” and can be ignored. The quantitative test is the ratio  $\alpha/\theta^5$ . The value of this ratio indicates how much stable the body at a specific position. Most of the calculations could be done numerically.

The core of this idea mentioned in the introduction and it will be expanded here. There are two possibilities one with  $\alpha < \theta$  shown in brown in Fig. 8.7 and two with  $\alpha > \theta$  shown in purple in Fig. 8.7. The angle is measured from the gravity centroid with one wing at the old coordinate and second wing is line from the gravity centroid to the new buoyancy center. For large  $\alpha$  the buoyancy center rotates back to restore the body toward the original state. In other words, it creates a moment that return original angle.

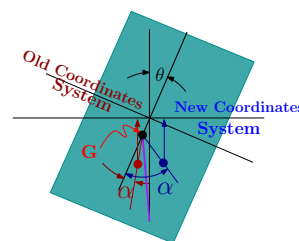


Fig. 8.7 – Arbitrary body rotates in  $\theta$  and the buoyancy centroid rotates in  $\alpha$ . The brown  $\alpha$  shows the case of stable scenario. The purple depicts the large  $\alpha$  not stable case.



(a) Foam on solid surface all stable

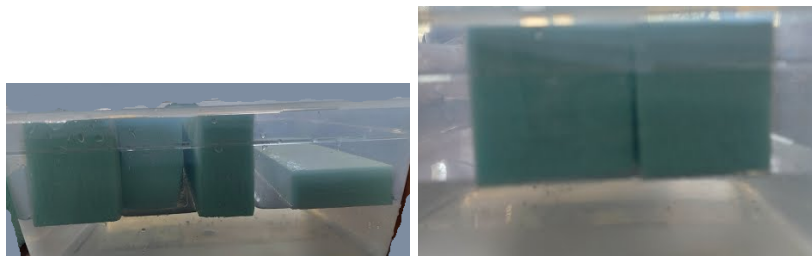
<sup>5</sup>it must be noted that this ratio really does not require finding either  $\theta$  or  $\alpha$ .





(a) Cedar blocks on solid surface      (b) Cedar blocks in liquid

Fig. 8.8 – Four blocks with different ratios of  $D/b = 1:1, 1:0.666, 1:1.2, 1:2$  depicted in the figure. It can be observed that 1:2 and 1:5 are stable while the boundary ratio of 1:1.2 is slightly moving from the flat situation. The material used in this experiment is old grow cedar on water. The density ratio is about 0.55. This result is with agreement with the model.



(b) Smaller foam blocks in liquid      (c) Larger foam blocks in liquid

Fig. 8.10 – Six blocks with different ratios of  $D/b = 1:1, 1:0.89, 1:0.789, 1:0.61, 1:0.537,$  and 0.39. It can be observed that all with the exception of 1:0.39 are stable while the boundary ratio of 1:0.39 tend to be unstable. The material used in this experiment is polystyrene on water. The density ratio is about 0.05. This results is with agreement with the model.

**Example 8.1: Minimum Condition for 3D**

Level: Intermediate

What are the minimum conditions for 3D effects?

**Solution**

The cause of 3D effect is the asymmetry in two directions “opposite to the motion at question.” That is a ship that perfectly symmetrical along the length of the ship but “front” (bow) and “back” (stern) are asymmetrical (for various reasons) the centroid of the ship move along back and forth (between the bow and the stern) as result ship has yaw rotation. (that is for example, roll creates yaw).

It was shown that Eq. (4.15) the relationship is

$$\alpha = \tan^{-1} \frac{\overbrace{\tan \theta}^{\tan \theta \sim \theta} \frac{I_{xx}}{V_0}}{GB'} \cong \frac{\theta \frac{I_{xx}}{V_0}}{GB'} \quad (8.3)$$

Where point  $A$  is the location of the intersection of the vertical line that goes through the buoyancy centroid and intersect with liquid surface/plane.

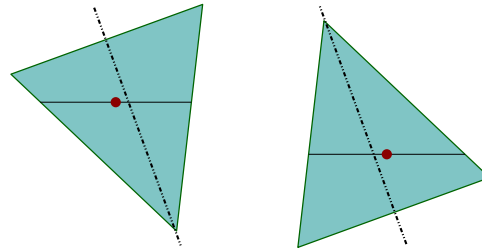


Fig. 8.11 – The change of centroid in a tilde body. The tilde increase the angle (right triangle),  $\alpha$  and hence make the body more stable.

Equation Eq. (8.3) is written for a very small angle  $\theta$  when the change of  $y$  is very small. And for practical application the stability condition is

$$\text{Stability Governing} \quad \mathbf{GB} \leq \frac{I_{xx}}{V_0} \quad (8.4)$$

This equation is correct only under the condition that point  $\mathcal{A}$  is fixed. Point  $\mathcal{A}$  is fixed only when floating body when the area does not change with the height. For example, if body is extruded rectangular or cylinder then  $\mathcal{A}$  is fixed. However, floating bodies like extruded triangle or a cone do not have a fixed centroid (at the liquid level). The analysis of such situation will be in the next version of the book. Nevertheless, a simple motivation is provided. Grossly, there are two zones separated by a border that the previous discussion dealt with. In other words, the border is the bodies that centroid are fixed (body with constant area as a function of the height). Bodies that the cross section is increasing with the height like cone that its pinnacle is in the liquid. In that case, centroid of the cross section area is moving with the angle. To illustrate this point, consider the extruded triangle depicted in Fig. 8.11. On the left the centroid moves to the left and on the right the centroid moves to right. The movement of the centroid to the right (right body in Fig. 8.11) increases the angle,  $\alpha$ , which make the body more stable and vise versa. Note, for case where the cross section area decreases with the height, a regular analysis is sufficient because the shifting of point  $\mathcal{A}$  increase the stability.

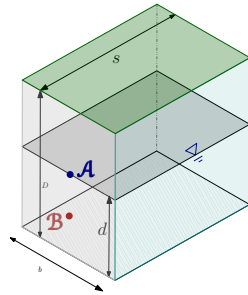


Fig. 8.12 – Rectangular body floating in a liquid for stability analysis.

**Example 8.2: Condition for extruded rectangular**
**Level: Intermediate**

What are the conditions that extruded rectangular shape will be floating stable in a liquid (see Fig. 8.12). Assume that the dimensions of the rectangular are  $s \gg b$  long and the cross section is  $b$  the width and  $D$  the height are same magnitude. The example is repeat one of the previous example.

**Solution**

The governing equation Eq. (8.4) determines the stability conditions. In this case,  $\mathcal{GB}$  is given by  $D/2 - d/2$  the moment of inertia given in the book  $b^3 s/12$ . The volume is  $V_0 = db s$ .

$$\frac{D}{2} - \frac{d}{2} \leq \frac{12}{d b s} \quad (8.2.a)$$

rearrange Eq. (8.2.a) reads

$$6(D - d) \leq \frac{b^2}{d} \quad (8.2.b)$$

The relation between the different heights (Archimedes' law) is

$$\rho_\ell d = \rho_s D \quad (8.2.c)$$

Substituting Eq. (8.2.c) into Eq. (8.2.a) reads

$$6 \frac{\rho_s D}{\rho_\ell} \left( D - \frac{\rho_s D}{\rho_\ell} \right) \leq b^2 \rightarrow 6 \frac{\rho_s D}{\rho_\ell} \left( D - \frac{\rho_s D}{\rho_\ell} \right) \geq b^2 \quad (8.2.d)$$

Eq. (8.2.e) can be rearrange to be written as

$$\frac{b}{D} \geq \sqrt{6 \frac{\rho_s}{\rho_\ell} \left( 1 - \frac{\rho_s}{\rho_\ell} \right)} \quad (8.2.e)$$

End of Ex. 8.2

The results of Eq. (8.2.e) are depicted in Fig. 8.13. It can be noticed that (as expected) for large values of  $b/D$  the body is stable. However, when the density ratio is very small ( $\frac{\rho_s}{\rho_\ell} \rightarrow 0$ ) or very large ( $\frac{\rho_s}{\rho_\ell} \rightarrow 1$ ) (solid density is close to liquid density) even for small value the ratio of geometries the body is stable (not intuitive). In the mid range of densities requires a larger ratio of  $b/D$ . Note that edge close range  $\rho_s/\rho_\ell \rightarrow 0$  or  $\rho_s/\rho_\ell \rightarrow 1$  this analysis is not applied.

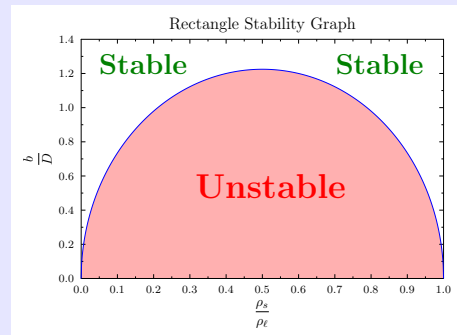


Fig. 8.13 – Extruded rectangular body stability analysis.

#### 8.4.0.1 1, 2, and 3 corners in the liquid

This topic should be part of dimensional analysis. In view to make part of the coherent flow of material it is present here. Nevertheless, it is add to here as a temporary place holder.

This discussion deals with uniform density. When extruded rectangular floating body in liquid there is two regimes. These two regimes are separated by half point ( $\rho_s = 0.5\rho_\ell$ ). At this limiting case when a square turning to  $45^\circ$  there are three corners (or one if half corner is considered to be out) in the liquid. Otherwise, there are two corners in the liquid at all time. When ( $\rho_s > 0.5\rho_\ell$ ) then there are situations where two corners or three corners inside the liquid. There are no situation with only one corner. Conversely, in the case ( $\rho_s < 0.5\rho_\ell$ ) there are only one corner or two corners in the liquid.

**Example 8.3: Extruded Triangle**

**Level: Easy**

A long extruded isosceles triangle is placed base down in a liquid (shown in Fig. 8.14). Analyze the stability for this case. Assume that the base and the height of the triangle are provided. A long extruded isosceles triangle is placed base down

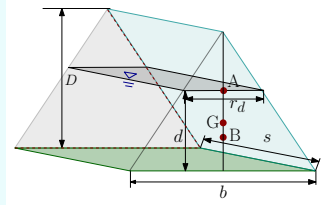


Fig. 8.14 – Floating base down triangle in liquid. The Points **A** and **B** are representation to actual location which is at the center.

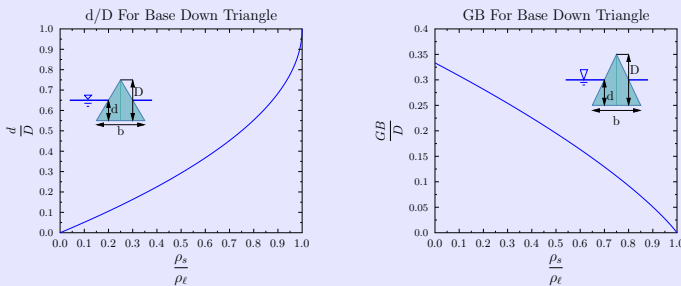
**Solution**

The mass centroid of the triangle is 1/3 of the height for upside down triangle. The governing equation for straight cross section area requires that

$$GB \leq \frac{I_{xx}}{V_0} \tag{8.3.a}$$

It was shown in Ex. 7.13 the relation between the density ratio and the height is given by Eq. (7.13.f) and provides

$$\frac{\rho_s}{\rho_\ell} = 1 - \left( \frac{D-d}{D} \right)^2 \longrightarrow \frac{d}{D} = 1 - \sqrt{1 - \frac{\rho_s}{\rho_\ell}} = f \left( \frac{\rho_s}{\rho_\ell} \right) \tag{8.3.b}$$



- (a)  $d/D$  as function for base down triangle.
- (b)  $GB$  as a function of  $\frac{d}{D}$ . The figure demonstrates that relationship can be approximated by a straight line. Notice that the functions do not go to one by geometrical considerations.

Fig. 8.16 – intermediate stages

In this case, the cross section decreases with the height of the triangle and hence if the triangle is stable without considering the movement of point **A** it will be more stable.

continue Ex. 8.3

The point  $\mathcal{B}$  is centroid of trapezoid and it is given by Eq. (4.20b) (the notations have to be fixed). Point  $B$  after notations converted is

$$B = \frac{d}{3} \left( \frac{2 \left( \frac{(D-d)b}{D} \right) + b}{\frac{(D-d)b}{D} + b} \right) \longrightarrow \frac{B}{D} = \frac{d}{3D} \left( \frac{2 \left( \frac{(D-d)}{D} \right) + 1}{\frac{D-d}{D} + 1} \right) \quad (8.3.c)$$

Observing the Fig. 8.15b that function can be approximated as a linearly function. As approximation Eq. (8.3.c) can be written as a straight line The distance  $\mathcal{GB}$  is

$$\frac{\mathcal{GB}}{D} = \frac{G}{D} - \frac{B}{D} \longrightarrow \frac{\mathcal{GB}}{D} = \frac{1}{3} - \frac{d}{3D} \left( \frac{2 \left( \frac{(D-d)}{D} \right) + 1}{\frac{D-d}{D} + 1} \right) \quad (8.3.d)$$

The moment of inertia is of rectangular is  $r_d^3 s/12$  and the immersed volume is

$$V = s \left( \frac{bD}{2} - \frac{(D-d)r_d}{2} \right) \longrightarrow V = \frac{bDs}{2} \left( 1 - \left( \frac{(D-d)}{D} \right)^2 \right) \quad (8.3.e)$$

The moment of inertia is

$$I_{xx} = \frac{\left( \frac{(D-d)b}{D} \right)^3 s}{12} \longrightarrow I_{xx} = \frac{s}{12} \left( \frac{(D-d)b}{D} \right)^3 \quad (8.3.f)$$

Substituting the various values into the governing equation

$$\frac{D}{3} \left( 1 - \frac{d}{D} \left( \frac{2 \left( \frac{(D-d)}{D} \right) + 1}{\frac{D-d}{D} + 1} \right) \right) \leq \frac{\frac{s}{12} \left( \frac{(D-d)b}{D} \right)^3}{\underbrace{\frac{bDs}{2} \left( 1 - \left( \frac{(D-d)}{D} \right)^2 \right)}_{\rho_s / \rho_\ell}} \quad (8.3.g)$$

Or after rearrangement it can be written as

$$2 \left( 1 - \frac{d}{D} \left( \frac{2 \left( \frac{(D-d)}{D} \right) + 1}{\frac{D-d}{D} + 1} \right) \right) \frac{\left( \frac{\rho_s}{\rho_\ell} \right)}{\left( \frac{(D-d)}{D} \right)^3} \leq \frac{b^3}{bD^2} = \left( \frac{b}{D} \right)^2 \quad (8.3.h)$$

The results of Eq. (8.3.h) are drawn in the following exhibit.

End of Ex. 8.3

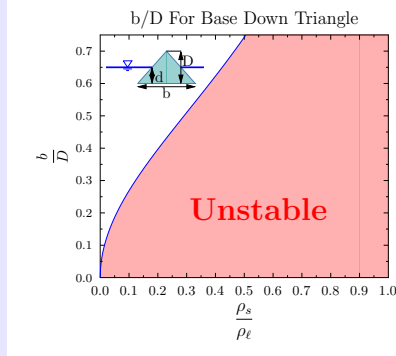


Fig. 8.17 – Base down triangle stability diagram.

The Fig. 8.17 exhibits the stability diagram of the base down triangle. As oppose to the extruded rectangular which has a dome, the base down triangle has no ability to be stable at high density ratio. The reason for it is that moment of inertia approaches to zero while the volume is increasing. On the other side, when the peak was in the liquid then at high density ratio the body is more stable. The position that body will be have that two conditions are satisfied. That it was studied yet and the significance was not explored.

**Example 8.4: Cylinder Wedge**

Level: Intermediate

A cylinder is floating on a liquid when z coordinate is upright. Under what conditions the cylinder is stable. Is 3-D effects appears in the stability analysis of the cylinder under the condition in this question.

**Solution**

There is no 3-D effects because the cylinder is symmetrical in both directions around the x axis and the y axis. The condition for stability is

$$GB \leq \frac{I_{xx}}{V_0} \tag{8.4.a}$$

The moment of inertia of circle is given in table 4.2  $I_{xx} = \pi r^4/4$ . The volume of the submerged part is  $\pi r^2 d$ . The location of point  $A = D/2$  and the location of  $B = d/2$ . The last part is to related between submerged volume to total volume as

$$d \rho_l = D \rho_s \tag{8.4.b}$$

Armed with all the components Eq. (8.4.a) can be written as

$$\frac{D}{2} - \frac{d}{2} \leq \frac{\frac{\pi r^2}{4} r^2}{\pi r^2 d}$$

End of Ex. 8.4

which can be rearranged as

$$\frac{D}{2} \left(1 - \frac{\rho_s}{\rho_\ell}\right) \leq \frac{r^2}{4d} = \frac{r^2}{4 \frac{D \rho_s}{\rho_\ell}}$$

and finally get the form as

$$\frac{r}{D} \geq \sqrt{\frac{2 \rho_s}{\rho_\ell} \left(1 - \frac{\rho_s}{\rho_\ell}\right)} \tag{8.4.c}$$

It can be observed that the smallest possible value of the Eq. (8.4.c) when the ratio  $(\rho_s/\rho_\ell = 0.5)$  and in that case,  $r > \sqrt{2}D$ . The results are presented in Fig. 8.18 The strange fact is the stability line appears symmetrical as the rectangular shape in regard to densities ratio.

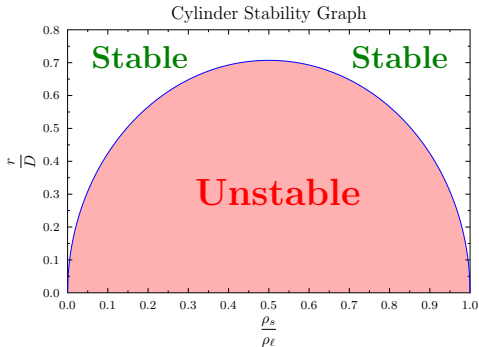


Fig. 8.18 – Cylinder in upright position stability line.



(a) Foam on solid surface all stable



(b) Smaller foam blocks in liquid



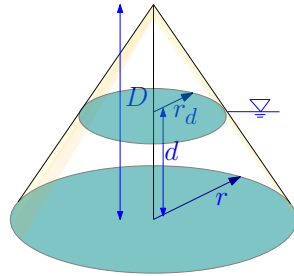


Fig. 8.20 – Cone stability calculations showing various parameters.

**Example 8.5: Triangle Up****Level: Intermediate**

Repeat Ex. 8.3 when the base is in the liquid.

**Solution****Example 8.6: Cone****Level: Advance**

Calculate the requirements for symmetrical cone which base on the bottom to be stable floating on liquid.

**Solution**

The cone has several properties that are well documented which include volume:  $V = \pi r^2 D/3$  the centroid:  $D/4$  The location of  $\mathcal{A}$  is independent of the liquid and is at the centroid which is at  $D/4$ . The location of the buoyancy center (centroid of truncated cone) conical frustum is at

$$B = \frac{D \left( 1 + \frac{2r_d}{r} + 3 \left( \frac{r_d}{r} \right)^2 \right)}{4 \left( 1 + \frac{r_d}{r} + \left( \frac{r_d}{r} \right)^2 \right)} = \bar{C}_1 \left( \frac{r_d}{r} \right) \quad (8.6.a)$$

The centroid is similar to complete cone with a correction factor. The ratio of  $r_d/r$  can be in The ratio of  $r_d/r = 1 - d/D$  results in

$$B = \frac{D \left( 6 - \frac{8d}{D} + 3 \left( \frac{d}{D} \right)^2 \right)}{4 \left( 3 - 3\frac{d}{D} + \left( \frac{d}{D} \right)^2 \right)} = C_1 \left( \frac{d}{D} \right) \quad (8.6.b)$$

The volume is the same

$$V_0 = \frac{\pi D r^2}{3} \left( 1 + \frac{r_d}{r} + \left( \frac{r_d}{r} \right)^2 \right) = \bar{C}_2 \left( \frac{r_d}{r} \right) \quad (8.6.c)$$

or substituting  $r_d/r = 1 - d/D$  results in

$$V_0 = \frac{\pi D r^2}{3} \left( 3 - 3 \frac{d}{D} + \left( \frac{d}{D} \right)^2 \right) = C_2 \left( \frac{d}{D} \right) \quad (8.6.d)$$

The correction factors,  $C_1$  and  $C_2$ , approach zero when  $r_d \rightarrow 0$ . The cone weight and the displaced liquid are identical hence

$$\rho_s V_{cone} = \rho_\ell V_0 \quad (8.6.e)$$

or explicitly

$$\rho_s \frac{\pi D r^2}{3} = \rho_\ell \frac{\pi D r^2}{3} \left( 3 - 3 \frac{d}{D} + \left( \frac{d}{D} \right)^2 \right) \quad (8.6.f)$$

which can be manipulated to be

$$\frac{d}{D} = \frac{3 + \sqrt{9 + \frac{4\rho_s}{\rho_\ell}}}{2} \quad (8.6.g)$$

when  $\rho_s \rightarrow 0$  then  $d = 0$  which was expected. The solution with the negative sign was rejected for physical reasons.

The condition for stability expressed in Eq. (8.3) can be simplified as

$$GB = \frac{I_{xx}}{V_0} \quad (8.6.h)$$

These two last equations related the densities to the displacement.

$$\frac{D}{4} - C_1 \left( \frac{d}{D} \right) = \frac{\frac{\pi r_d^4}{4}}{C_2 \left( \frac{d}{D} \right)} \quad (8.6.i)$$

The solution of equations (8.6.i) provide the relationship between  $r$  and  $D$  and densities ratio so limiting case of stability is established. For this chapter this point is a stopping point. The relationships found here can be drawn.



(a) Cedar block in cotton seed oil



(b) Douglass fir cylinder in cotton seed oil



(c) Foam block in cotton seed oil

Fig. 8.22 – The same block that were used in previous experiments with water depicted in the current figure. For the cedar blocks it shows that they are more stable. While for the foam block becomes less stable. I can be observed that 1:2 and 1:5 are stable while the boundary ratio of 1:1.2 is slightly moving from the flat situation. The material used in this experiment is old grow cedar on water. The density ratio is about 0.55. This result is with agreement with the model.

The examination up to this point was whether the body is stable or not. The question in which angle the body is stable was not discussed. The trial and error and methods can be used however, another methods can be used. For example, the extruded rectangular has moment of inertia which is increases with the inclination angle,  $\beta$  ( $1/\cos^3 \theta$ ). In some situation as depicted in the figures the inclination change to some degree or point and no further. To sharpen the question, again more tilding the rectangular results in larger moment of inertia while the volume remain the same. Thus, according to this argument all the body should continue to rotate maximum moment of inertia. In fact, the experiments that this author has done show that body is stable around that several points and they not continues (with the exception of cylinder “type” on the side). The reason for singularly of these points is that another condition has to be fulfilled. The buoyancy centroid must be on the same line vertical line with the buoyancy centroid. So the question should be locate all the angle in which the buoyancy and gravity centroids are on the same line. Then check if that point is stable. For instance, in the case of rectangular and case of  $\rho_s/\rho_e ll$  is the same line are on angles  $0^\circ$  and  $45^\circ$ .

**Example 8.7: Moment with Angel**
**Level: Intermediate**

Express the moment of inertia for rectangular body as a function of the inclination. Start with complete vertical position. Assume that the trapezoid shape remains through process. Note that the moment of inertia of rectangular is  $bh^3/12$ .

**Solution**

The surface has depth of  $s$  into the page and the width of the surface is  $b/\cos \theta$ . The

End of Ex. 8.7

location of  $\mathcal{A}$  is independent of inclination. The buoyancy centroid at  $B$  can

$$\frac{\Delta y}{b} = \left(\frac{b}{d}\right)^2 \frac{\tan^2 \theta}{12} \quad (8.7.a)$$

$$\frac{\Delta y}{b} = \left(\frac{b}{d}\right) \frac{\tan \theta}{6} \quad (8.7.b)$$

$$\alpha = \tan^{-1} \left( \frac{2 \tan \theta}{6 \left(\frac{d}{b}\right) - \tan^2 \theta} \right) \quad (8.7.c)$$

The location of buoyancy centroid is at

$$GB' = \sqrt{\left(\frac{d}{2} - \Delta y\right)^2 + (\Delta x)^2} \quad (8.7.d)$$

The critical length  $GB''$  is obtained by

$$GB'' = GB' \cos |\alpha - \theta| \quad (8.7.e)$$

The moment of inertia can be expressed as

$$I_{xx} = \frac{s b^3}{12 \cos^3 \theta} \quad (8.7.f)$$

### Example 8.8: Two triangles attached

Level: Easy

One way to make a ship to be more hydrodynamical is by making the body as narrow as possible. Suppose that two opposite sides triangle (prism) is attached to each other to create a long “ship” see Fig. 8.23. Supposed that  $b/h \rightarrow \sim 0$  the body will be unstable. On the other side if the  $b/h \rightarrow \sim \infty$  the body is very stable. What is the minimum ratio of  $b/h$  that keep the body stable at half of the volume in liquid (water). Assume that density ratio is  $\rho_s/\rho_\ell$ . Is there limits on the density ratio.

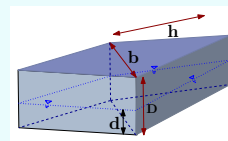


Fig. 8.23 – Stability of two triangles put together.

### Solution

The answer the question requires solve the governing equation. In this case the moment of inertia of half triangle is  $hb^3/24$  for rotating triangle around the base. There are four half triangles and thus the total moment of inertia is  $hb^3/6$ . The

**End of Ex. 8.8**

immersed volume of the body is

$$V_0 = 2 \left( \frac{b h d}{2} \right) = b h d$$

The relationship of the density ratio to heights (Archimedes Law) is

$$d = \frac{D \rho_s}{\rho_\ell} \tag{8.5}$$

The governing equation is

$$\frac{D}{2} - \frac{d}{2} = \frac{\frac{b^2}{h}}{b h d} \rightarrow \frac{D}{2} \left( 1 - \frac{\rho_s}{\rho_\ell} \right) = \frac{1}{6} \frac{b^2}{d}$$

or in a dimensional form as

$$\left( 1 - \frac{\rho_s}{\rho_\ell} \right) \frac{\rho_s}{\rho_\ell} = \frac{1}{3} \left( \frac{b}{D} \right)^2 \tag{8.6}$$

According to Eq. (8.6) the stability is affected by the density ratio and by the ratio of  $b/D$ . Furthermore, according to this model the stability is not function of length,  $h$ . However, according to this model it is build on the idea that that there in marine terminology there is no pitch (rotating in the other direction). If  $h$  is the same order as  $b$  then this model is fail to take into account the stability of the other direction. Hence, with these hand waving argument  $h$  must at least two time of  $b$ .

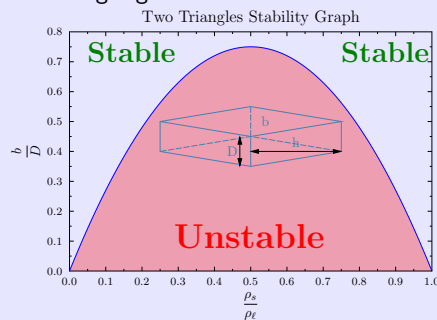


Fig. 8.24 – Two attached triangles stability diagram. Shown that minimum  $b$  is at least in the same size of  $D$ .

**Example 8.9: Extruded Triangle angle****Level: Intermediate**

This example is expansion the question 3.4 by Lautrup (Lautrup 2011). Extruded triangle with a shape pointed point down is placed in a liquid. When alpha angle is very sharp the triangle is not very stable. On the other hand, when the angle is very wide the triangle is stable. Assuming that the area change does not affect the analysis. what is the minimum angle for with the body is stable for arbitrary density ratio ( $0 \leq \rho_r \leq 1$ ). This example only provide the trends and it sufficient of the make of the point of the range where to expect the stable zone.

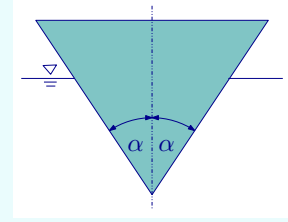


Fig. 8.25 – Triangle With The Peak Down stability analysis.

**Solution**

The Symmetry dictated that it is sufficient to balance half the triangle. The density ratio related to the draft as

$$\rho_s \frac{D D \tan \alpha}{2} = \rho_\ell \frac{d d \tan \alpha}{2} \longrightarrow \frac{\rho_s}{\rho_\ell} = \left( \frac{d}{D} \right)^2 \quad (8.9.a)$$

The moment of inertia of the of the cross section area is

$$I_{xx} = \frac{s (2 d \tan \alpha)^3}{12} \quad (8.9.b)$$

The volume is

$$V_0 = \frac{\frac{1}{2} s d^2 \tan \alpha}{\frac{1}{2}} \longrightarrow V_0 = s d^2 \tan \alpha \quad (8.9.c)$$

The governing equation is

$$G - B = \frac{I_{xx}}{V_0} = \frac{\frac{2 \frac{1}{2} (d \tan \alpha)^3}{3}}{\frac{1}{2} d^2 \tan \alpha} = \frac{2 d \tan^2 \alpha}{3} \quad (8.9.d)$$

The distance  $G - B$  the difference between the two triangles. Hence

$$\frac{2D}{3} - \frac{2d}{3} = \frac{2d \tan^2 \alpha}{3} \longrightarrow \left( 1 - \frac{d}{D} \right) = \frac{d}{D} \tan^2 \alpha \quad (8.9.e)$$

The value of limit  $\alpha$  is

$$\alpha = \tan^{-1} \sqrt{\left( 1 - \frac{d}{D} \right) \frac{D}{d}} \quad (8.9.f)$$

This results is exhibited in the following graph. It is caution that Eq. (8.9.f) represents the trends only and additional correction has to be taken into account.

End of Ex. 8.9

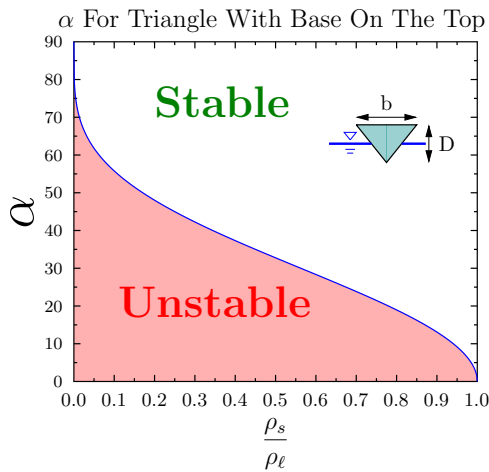


Fig. 8.26 –  $\alpha$  as function of density ratio.

It can be observed from Fig. 8.26 that for small density ratios a large angle is required. This results is obvious and it is based on intuitive. However, it expect that required angle will have "jump" at the initial and the final zone. For large range the change is approaching a linear. Again, it must be noted that this results is not the final because the change of the rotation line and large tilde angle is applied then the triangle might not be stale.

### 8.4.1 Usage of Stability to find Centroid

The stability arguments can be used to find the centroid of fraction of circle. A cylinder is naturally stable and no matter what the value of the density ratio (location of liquid level). Hence, this calculation is based on this concept/idea, the extruded circle must obey the stability equation equation as

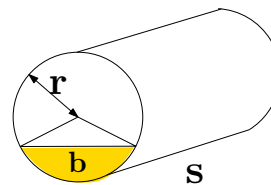


Fig. 8.27 – Circular shape for finding centroid.

$$\frac{I_{xx}}{V_0} = G - B \tag{8.7}$$

Since  $G$  is fix for all circles while  $B$  is unknown. The left side of the equation can be reduced. For example, for  $\rho_s/\rho_l = 0.5$  it becomes

$$\frac{I_{xx}}{V_0} = \frac{\frac{1}{2}(2r)^3}{\frac{1}{2}\pi r^2} = \frac{4r}{3\pi} \tag{8.8}$$

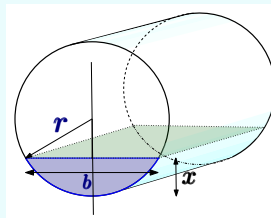
Thus, the centroid is found and the depth ( $s$ ) is irrelevant but to the results but is useful tool in calculation the centroid of any fraction of the circle. In general, the moment of inertia of and area of the golden area can be evaluated and the direct expression is

$$\mathcal{B} = \mathcal{G} - \frac{I_{xx}}{V_0} = \frac{r}{2} - \frac{b^3/12}{A} \quad (8.9)$$

The area is readily available and hence center of buoyancy can be calculated.

### Example 8.10: Centroid of Quarter Segment

Level: Easy



Calculate the centroid of the circular segment with the height of  $x$  as shown in the Fig. 8.28. In particular calculate the case where  $x = r$ . Notice that cylinder is imaginary and it should be used to calculate the moment of inertia and volume.

Fig. 8.28 – Centroid of circular shape. Notice that cylinder is used to calculate Circular shape centroid (transport dark blue).

### Solution

The equation to calculate the centroid is

$$y_c = r - \frac{(I_{xx}/s)}{A} \quad (8.10.a)$$

The moment of inertia of line (actually the green area) is  $I/s = (b)^3/12$  and the area

$$A = \frac{r^2}{2} \left( \sin^{-1} \left( \frac{r-x}{r} \right) - \frac{r-x}{r} \right)$$

The relationship between  $b$  and  $x$  is based on Pythagoras theorem as

$$b = 2 \sqrt{r^2 - (r-x)^2} = 2 \sqrt{2xr - x^2}$$

with the given properties the centroid can be determined.

$$y_c = r - \frac{\frac{8(2xr - x^2)^{3/2}}{12}}{\frac{r^2}{2} \left( \sin^{-1} \left( \frac{r-x}{r} \right) - \frac{r-x}{r} \right)} \quad (8.10.b)$$



End of Ex. 8.10

or in dimensionless form by introducing  $\tilde{x} = x/r$  to be

$$y_c = r - \frac{8r(2\tilde{x} - \tilde{x}^2)^{3/2}}{12} = \frac{r}{2} (\sin^{-1}(1 - \tilde{x}) - (1 - \tilde{x}))$$

$$r \left( 1 - \frac{4(2\tilde{x} - \tilde{x}^2)^{3/2}}{3(\sin^{-1}(1 - \tilde{x}) - (1 - \tilde{x}))} \right) \quad (8.10.c)$$

and the equation can be dimensionless as

$$\frac{y_c}{r} = 1 - \frac{4(2\tilde{x} - \tilde{x}^2)^{3/2}}{3(\sin^{-1}(1 - \tilde{x}) - (1 - \tilde{x}))} \quad (8.10.d)$$

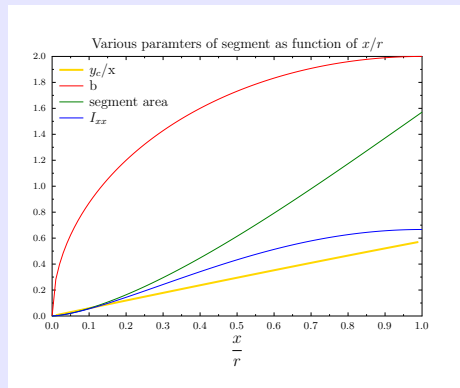


Fig. 8.29 – Various dimensions relate to segment.

The end values can be observed easily. When  $\tilde{x}$  is equal to one  $b$  must be 2 etc.

The question that must follow this discussion is whether the circular contour is some kind of boundary between stable and unstable contours? The investigation so far shows and indicates that it is the case but there should be additional stipulation which was not formulated yet. For example, a removal of a small section of the circle makes the body unstable and the “circle” will turn away.

## 8.5 Potential Energy Approach

This method was suggested by Erdős at el and was slightly improved by Abolhassani. This method is based on the idea that a derivative of potential energy can provide a

location or locations where a system has a minimum (or maximum) and thus it is potential of stability point<sup>6</sup>. The energy used in this scenario is the gravitational energy that is expressed as

$$U_{sys} = (M + m) g h_{M+m} = g (m h_m + M h_M) \quad (8.10)$$

Where subscript *sys* referred to the entire system. The *m* is referred to floating body and *M* is referred to the displaced liquid in other words to the mass if the liquid was filling the submerged volume. The logic to the last definition is that it represents the potential of the buoyancy force acting in the center immersed part. The change in the potential is due to the change in the angle

$$\frac{dU_{sys}}{d\theta} = 0 \quad (8.11)$$

The condition that angle,  $\theta$  is by checking the second derivative if it positive or negative. In away doing example it will repetitive of the moment method converting it to potential and going over the mathematics. This book is more focus on the physics and therefore it not presented.

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<sup>6</sup>This topic should be discussed elementary physics class and not fluid mechanics textbook. However, if there will be a significant request it will be briefly discussed.

# 9

## Metacenter

### 9.1 Introduction

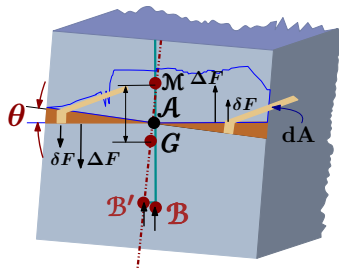


Fig. 9.1 – Stability analysis of floating body.

The last chapter two methods were discussed: the Direct Examination and the abstract Potential Energy. The cousin of the direct examination is the Metacenter method. In Metacenter method, the ship or the floating body is given a small rotation or tilde (as in Direct Examination) and instead of resulting angle, a distance from the gravity centroid to new imaginary location (Metacenter) is measured. If the distance  $G$  to  $M$  is point up it considered to be possible and vice versa. If the distance is positive then the body is stable and if it is negative the body is unstable. The advantages and short falls will be discussed at the end of the Chapter.

Assuming the general body is floating and it is at a certain configuration. The body is tilted by a given small angle,  $\theta$ , and the resulting distance is measured. As opposed to the perceived public opinion, the tilting point is around point **A** and not around imaginary point **M**<sup>1</sup>. If the body will be rotating around point **M** it will not keep the displaced volume constant. The immersed part of the body centroid translates to a new location, **B'** as shown in Fig. 9.1. The mass (gravity) centroid is still at the same old location since the body itself does not change (on moving gravity center later). This deviation of the buoyant centroid from the old buoyant centroid, **B**, is similarly as it was done in the Direct Examination. The right brown area (volume) in Fig. 9.1 is displaced by the same area (really the volume) on left since the weight of the body didn't change<sup>2</sup> so the total immersed area (volume) is constant.

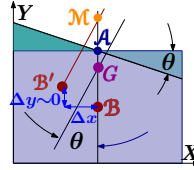


Fig. 9.2 – Description to demonstrate what is Metacenter.

In the previous chapter it was shown that  $\Delta x$  or  $\overline{BB'}$  is

$$\overline{BB'} = \frac{I_{xx} \tan \theta}{V_0} \quad (9.1)$$

the point where the gravity force direction is intersecting with the center line of the cross section is referred as Metacenter point, **M**. The location of the Metacenter point can be obtained from the geometry as Metacenter point! definition

$$\overline{BM} = \frac{\overline{BB'}}{\sin \theta} \quad (9.2)$$

And combining equations (9.1) with (9.2) yields

$$\overline{BM} = \frac{I_{xx} \tan \theta}{V_0 \sin \theta} \quad (9.3)$$

For small angle ( $\theta \sim 0$ )

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\tan \theta} \sim 1 \quad (9.4)$$

<sup>1</sup>For fixed **A** point.

<sup>2</sup>It is correct to state: area only when the body is extruded. However, when the body is not extruded, the analysis is still correct because the volume and not the area should be used.

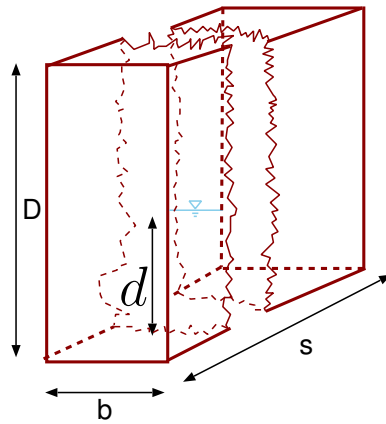


Fig. 9.3 – Cubic body dimensions for stability analysis.

It is remarkable that the results is independent of the angle. Looking at Fig. 9.2, the geometrical quantities<sup>3</sup> can be related as

$$\overline{GM} = \frac{\overline{BM}}{V_0} - \overline{GB} \tag{9.5}$$

Eq. (9.5) can also written for solid body as

$$\overline{GM} = \frac{\overline{BM}}{\rho_s V_{body}} - \overline{GB} \tag{9.6}$$

To understand these principles consider the following examples.

### 9.2 Application of $\overline{GM}$

All the terms in Eq. (9.5) normally provided and it is simply plugging them into the Eq. (9.5) and obtaining the results. Illustrate these points an extensive example is provided.

**Example 9.1: Cylinder Wedge**

**Level: Intermediate**

In Fig. 9.3 depicts the extruded rectangular with various dimensions. Assume that the body is solid with density below the liquid density, calculate the  $\overline{GM}$  for various dimensions.

<sup>3</sup>Alternative explanation will be provided at the end of the chapter.

## Solution

The governing equation is

$$\overline{GM} = \frac{I_{xx}}{V_0} - \overline{GB} \quad (9.1.a)$$

As before the densities is used to related

$$V_0 \rho_\ell = V_{body} \rho_s \longrightarrow d \rho_\ell = D \rho_s \quad (9.1.b)$$

Point  $G$  is located at  $D/2$  and point  $B$  is located at  $d/2$ . Moment of inertia is  $I = b^3 s/12$  and the volume is  $V_0 = d s b$  Armed with these data Eq. (9.1.a) becomes

$$\overline{GM} = \frac{b^3 s}{12 d s b} - \left( \frac{D}{2} - \frac{d}{2} \right) \quad (9.1.c)$$

or in dimensionless form as

$$\frac{\overline{GM}}{D} = \frac{1}{12} \left( \frac{b}{D} \right)^2 \left( \frac{\rho_\ell}{\rho_s} \right) - \frac{1}{2} \left( 1 - \frac{\rho_s}{\rho_\ell} \right) \quad (9.1.d)$$

Plotting the results of various density and  $b/D$  provides the following figure

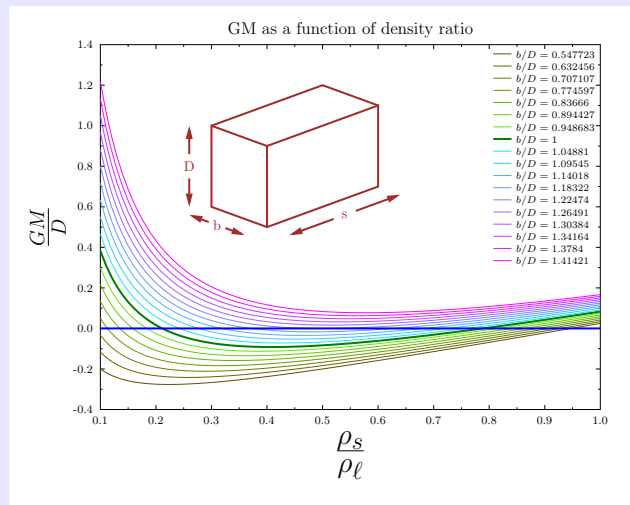


Fig. 9.4 –  $\overline{GM}$  of Rectangular shape with various dimensions.

The rectangular has larger  $\overline{GM}$  when floating on very heavy liquid. It is more stable if it is lighter. The blue line differentiate between positive and to negative  $\overline{GM}$  values. The Fig. 9.4 exhibits the  $\overline{GM}$  as function of the density ratio for various ratio of  $b/D$ . The figure demonstrates that there is a minimum with every graph that is around the  $\rho_s = 0.5\rho_\ell$ . For some ratios of  $b/D$  the figure demonstrates that  $\overline{GM}$  is negative. As solid density approaches to liquid density, the body becomes more stable and even with positive  $\overline{GM}$  for some  $b/D$  ratios. At mid range density range the body is less stable.

**Example 9.2: What to do to save the Ship****Level: Easy**

Assume that you are on a floating body (boat or ship) and it is about to turn to its side. What should you do in order to save the floating body? Throw items overboard or bring more things to ship like your raft that is normally tied to your boat?

**Solution**

If the ship or the boat is light that throwing items will make it more stable. On the other hand, if the boat is almost full and you should add more items and make it as heavy as you can (even pump water into the ship). It is common to have a maximum load marking on the ship or boat. Normally this point should be designed in about 30% of the ship's displacement. Thus, if the convention is applied, it is better to throw as much as possible. The reason that maximum marks exist is or should be for stability reasons. Loads above that point will make the ship unstable (below safety factor).

An anecdote of this author, on his ship mechanic duty exam (on a missile boat) a common question was what to do when the ship shows signs of turning. The proper answer was to pump and throw overboard everything as possible out. The question was originated by someone who experienced it first hand without any theoretical understanding.

**Example 9.3:  $GM$  of cylinder****Level: Easy**

A cylinder with a radius,  $r$  and a length  $D$  is floating on a liquid. Calculate the  $GM$  for various density ratios and ratios of  $r/D$ .

**Solution**

This example basically repeats Ex. 9.1 for a cylinder. The immersed volume is  $\pi d r^2$ . The moment of inertia of a circular shape is  $\pi r^4/4$ .

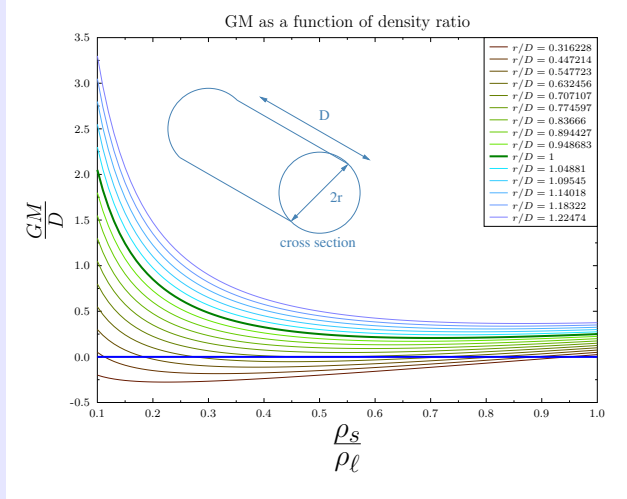
$$GM = \frac{\pi r^4/4}{\pi d r^2} - \frac{D}{2} \left( 1 - \frac{\rho_s}{\rho_\ell} \right) \quad (9..a)$$

or in a clear form

$$GM = \frac{r^2}{4d} - \frac{D}{2} \left( 1 - \frac{\rho_s}{\rho_\ell} \right) \quad (9..b)$$

Or in a dimensionless form as

$$\frac{GM}{D} = \frac{1}{4} \left( \frac{r}{D} \right)^2 \frac{\rho_\ell}{\rho_s} - \frac{1}{2} \left( 1 - \frac{\rho_s}{\rho_\ell} \right) \quad (9..c)$$

Fig. 9.5 –  $\overline{GM}$  as a function of density ratio for various  $r/D$ .

### 9.2.1 Unstable Bodies, Negative $\overline{GM}$

What happens when the  $\overline{GM}$  has negative values or in other words the height (draft) increases beyond the critical height ratio? The body will flip into the side and turn to the next stable point (angle). This is not a hypothetical question, but rather practical. This stable point (that is body at new angle) will be point where body will be rotating around. At that point, the possible rotation swing to the deck is smaller and thus the ship is less stable.

This situation happens when a ship is overloaded with containers above the maximum height. In commercial ships, the fuel is stored at the bottom of the ship and thus the mass centroid (point  $G$ ) is changing during the voyage. So, the ship that was stable (positive  $\overline{GM}$ ) leaving the initial port might become unstable (negative  $\overline{GM}$ ) before reaching the destination port. It is the responsibility of the officer (captain) to be aware of this pitfall.



### 9.3 The $\overline{GZ}$ diagram

A typical approach in stability instead of looking at the value of  $\overline{GM}$  a different imaginary distance is checked. This distance is what called the right hand arm. This refers to distance between the gravity and buoyancy forces. In Fig. 9.6 depicts the distance  $\overline{GZ}$ . This distance is perpendicular to the line  $\mathbf{B}'M$ . The line  $\mathbf{GB}_e$  is parallel to  $\mathbf{B}'M$ .  $\mathbf{B}_e$  is the buoyancy point that obtained when the equilibrium is achieved. In other words, this point is the point the regular  $B'$  that travels through until the stability point. It can be noticed that his change occurs when the body is rotated and it no question stability. In this analysis, the question is the location of buoyancy centroid. This examination is based on the size of the right turning arm. When the arm length is zero the floating body is stable or that is point like watershed line after which the body will turn to next stable point.

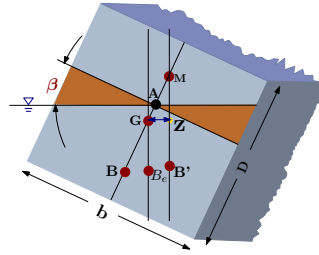


Fig. 9.6 – The right arm distance is display as  $\overline{GZ}$  as a function of  $\beta$

In this analysis, the buoyancy centroid will be determined. The simple way to explain this point is by example. First body that will be dealt is the extruded rectangular like the one show in Fig. 9.6. Utilizing Eq. (4.20) provides the buoyancy centroid. For simplicity, first the vertical distance between the buoyancy centroid to the liquid of the water is calculated. Later, the horizontal distance between the gravity centroid and the buoyancy centroid.

The vertical distance can asses by finding first distance  $\mathbf{B}'L$  which can ascertained by

$$\mathbf{B}'L = C_\xi - LN = C_\xi - C_\eta \tan \beta \tag{9.7}$$

The vertical distance is then

$$\overline{VD} = \mathbf{B}'L \cos \beta = (C_\xi - C_\eta \tan \beta) \cos \beta \tag{9.8}$$

Dividing equation by draft,  $D$  yields

$$\frac{\overline{VD}}{D} = \left( \frac{C_\xi}{D} - \frac{C_\eta}{D} \tan \beta \right) \cos \beta \tag{9.9}$$

Where  $C_\xi/D$  can be obtained from Eq. (4.21) plus utilizing Archimedes's formula as

$$\frac{C_\xi}{D} = \frac{\rho_r}{2} + \frac{b \tan \beta}{D} - \frac{\tan^2 \beta}{24} \left( \frac{b}{D} \right)^2 \frac{1}{\rho_r} \tag{9.10}$$

Where  $\rho_r$  is the density ratio of the solid body over the liquid. Utilizing Eq. (4.22d)

$$\frac{C_\eta}{D} = \frac{b}{2D} - \frac{1}{12} \left( \frac{b}{D} \right)^2 \frac{\tan \beta}{\rho_r} \tag{9.11}$$

Before the results are presented the maximum angle  $\beta$  has to be addressed. It can be noticed that the displaced volume is fixed and independent of the angle of inclination. The maximum angle that the body can turn before it gets over the edge is

$$\tan \beta_{max} = \begin{cases} \frac{2d}{b} = \frac{2\rho_r}{b}, & \text{for } \rho_r \leq 0.5. \\ \frac{2(D-d)}{b} = \frac{2(1-\rho_r)}{b}, & \text{otherwise.} \end{cases} \quad (9.12)$$

The results are exhibited in Fig. 9.7. The maximum angle has a maximum at  $\rho_r = 0.5$  with the result that the other corner is going to be immersed in the liquid for a large value of density ratio. While, this point could be seen as trivial, it is very important even with the calculations. At this point, the rate of change for moment of inertia obtains its minimum. This is a suspicious point in which it could be a stable point (e.g. point where the body rotates around).

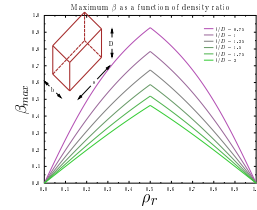
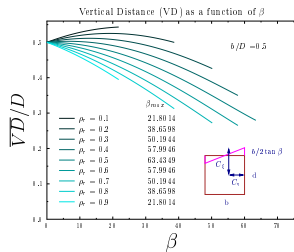
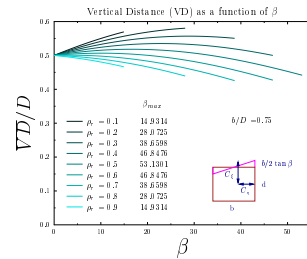


Fig. 9.7 – The maximum beta angle that occurs before the corner is immersed in the liquid

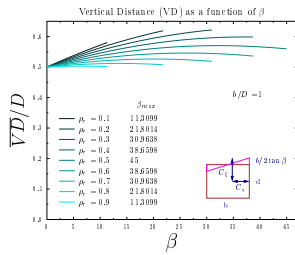
At this point the discussion can turn back to vertical and horizontal distances. The horizontal distance  $GZ$  has to go through several stages. First the vertical distance from the buoyancy (referred to as  $VD$ ).



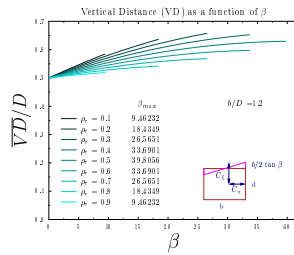
(a) Vertical distance for  $b/D = 0.5$



(b) Vertical distance for  $b/D = 0.75$



(c) Vertical distance for  $b/D = 1.0$



(d) Vertical distance for  $b/D = 1.2$

Fig. 9.9 – Vertical distance for rectangular extruded shape showing for 4 different  $b/D$  ratios as function of the turning angle  $\beta$ .

Fig. 9.9 shows the results of the vertical distance as a function for various density ratio and various  $b/D$  ratio. For small values value of  $b/D$  the vertical distance become smaller. For example, for  $b/D = 0.5$  almost all the values are trending down. While the values treading up for large values see Fig. 9.8d. This trend is consistent with the Potential

Energy is based on. According to this method, the shorter distance of  $\overline{BM}$  the more stable the situation is. Here, the stable situations will have a trend going up because they are stable and vice versa. The intermediate stage has a mix situation where part trending up and part trending down which indication that the body is stable only under certain conditions which is consistent with previous observations.

The distance  $\overline{GZ}$  is calculated utilizing the nomenclatures that used in Fig. 9.10. The distance  $\overline{GJ}$  is obtained by

$$\overline{GJ} = d - \overline{JK} - \overline{AG} \tag{9.13}$$

Where these distances are evaluated as  $\overline{JK} = d + \Psi - C_\xi$  and  $\overline{AG} = d - D/2$ . Utilizing these definitions yields

$$GJ = \frac{D}{2} + C_\xi - (\Psi + d) \tag{9.14}$$

The dimensionless value of  $C_\xi$  is given by Eq. (9.10). Utilizing the  $GJ$  to obtain the

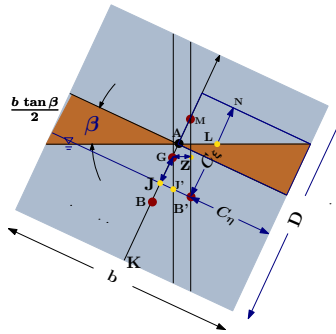


Fig. 9.10 – Schematic for naming for the  $\overline{GZ}$  for vertical and horizontal distances. Notice, some of the nomenclatures are adapted from page 48.

next value in the chain of values as

$$\overline{JJ'} = \overline{GJ} \tan \beta \tag{9.15}$$

with this value

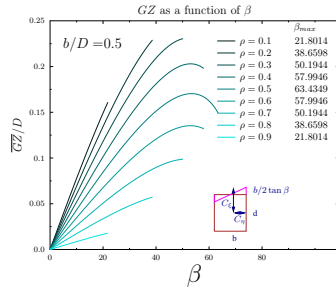
$$\overline{B'J'} = \overline{JB'} - \overline{JJ'} \tag{9.16}$$

where the value of  $\overline{JB'}$  can be observed from the diagram as

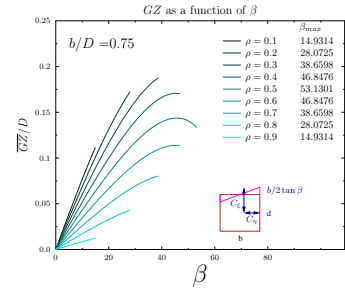
$$\overline{JB'} = \frac{b}{2} - C_\eta \tag{9.17}$$

and finally

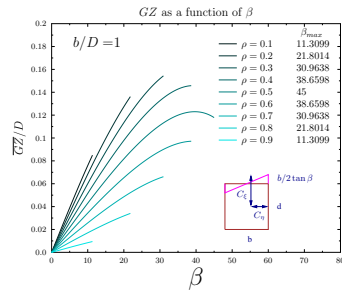
$$\mathbf{B}'J'_\perp = \mathbf{GZ} = \mathbf{B}'J' \cos \beta \tag{9.18}$$



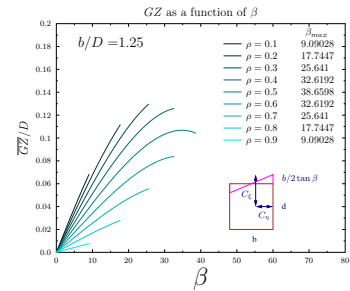
(a)  $\mathbf{GZ}$  as a function  $\beta$  for  $b/D=0.5$



(b)  $\mathbf{GZ}$  as a function  $\beta$  for  $b/D=0.75$



(c)  $\mathbf{GZ}$  as a function  $\beta$  for  $b/D=1.0$



(d)  $\mathbf{GZ}$  as a function  $\beta$  for  $b/D=1.25$

Fig. 9.12 –  $\mathbf{GZ}$  as a function of the  $\beta$  for various  $b/D$ . It can be noticed that the angle is smaller for larger density ratio.

Several trends can be observed in Fig. 9.12. First it must be noted that of the abscissa scale is different for these graphs. As the ratio of  $b/D$  incenses the distance  $\mathbf{GZ}$

decreases. The maximum is achieved earlier for larger  $b/D$  values. After the body reaches the maximum angle, new equation describes by the new equations and it is not shown in the figure. The change will manifest itself in the direction but still will be continues that is no jump in the angle.

### 9.4 Stability of Body with Shifting Mass Centroid

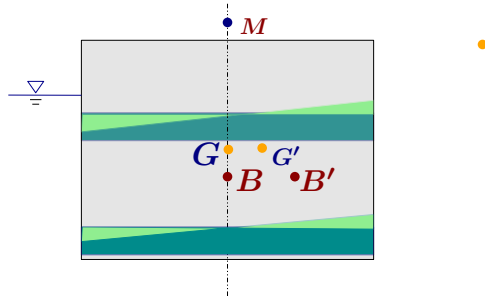


Fig. 9.13 – The effects of liquid movement on the  $\overline{GM}$ .

Ships and other floating bodies carry liquid or have a load which changes the mass location during tilting of the floating body. For example, a ship that carries wheat grains where the cargo is not properly secured to the ship. The movement of the load (grains, furniture, and/or liquid) does not occur in the same speed as the body itself or the displaced outside liquid. Sometimes, the slow reaction of the load, for stability analysis, is enough to be ignored. Exact analysis requires taking into account these shifting mass speeds. However, here, the extreme case where the load reacts in the same speed as the tilting of the ship/floating body is examined. For practical purposes, it is used as a limit for the stability analysis. There are situations where the real case approaches to this extreme. These situations involve liquid with a low viscosity (like water, alcohol) and ship with low natural frequency (later on the frequency of the ships). Moreover, in this analysis, the dynamics are ignored and only the statics is examined (see Figure 9.13).

A body is loaded with liquid “L” and is floating in a liquid “F” as shown in Figure 9.13. When the body is given a tilting position the body displaces the liquid on the outside. At the same time, the liquid inside is changing its mass centroid. The moment created by the inside displaced liquid is

$$M_{in} = g \rho_L \theta I_{xxL} \tag{9.19}$$

Note that  $I_{xxL}$  isn't the identical to the moment of inertia of the outside liquid interface.

The change in the mass centroid of the liquid “L” then is

$$\overline{G_1 G'_1} = \frac{g \rho_L \theta I_{xxL}}{\underbrace{g V_L \rho_L}_{\substack{\text{Inside} \\ \text{liquid} \\ \text{weight}}}} = \frac{I_{xx} \theta_L}{V_L} \tag{9.20}$$

Equation (9.20) shows that  $\overline{GG'}$  is only a function of the geometry. This quantity,  $\overline{G_1G'_1}$ , is similar for all liquid tanks on the floating body.

The total change of the vessel is then calculated similarly to center area calculations.

$$g m_{total} \overline{GG'} = g m_{body} \overline{GG'} + g m_f \overline{G_1G'_1} \tag{9.21}$$

For more than one tank, it can be written as

$$\overline{GG'} = \frac{g}{W_{total}} \sum_{i=1}^n \overline{G_iG'_i} \rho_i V_i = \frac{g}{W_{total}} \sum_{i=1}^n \frac{I_{xxb_i}}{V_{b_i}} \tag{9.22}$$

A new point can be defined as  $G_c$ . This point is the intersection of the center line with the vertical line from  $G'$ .

$$\overline{GG_c} = \frac{\overline{GG'}}{\sin \theta} \tag{9.23}$$

The distance that was used before  $\overline{GM}$  is replaced by the criterion for stability by  $\overline{G_cM}$  and is expressed as

$$\overline{G_cM} = \frac{g \rho_A I_{xxA}}{\rho_s V_{body}} - \overline{GB} - \frac{1}{m_{total}} \frac{I_{xxb}}{V_b} \tag{9.24}$$

If there are more than one tank partially filled with liquid, the general formula is

$$\overline{G_cM} = \frac{g \rho_A I_{xxA}}{\rho_s V_{body}} - \overline{GB} - \frac{1}{m_{total}} \sum_{i=1}^n \frac{I_{xxb_i}}{V_{b_i}} \tag{9.25}$$

One way to reduce the effect of the moving mass center due to liquid is done by substituting a single tank with several tanks. The moment of inertial of the combine two tanks is smaller than the moment of inertial of a single tank. Increasing the number of tanks reduces the moment of inertia. The engineer could design the tanks in such a way that the moment of inertia is operationally changed. This control of the stability,  $\overline{GM}$ , can be achieved by having some tanks spanning across the entire body with tanks spanning on parts of the body. Movement of the liquid (mostly the fuel and water) provides way to control the stability,  $\overline{GM}$ , of the ship.

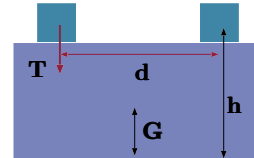


Fig. 9.14 – Measurement of GM of floating body.

### 9.4.1 Neutral frequency of Floating Bodies

Please ignore this section as it will completely be modified.

## Meta

The examination of the  $\overline{GZ}$  can be treated as linear for small range. Hence, this fact can be basically making the ship or the floating body act as it is a pendulum in some senses. Yet at the present case the mass or the moment of inertia is not easy to calculate. In movement of pendulum in air the added mass and the add moment of inertia are hard to calculated and is not negligible. The governing equation for the pendulum is

$$\ell \ddot{\theta} - g \theta = 0 \quad (9.26)$$

Where here  $\ell$  is length of the rode (or the line/wire) connecting the mass with the rotation point. Thus, the frequency of pendulum is  $\frac{1}{2\pi} \sqrt{\frac{g}{\ell}}$  which measured in  $Hz$ . The period of the cycle is  $2\pi \sqrt{\ell/g}$ . Similar situation exists in the case of floating bodies. The basic differential equation is used to balance and is

$$\overbrace{I \ddot{\theta}}^{\text{rotation}} - \overbrace{V \rho_s \overline{GM} \theta}^{\text{rotating moment}} = 0 \quad (9.27)$$

In the same fashion the frequency of the floating body is

$$\frac{1}{2\pi} \sqrt{\frac{V \rho_s \overline{GM}}{I_{body}}} \quad (9.28)$$

and the period time is

$$2\pi \sqrt{\frac{I_{body}}{V \rho_s \overline{GM}}} \quad (9.29)$$

In general, the larger  $\overline{GM}$  the more stable the floating body is. Increase in  $\overline{GM}$  increases the frequency of the floating body. If the floating body is used to transport humans and/or other creatures or sensitive cargo it requires to reduce the  $\overline{GM}$  so that the traveling will be smoother.

## Meta End

### 9.4.2 Limitations

This topic is dealing with several aspects of fluid mechanics like wave propagation on shallow liquid and the add mass. Admittedly these topics are very advance and require considerable amount of material to explain them. Simplified explanation is provided. The take away of this section to illustrate that the analysis is provides the limit and the actual movement are some what different. When the ship or the floating body undergoes rolling, the shape of the body is fixed (to very large degree) while the shape or the configuration of the liquid in the container or the tank is not. The actual time to get

the shape that should be assumed to be in a steady state. Steady state refers to a state which obtained when the particles stay at that location for ever (again approximate definition). How long it takes for the liquid to reach a steady state is very hard to calculate. However, it (to ascertain) is a much easier to do. First, it has to define the characteristic time (the typical time to for liquid to move from side to side).

The movement of the liquid from side to side is done by a wave (this claim is without a proof and more about it will appear in the book of "Basic of Fluid Mechanics" by this author). The speed of the wave over liquid in a shallow liquid is approximated by gravity and the liquid depth.

$$U \sim \sqrt{hg} \quad (9.30)$$

This equation provides a reasonable estimate of the velocity. The time required is then

$$t_c = \frac{b/2}{\sqrt{hg}} \quad (9.31)$$

When writing Eq. (9.31), it was assumed that the average distance is about the half the width ( $b/2$ ) (liquid is moving side to side). Several additional assumptions have to be made to find the characteristic time liquid movement. What is the

When the body or the ship is rolling the

### 9.5 Metacentric Height, $\overline{GM}$ , Measurement

The metacentric height can be measured by finding the change in the angle when a weight is moved on the floating body.

Moving the weight,  $T$  a distance,  $d$  then the moment created is

$$M_{weight} = T d \quad (9.32)$$

This moment is balanced by

$$M_{righting} = W_{total} \overline{GM}_{new} \theta \quad (9.33)$$

Where,  $W_{total}$ , is the total weight of the floating body including measuring weight. The angle,  $\theta$ , is measured as the difference in the orientation of the floating body. The metacentric height is

$$\overline{GM}_{new} = \frac{T d}{W_{total} \theta} \quad (9.34)$$

If the change in the  $\overline{GM}$  can be neglected, equation (9.34) provides the solution. The calculation of  $\overline{GM}$  can be improved by taking into account the effect of the measuring weight. The change in height of  $G$  is

$$g m_{total} G_{new} = g m_{ship} G_{actual} + g T h \quad (9.35)$$

Combining equation (9.35) with equation (9.34) results in

$$\overline{GM}_{actual} = \overline{GM}_{new} \frac{m_{total}}{m_{ship}} - h \frac{T}{m_{ship}} \quad (9.36)$$

The weight of the ship is obtained from looking at the ship depth.



## 9.6 Stability of Submerged Bodies

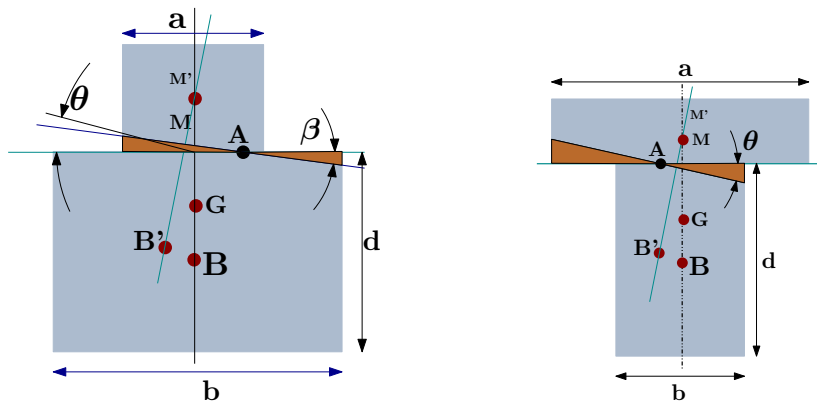
The analysis of submerged bodies is different from the stability when the body lies between two fluid layers with different density. When the body is submerged in a single fluid layer, then none of the changes of buoyant centroid occurs. Thus, the mass centroid must be below than buoyant centroid in order to have stable condition. Stable conditions

However, all fluids have density varied in some degree. In cases where the density changes significantly, it must be taken into account. For an example of such a case is an object floating in a solar pond where the upper layer is made of water with lower salinity than the bottom layer (change up to 20% of the density). When the floating object is immersed into two layers, the stability analysis must take into account the changes of the displaced liquids of the two liquid layers. The calculations for such cases are a bit more complicated but based on the similar principles. Generally, this density change helps to increase the stability of the floating bodies. This analysis is out of the scope of this book (for now).

## 9.7 Stability of None Systematical or “Strange” Bodies

This topic was address somewhat in the previous chapter and will be expanded in the following versions of this book. Previously, the discussion dealt with smooth transition in which the cross section area is a smooth function of the height. In section the discussion deals with abrupt change in the cross section. The change is more profound. Consider two extreme cases, one where narrow immersed body and two the wide part immersed. The rotation point A is not rotating and move in two different directions depending on cross section area (as function of the height) direction. Clearly without calculating any data, one can observed that the body with the wide immersed part rotating point A moves to the right while the other case the rotating point A moves to left. In, the change if direct of the rotating point change the way calculations are carried out. Never the lest it is clear wide immersed body is more stable.

While most floating bodies are symmetrical or semi-symmetrical, there are situations where the body has a “strange” and/or un-symmetrical body. The calculations will appear in the following versions. These are on the same considerations that that were discussed in the previous chapter.



(a) Floating Body with abrupt transition to small on the top  
 (b) Floating Body with abrupt transition to large on the top

Fig. 9.15 – Calculations of  $\overline{GM}$  for abrupt shape body.

# 10

## Variable Density Floating Bodies

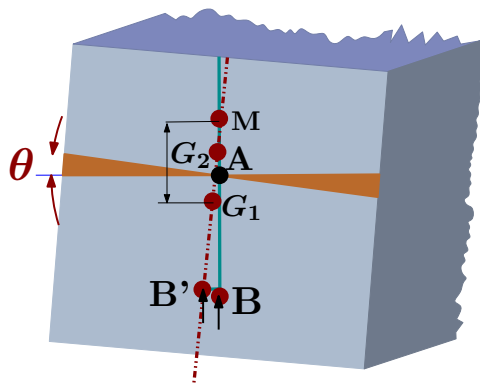


Fig. 10.1 – Body with movable gravity centroid.

Ship and other floating bodies do not have a constant density. In other words, the gravity centroid moves depend on the loading to a different location in the ship or body. That point is different from the centroid of the entire body. However, the buoyancy centroid is in the same location. located before as it depend on the displaced volume. If the gravity centroid is below the buoyancy centroid then the body is stable and no further calculations are needed. However, for most cases the issue to find how much to change the location of the gravity centroid to the vessel is stable. This chapter is dealing with the requirement on location of the gravity centroid for given geometry.

Previously the gravity location was given for the question was given a rotation of the body by  $\theta$  what is the resulting  $\alpha$ . Now the question is more complicated. Given

the geometry, what are the requirements on the location of the gravity centroid to make the body stable. A new triangle can be build that is now made from three points  $B$ ,  $B'$  and  $G$ .

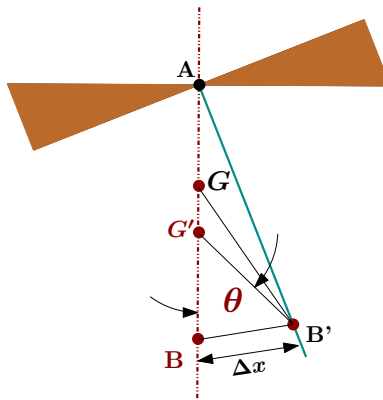


Fig. 10.2 – Diagram to explain the calculation for safe  $G$ .  
The geometry (law of sines) it can be written and utilizing Eq. (4.15)

$$\frac{\Delta x}{\sin \theta} = \frac{BG'}{\sin 90^\circ} \longrightarrow \frac{\tan \theta \frac{I_{xx}}{V_0}}{\sin \theta} = BG' \quad (10.1)$$

Or in a clear form

$$BG' = \frac{I_{xx}}{V_0} \quad (10.2)$$

The location of gravity in solid body (or floating body) is at  $G$  and normally is at the centroid of the entire body. The distance required to lower the gravity to make the body stable is  $GG'$ . Hence,

$$GG' = BG - BG' \longrightarrow GG' = BG - \frac{I_{xx}}{V_0} \quad (10.3)$$

After the required distance is calculated, a weight needed to be attached, to bottom, or other place. The weight size is based on the calculations explained in Chapter 4.

#### Example 10.1: Missing GM for Ship

Level: Easy

In Ex. 8.2 dealt with rectangular shape which has zone with were unstable. Here in this example, a possible correction will be calculated ( $GG'$ ).

#### Solution

The Fig. 8.13 shows the shape where the rectangular is unstable and for that range

## End of Ex. 10.1

the required change is calculated. Eq. (8.2.e) is copied here for the boundary.

$$\frac{b}{D} = \sqrt{6 \frac{\rho_s}{\rho_\ell} \left(1 - \frac{\rho_s}{\rho_\ell}\right)} \quad (10..a)$$

and

$$\frac{d}{D} = \frac{\rho_s}{\rho_\ell} \quad (10..b)$$

In this case  $BG = D/2 - d/2$  and thus the  $GG'$  is

$$GG' = \left(\frac{D}{2} - \frac{dD}{2}\right) - \frac{1}{12} \frac{b^2 D}{D^2} \frac{\rho_\ell}{\rho_s} \quad (10..c)$$

and can be combined

$$\frac{GG'}{D} = \frac{1}{2} \left(1 - \frac{\rho_s}{\rho_\ell}\right) - \frac{1}{12} \left(\frac{b}{D}\right)^2 \frac{\rho_\ell}{\rho_s} \quad (10..d)$$

It can be observed that the largest change should be made when the density is when the liquid density is relatively very low. Most large ships are with density ratio on the upper limit has the change is relatively small.

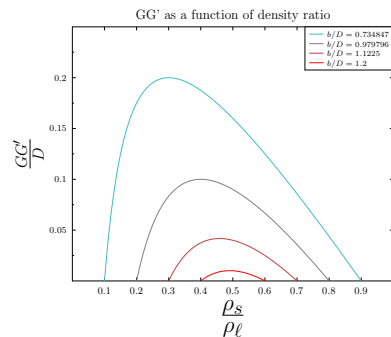


Fig. 10.3 – Requirements on gravity centroid for rectangular shape body.

The next is similar for the cylinder that was dealt before.

### Example 10.2: What cylinder is stable

Level: Hard

Earlier in the book in Fig. 8.24 the stability of cylinder was discussed. Here, the correction that need to be applied so that cylinder will be stable ( $GG'$ ).

### Solution

The Fig. 8.13 shows the shape where the rectangular is unstable and for that range

End of Ex. 10.2

the required change is calculated. stability zone.

$$\frac{r}{D} = \sqrt{2 \frac{\rho_s}{\rho_\ell} \left(1 - \frac{\rho_s}{\rho_\ell}\right)} \quad (10..e)$$

and

$$\frac{d}{D} = \frac{\rho_s}{\rho_\ell} \quad (10..f)$$

In this case  $BG = D/2 - d/2$  and thus the  $GG'$  is

$$GG' = \left(\frac{D}{2} - \frac{dD}{2D}\right) - \frac{1}{4} \frac{r^2 D}{D^2} \frac{\rho_\ell}{\rho_s} \quad (10..g)$$

and can be combined

$$\frac{GG'}{D} = \frac{1}{2} \left(1 - \frac{\rho_s}{\rho_\ell}\right) - \frac{1}{4} \left(\frac{r}{D}\right)^2 \frac{\rho_\ell}{\rho_s} \quad (10..h)$$

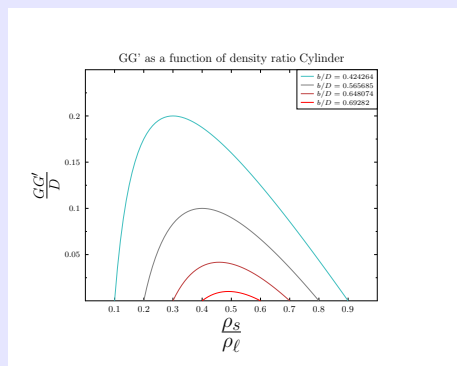


Fig. 10.4 – Requirements on gravity centroid for cylinder shape body. The Fig. 10.4 is seem to the replica (hopefully) not a mistake.

### Example 10.3: EverGiving GM

Level: Easy

The famous ship Ever Giving for this excise can be consider as extruded square box with the following dimension  $L = 400[m]$   $b = 60[m]$  and the  $D = 60[m]$ . The maximum draft (the depth) the ship can be in the liquid is  $16[m]$ . What is the minimum requirement on gravity centroid.

#### Solution

The ratio of  $b/D = 1$  in this case. From the value of  $d$  and  $D$  the ratio of density can be estimated as  $16/60 = 0.267$  and observing from Fig. 10.3 the value of  $GG'/D = 0.075$  thus, the value of  $GG' = 60[m] \times 0.075 \sim 4.5[m]$  For uniform body the gravity centroid

**End of Ex. 10.3**

has to be at half of  $D$  thus it is has to be at  $30[m]$ . Thus, subtracting the correction it became that gravity centroid has to be at least below the  $25[m]$  from the bottom of the ship.





# 11

## Added Mass (Moment of Inertia) and Transfer Properties

This chapter is dedicated to my teacher, Dr. Touvia Miloh.

Genick Bar-Meir

Traditionally this topic is not covered by an introductory class on fluid mechanics nor ship stability or marine architecture. This topic is considered to be too advanced for regular fluid mechanics students or marine architecture. Here, it is advocating to introduce the material conceptually without actually doing the actual calculations of the added mass and added moment of inertia. The reason for this approach is to make students aware of these phenomena and/or at least to the practitioners in the field.

It is the experience of this undersign, that it is common that people not aware of these properties and make calculations that have very little to do with reality. The most obvious, a typical usage of both properties is in the marine or ship calculations issue dealing with interaction of fluid with solid structures and energy harvesting. Yet, many engineers who start to work in these areas are clueless on these issues. Not only entry engineers are in lack understanding even researchers who work in this area of added mass. Some of the these researchers are making calculations that are not rooted on reality or break the meaning of the added mass. For instance, some papers dealing with numerical calculations of added mass but they include turbulence effects while turbulence should not account by the calculations of the added mass (Kianejad, Enshaei, and Ranmuthugala 2017)<sup>1</sup>. For another example Brennen a main/premier researcher stated that (Brennen 1982, p 6 second paragraph)

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<sup>1</sup>Kianejad et al also include among other thing large angles (strange) and several other parameters. It is not unique and many other researchers include these parameters. None of them seem to be able to explain this inclusion. It is a case of more is actually less

One other complication will emerge in the following section when the complete added mass matrix is defined, namely that the force on the body due to acceleration is not necessarily in the same direction as the acceleration. For an unsymmetnic body acceleration in one direction can give rise to an "added mass" effect resulting in a force which has a component in a direction perpendicular to the direction of acceleration. If, for example, one were lifting a body from the ocean bottom by means of a cable then an increase in the lift rate could produce a lateral action of the body.

This statement is wrong! Brennen was not aware to the existence of the transfer properties. He conflates the transfer properties with the added properties which will be discussed down below. The added mass properties are related to the acceleration.

In fact, since this material has been changed dramatically the current state of knowledge, it should be brought even to the attention of the people which deal with added mass research, mass transfer transfer (bubble flow) or multi-phase (especially with liquid solid flow or large change in density)<sup>2</sup>. Even the most fundamental part like difference between the added mass coefficients vs the added mass was not clear in any book or research to this point even to this author before the writing.

This chapter is introductory which emphasis the understanding of the concept rather than actual calculations. Hence, this chapter provides examples where people mistakenly claim that something is results of added mass effect while it is not. Thus, a discussion will be focusing on the definition and character, later attempt is made to differentiate what is and what is not added mass (or related to it or results of it).

One can wonder how long it will take this information to penetrate to premier universities, USA government and other organizations?<sup>3</sup>

## 11.2 History

### Meta

The added mass was introduced as early as 1776 by Du Buat (Du Buat 1786) in first edition (copy of the book strangely can be found the University of Michigan call number QC151.D8211816). The time and gravity calculations required more precision which was prompted notice added mass (time for the pendulum period). His measurements showed that the added mass of sphere was value in the range 0.45-0.67. Consider that theoretical value is 0.5 which is a great success. Later Green studied the added mass in 1833 for the same reason and George Stokes (Stokes et al. 1851). In parallel Friedrich Bessel proposed the concept of added

<sup>2</sup>It so strange to write something and while you doing it to realize that so little is based on science. Everything has to be questioned so it could be understand because the common believe is full with misconceptions.

<sup>3</sup>Some individuals believe that it will take about 10 years to spread the information. Other believe that it will be about 40 years. This author contacted several individuals who work with the USA government or with association with the government yet they did not bother to reply. This fact makes one to believe that it will take long time. Yet two discussions on the invitation to give a workshop makes one more optimistic.

mass in 1828 (no real reference only claims of second hand). Even Charles Darwin (Darwin 1953) have worked on showing that the particle actually moving during the movement of the body. The sphere shape was solved on 1833 by Green (Lamb 1924). The generalization of the added properties was done by Stokes where investigate the affects of the boundaries.

## Meta End

### 11.3 What is the Added Mass?

A ball placed in a close container and via exterior (or interior) force such as magnetic force is propelled forward. A ball in a container full with fluid is shown in Fig. 11.1.

The container is closed and hence no material leaves or enters the container. First, it is assumed that fluid in the container is incompressible. Some of the material must move to the right when the ball moves to the left (conservation of mass). The mass conservation requires that if a change in ball's location occurs then the first derivative (velocity) and the second derivative (acceleration) etc of the ball must be compensated as well. These compensations of such movement requires the fluid to

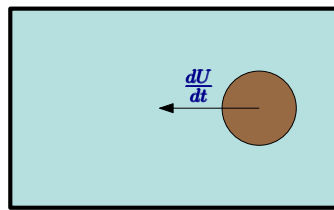


Fig. 11.1 – Accelerating ball in a close container.

displace volume not occupying by the ball. These changes in the fluid field require that a force or in other words, the change in the field energy must occur. Namely, additional force is needed to give the body a certain acceleration beside the regular  $F = m_{body} a$ . This additional force, can be written as  $F_a = m_a a$  where  $a$  is the same acceleration of the body,  $F_a$  is the required added force, and  $m_a$  is the added mass. Without the detail, it must recognized the (exterior not required) force also to the right (even though the liquid moves the left). Thus, if the force can be calculated, the added mass can be calculated. The flow field is elliptical character and hence the added force should linearly depend on the body acceleration. The term elliptical refers to the superposition possibility (see equation ??). Body in a flow in which obeys these kind of equations can have a segregate added force and the added mass (There are those who oppose to this statement.). These complicated calculations of those forces and moments due to the fact that they requires two stages (at least). The first stage is to find velocity field and the second to find how this velocity field is changed with the body acceleration.

The added mass is not defined by actual material. Furthermore, it does not confined/defined to the material that has a certain acceleration. The definition of the added simply is based on the added force. The added mass, like all the masses, represents the resistance to the acceleration. In "normal" situation the added mass confined to specific material. In this case, no specific material can be assigned or confined. This

imaginary material is material that would be at the origin coordinate system used for the calculations and mass value is ratio of the force and the acceleration. In fact this mass is attributed to liquid (with constant density) and hence it can be defined by the liquid volume (utilizing the liquid density). This volume is not the same as the volume of the body but a strong function of the body geometry.

Suppose that these calculations can be carried (somehow easily), the change of the force depends on the container. Different containers require different forces. As engineers always have done, the documentations/calculations are broken into two parts. The first part, calculations/documentations are carried for a moving body in a infinite container. Second part done when the added mass is calculated/documented for body, then the change in the container size effects are calculated and documented. The question, can the container effects be transferable from one body to another. At first glance, the problem is elliptic in nature (allows superposition) and it seems that it should be correct. Yet, this topic for some strange reasons is not conclusive and there is a dispute on accuracy of the idea (even though the proof in the appendix of this chapter it is still debatable<sup>4</sup>). Of course, the main opposition rest with those how believe that turbulence and velocity (Reynolds number) should entered into the calculations. This author takes the position that is reasonable to apply superposition<sup>5</sup>.

The added mass is important in several situations specialty when the added properties are in same size as the forces or the moments to the only the body. The typical situations where the added properties important are the marine (ship or floating bodies), high rise building, and propellers.

Two topics are to be mentioned the compressibility, free surface, and the case of the two phase materials like water and air. The second topic is a typical issue for ship stability and even for multiphase flow. Compressible substance such as air, often have much lower density then the solid (or semisolid) body and hence can be ignored for most applications in this introductory material. The calculations of the added mass for free surface suffered from additional complications. The surface is deformed and it added complexity to the calculations. At this stage a little can be added (to free surface) that is not too complicate.

The idea of added mass is initially build for high Reynolds numbers (ideal flow). Clearly the added mass exist in different Reynolds number range yet the separation is harder. It implies that added mass is somehow depends on Reynolds seem a bit strange and again this point is not conclusive. It seems that it like turbulence, and viscosity should not be accepted to be part the added mass topic. If accepting the Reynolds number effect, it means that the added mass depends somehow on the Reynolds number among other parameters in situations like Stokes flow. Thus, the reading the "fine print" is important. That is, at what ranges and other limiting factors, it is applicable (is viscosity and other effects are really segregated).

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<sup>4</sup>The meaning dispute in this context is that there are many publications dealing with different boundary conditions for different bodies.

<sup>5</sup>If someone can provided the proof to either way that should be the case. The proof at the end of this chapter convince some but not all.

### 11.4 The Added Mass Matrix of a Body

The velocity potential of inviscid flow is defined in (Bar-Meir 2021) as

$$U_x = \frac{\partial \phi}{\partial x} \quad U_y = \frac{\partial \phi}{\partial y} \quad (11.1)$$

$$U_z = \frac{\partial \phi}{\partial z} \quad (11.2)$$

Note the velocity due to the rotation is not defined in this book. The elliptic character (linearity) of the ideal flow (Bar-Meir 2021) allows combination of different potentials due to different velocity which can be combined as

$$\phi = \phi_x + \phi_y + \phi_z \quad (11.3)$$

where  $\phi$  is the potential function and the subscript  $i = x, y, z$  is due to the velocity in the  $i$  direction. Where the subscript  $i$  can be in any orthogonal coordinate system. While this description is not fully used in this section, it gives a hint how the potential function is used later. The added force is a single force that act on the body. In other words added force like all the forces is a vector. When a vector (with 3 elements) is divided by a vector (with 3 elements) then results is a matrix of  $3 \times 3$  (see the two vectors in Fig. 11.2). Thus dividing the force (a vector) by the velocity (a vector) create similar situation to the stress matrix (created by division of the force by area which is a vector also see Eq. (3.16)). The results are added mass coefficients which are **not** the added masses<sup>6</sup>. The added mass is a scalar in the same way the body mass which the reason it can be added to the body mass without any limitations. This added mass can be assumed to be located at the body gravity centroid (at least is acting on the action line of the gravity centroid). As opposed to the regular mass, the added mass value is not a constant and it depends on the direction of the acceleration. Hence, the added mass is **not** a vector in the regular meaning but a scalar with different value which depends on the direction<sup>7</sup>. The added mass is based on the added force and if the added force can be broken into three components,  $F_x, F_y,$  and  $F_z$  a new added mass can be defined as

$$m_i = \frac{F_i}{a_i} \quad i = x, y, z \quad (11.4)$$

The added mass that the body experience in the direction  $x$  depends on  $g$  acceleration and the total added force and cosine of the angle to  $x$ . This value is not added mass coefficients which are different creatures. The discussion on the added mass coefficients is provided in the appendix at the end of this chapter.

It is common believe, the added moment of inertia in many publications in the same vein as the added mass. The total added moment of inertia does exist, but unfortunately the common believe defined it as a moment inertia that the body rotation

<sup>6</sup>This is source of confusion that got even this author sin in accepting this point before.

<sup>7</sup>This author is not familiar with a similar physical concept. Notice that Force is in the same direction of the acceleration. Hence dimensional of the units is of [kg]

around the gravity centroid (well it would be true if the body indeed rotate around the gravity centroid). If the ship or the marine body was rotating around the gravity centroid it would be appropriate. However, as it was shown in this book, the floating body (ship) does not rotate around the gravity centroid. Furthermore, the rotating axes do not pass through the gravity centroid but depend on the geometry of the body location of the liquid line. The governing equations are needed to be solved when analyzed the marine body and have to use the actual added moment of inertia (or close to real value and not something that is hundreds of percents off the mark like today it is done). As it not enough, the axes for added moment of inertia have different origin. The ascertain the actual moment of inertia for marine body, the parallel theorem for added properties has to be used. This theorem is similar to the tensor parallel axis theorem which will discussed in the appendix.

It is the common believe that the added mass is a matrix of  $6 \times 6$ . The reason the number is six because there is 6 different movements. Assuming that the argument about the rotation potential is dismissed and the rotations are included. The elements of the matrix is made of added mass coefficients **not** the added masses. The matrix is made of the main diagonal (green color) which display the main added mass coefficients. The regular (black color) are secondary added mass coefficients. The secondary coefficients are the results of the division of the added mass by the velocity for example of  $x$  and taking the component in  $y$  direction. The main added mass components refers to the added mass created by dividing by  $x$  and the  $x$  element of that value. The values of main added mass are always positive they are never zero (in the common believe). The secondary values can be zero or a positive. Vininje (1989) claims to find negative values yet it generally rejected by all others and this author has no opinion on this point either way because it like twilight zone. In fact, it is very common to have zero for secondary added masses (on the non main part). Under the common believe is that, the secondary added mass appears when only when there is velocity the secondary direction. None seem to explain why this happen based any physical principle. No explanation, on how this added property can happen or even possible, was found in the literature. According to the common believe the matrix looks like is

$$m = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{pmatrix} \quad (11.5)$$

It can be noticed that some elements of the matrix have two kind units (actually three), mass and moment of inertia. According to the common believe this matrix is always symmetrical that is, for example,  $m_{32} = m_{23}$ . The appendix of this chapter offers a discussion when the conditions for this symmetry really happen<sup>8</sup>. This kind of writing allows to present the compact writing which supposed to reduce the clutter. It is

<sup>8</sup>This common believe is just like CNN or NBC or BBC never know when they are telling the true.

interesting to point out that people holding the common believe put large efforts to prove this idea.

According to the common method, the secondary added mass coefficients appear only when the secondary velocity appears. The real reason “secondary added mass” appears completely different from the reason that common believe adapted. The reason stems from the fact that the body has truly **only one** added mass (It is amazing that no one has noticed this basic fundamental fact.). To make things easy for calculations, people attribute the added mass to different velocity components. This process attribution is what create that added mass coefficients. It is the reverse is only the convenience that people are using to describe added mass in the velocity coordinate. actually has only a single velocity (strange that it has to be pointed out but trees are missed because the forest.). These claims that body has secondary added mass (such as  $m_{12}$ ) are simply wrong and they are not property of the body in the same sense as the common believe of this ideology. The added mass of a body is actually a **single** component as opposed to transfer property that will be discussed later. To understand it, consider a body that moves in the general direction  $\mathbf{U}$ . The floating body is accelerating in the arbitrary direction with the velocity  $\mathbf{U}$ . At that direction the body has only added mass  $m_U$  and body does not have any secondary added mass. This claim can easily verified by the fact that the added force is only a single component at the direction of the acceleration. The common believe internal conflict is exposed because there are **no** other velocity components and yet common believe claims that they exist. The body is only affected in the direction of the acceleration. According to the common believe, in the coordinate system shown in the Fig. 11.2 the body has to have two components main and two secondary added mass. Yet, the actual value of the added force is  $m_U dU/dt$ . The component of added force in the  $x$  is  $m_U dU/dt \cos \theta$  where  $\theta$  is the angle between the velocity and  $x$  coordinate. In same time, the added force in the  $y$  direction is  $m_U dU/dt \sin \theta$ . Thus, the added mass in the  $x$  direction must be  $m_U \cos \theta$  and the added mass in the  $y$  direction must be  $m_U \sin \theta$ . The common approach has different values for the added mass in  $x$  and  $y$  not to mention the secondary added mass. The difference is results of the conflating added mass with added mass coefficients.

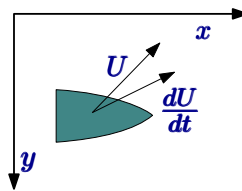


Fig. 11.2 – Ship in two dimensional general coordinate.

For example, Brennen suggested (that appeared earlier) that somehow there is added force perpendicular to the direction of boy acceleration is conflating two different concepts and conflicts with the definition of the added mass<sup>9</sup>. One concept is dealing with added mass and other concept deals with the transfer properties (related mostly to velocity). The distinguishing can be done by checking if only the acceleration affects the force or the velocity. In the case of asymmetrical body, the force appears even if there is no acceleration, hence it is transfer properties.

<sup>9</sup>Added mass is results of the acceleration.

**Example 11.1: Galina's First Question****Level: Intermediate**

Prof. Galina Yakovlevna Dynnikova expert on added mass suggested that suppose the body that moves a long the path of  $\mathbf{U}$  is asymmetrical than there must be a force in the direction perpendicular to the body's path. Hence there must be secondary added mass. Thus, this facts breaks the idea of uniqueness of the added mass on the direction of  $\mathbf{U}$ . Explain why the researcher's idea is not breaking uniqueness of added mass.

**Solution**

If the body moves along a line (and either accelerates or not), then the body asymmetry might create a force. First case, the body is moving with a constant velocity then there is only force perpendicular to movement. Hence there is no work (work is force in the direction of the velocity). Hence change in the energy of the velocity field is zero. If there is no acceleration, there is no work and again no added force and hence no added mass. Second case, the acceleration in the direction of body does no affect the force in perpendicular direction and hence no added force and therefore no added mass.

What cause these added masses in the different direction?<sup>10</sup> All the explanations that appear in the literature do not make sense or were clear to this author. For example, in MIT class 2.016 Hydrodynamics it is stated that

A good way to think of the added mass components,  $m_{ij}$ , is to think of each term as mass associated with a force on the body in the  $i^{\text{th}}$  direction due to a unit acceleration in the  $j^{\text{th}}$  direction.

All the explanations must obey the physics laws or discuss the intuition and state that it is intuition. It further has to be pointed that researchers who hold the common believe that the secondary added mass appears only when the velocity in the secondary direction appears. In reality, the body does not know about any coordinates system. People are thinking about velocity components as different velocities in  $x$  or  $y$  etc which not make it correct. The body moves only in one direction which is vectorially combination of the components but it is only one direction. And in the acceleration direction, the body only has one added mass, no other direction! The added mass can be broken into components. However, in the human imagination and convenience, the factual added mass can be broken into components for convenient of the calculations. The reason that added mass can be broken into component is because the added force can be broken into components. The second mistake of this statement is that word "**acceleration**" in the third line is not right and should be velocity. At this stage it is not clear if it because conceptional misunderstanding or just typo or not paying attention to wording (this author is famous for doing it all the time). Is there connection between the added mass coefficients in different coordinates to arbitrary direction? Or in other words, can added properties coefficients utilized to build the added mass? The answer is yes.

<sup>10</sup>This was explain before it is a rehash. It seems to a necessity because the previous explanation is not digested well with some individuals.



**Example 11.2: Using Added Mass Coefficients** Level: S. S. Advance

Extruded ellipse with big radius  $a$  is and small radius  $b$  shown in Fig. 11.3. The added mass coefficients are given for the small radius  $\pi b^2$  and for the large radius  $\pi a^2$ . Using the data calculate the added mass for a body that moves at angle  $\alpha$  as shown in Fig. 11.3. The calculations involve using transformation of coordinates which is advance material. Nevertheless, this material is provided here to demonstrate how the transformation is done and to teach linear algebra. Yet, the background will be provide in the appendix to the book.

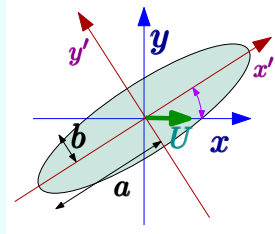


Fig. 11.3 – Calculation of added mass from added mass coefficients

**Solution**

The added mass coefficients is given two dimension matrix as

$$\mathbb{M}' = \rho \pi \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \tag{11.2.a}$$

It can be noticed that the main terms are similar cylinder as “seen” by the flow. The secondary terms are zero because symmetry. The acceleration in the prime coordinate is  $dU/dt \cos \alpha$  in the  $x'$  direction and the  $y'$  (note the negative direction)  $-dU/dt \sin \alpha$ .

$$\mathbb{F} = dU/dt \cos \alpha \hat{\mathbf{x}}' - dU/dt \sin \alpha \hat{\mathbf{y}}' \tag{11.2.b}$$

Writing it in a matrix form the added force in prime coordinate system is then

$$\mathbb{F}' = -\rho \pi \frac{dU}{dt} \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \cdot \begin{pmatrix} \cos \alpha \\ -\sin \alpha \end{pmatrix} \tag{11.2.c}$$

In coordinate  $x$ - $y$  the tensor of the added masses is

$$\mathbb{M} = \rho \pi \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \cdot \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \cdot \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \tag{11.2.d}$$

after the arithmetic one gets (hopefully)

$$\mathbb{M} = \rho \pi \begin{pmatrix} b^2 \cos^2 \alpha + a^2 \sin^2 \alpha & (a^2 - b^2) \cos \alpha \sin \alpha \\ -(a^2 - b^2) \cos \alpha \sin \alpha & b^2 \sin^2 \alpha + a^2 \cos^2 \alpha \end{pmatrix} \tag{11.2.e}$$

The added force is then

$$\mathbb{F} = \rho \pi \begin{pmatrix} b^2 \cos^2 \alpha + a^2 \sin^2 \alpha & (a^2 - b^2) \cos \alpha \sin \alpha \\ -(a^2 - b^2) \cos \alpha \sin \alpha & b^2 \sin^2 \alpha + a^2 \cos^2 \alpha \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11.2.f}$$

End of Ex. 11.2

or in simple terms as

$$\mathbb{F} = \rho \pi \frac{dU}{dt} \left( \begin{array}{c} -b^2 \cos^2 \alpha + a^2 \sin^2 \alpha \\ (a^2 - b^2) \cos \alpha \sin \alpha \end{array} \right) \quad (11.2.g)$$

The equation Eq. (11.2.g) shows the value of the added force and added mass will be without the acceleration.

**Example 11.3: Added Mass Calculations**

Level: S. Advance

The added mass in a given orientation has value of  $3.0 V_0$ . At this stage the units are ignored. Where  $V_0$  is the volume of the body. The body accelerates in a two dimensional coordinate system with 30 degree to  $x$  coordinate. The body does not experience any rotation. Calculate the added mass for the  $x$  coordinate and added mass for the  $y$  coordinate. Explain if the normalization process is required. Can the body has added mass that is 3 times the body volume?

**Solution**

First, the added mass has to be transferred to  $x$  and  $y$ . In  $x$  the added mass is

$$m_x = m_a \cos \theta = 3.0 V_0 \times \frac{\sqrt{3}}{2} \quad (11.3.a)$$

The transfer of the added mass to  $y$  is

$$m_y = m_a \sin \theta = 3.0 V_0 \times \frac{1}{2} \quad (11.3.b)$$

The discussion about the normalization refers to the formulation such that when multiply by  $U_0$  the actual added mass is added. The added force is  $F_a = m_a dU_0/dt$  and the added force in the  $x$  direction. The added force in the  $x$  direction is  $F_x = \cos \theta F_a$  or  $F_x = \cos \theta m_a dU_0/dt$ . Thus, the added mass in the  $x$  direction should be multiply by the actual velocity of the body.

The body can have a large added mass. For example, a thin body moving in the largest cross section has added mass several time the volume (times the liquid density).

**11.4.1 Added Moment of Inertia Coefficients**

A slightly similar situation to this one in linear motion happens in the rotations. The three rotations can be combined to form a single rotation if the rotations share common origin. If the common origin does not exist than the rotations components cannot be joined. Thus, only some circumstances rotations components can be combined. As writing this section, there is no known method to make general solution for it<sup>11</sup>. In the particular case where the rotation components have the same origin then the

<sup>11</sup>At the moment of this writing, this undersign is the only who accept this idea. And if this author has no a solution, no one has.

components can be combined to create a new axis around in which actuality the body is rotated. Then if the added moment of inertia known, it can be broken into specific coordinate. The arguments used the linear motion can be used for added moment of inertia as well.

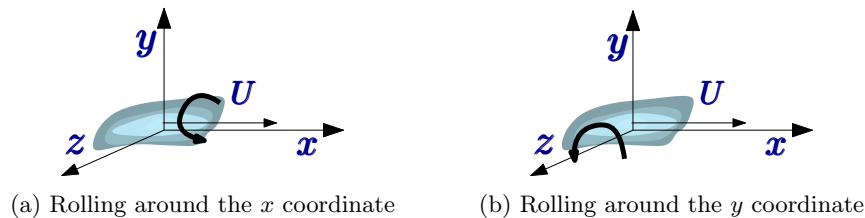


Fig. 11.5 – Rolling with two possibilities one around the  $x$  coordinate and two around  $y$  coordinate

After establishing the general principle concept for the linear and rotation added properties, at this stage the combination of two kinds is discussed. Two simple cases can be provided a taste of the situation and they are presented here. The first case shown in Fig. 11.4a and second case shown in Fig. 11.4b. In both cases, the body move in the  $x$  coordinate direction. However, the first case the body is rotating around the  $x$  coordinate and the second case the body is rotating around the  $y$  coordinate. In the first case the rotation does not change added mass nor it change the added moment of inertia of the body (this observation conflicts with the common believe ideology).<sup>12</sup> The reason for this “oblivion” behavior is that the fluid sees the same kind of body during the trajectory (direction). Along the path fluid, the body rotation does not change the added moment of inertia. It can be verified by trying to calculate the moment of inertia at any in body projection. For a body with a constant velocity in  $x$  direction and a rotation around  $x$ , according to the definitions in this book ( $dU/dt = 0$ ), will experience no added force and therefore no added mass in the direction of the movement. That is, the acceleration does not change the effect of rotation on the added mass in the  $x$  direction. For the second case, fluid sees a different body during the trajectory. The added moment of inertia does not change during that process because the rotation will cause the same energy change in fluid field. The change in the added mass in this case is some kind of periodic value (sinusoidal). As side note, it can be noticed that the cylinder causes added mass in all directions while moment of inertia in the special direction has a zero value.

## 11.5 Calculations of the Added Mass

There are several techniques used to evaluate the added masses and the added moment of inertia. The first technique is done by representing the body by potential flow elements. The word represents in this context means body that normal (perpendicular) velocity at the body border is zero relative to the body. Thus, the elements create a body

<sup>12</sup>This author did not find any explanation to this conflict. The rotation does not change the energy field of the fluid and has no change in the added mass according to the definition in this book.

with zero velocity at the surface (or in case of porous media some kind condition specific to boundary). Later, the potential flow elements are used, in various techniques, to evaluate the force and hence obtaining the added mass value. There those who conflate viscosity or viscous effects with added mass. Added mass is not result of viscosity. This process is very delicate and requires understanding the even and odd functions as a prerequisite (most of the time). This technique is out scope for introductory book<sup>13</sup>. The other method is numerical calculations by solving the Euler equation (This statement is disputed by some who believe in turbulence is of added mass. In this book definition turbulence is not part of the definition of added mass). There are two ways (actually more) to do it, one by looking at the forces forces and two by looking at energy considerations.

The acceleration is related to the kinetic energy. Furthermore, a change in the kinetic energy of the field requires power supply to be delivered to the body to make the change.

The kinetic energy in the field is

$$E = \frac{\rho_\ell}{2} \int_V ((u_1)^2 + (u_2)^2 + (u_3)^2) dV \quad (11.6)$$

where here  $E$  denotes the energy of the entire field and  $u_1$  is the locale velocity in let say  $x$  the same for the other components ( $u_2$  velocity in  $y$  and  $u_3$  velocity in  $z$  or any other orthogonal coordinate system). Note, this statement is correct for any orthogonal system. The velocity of the body is denoted as  $U_0$  and can be pulled out the integral as

$$E = \frac{\rho_\ell (U_0)^2}{2} \int_V \left( \left( \frac{u_1}{U_0} \right)^2 + \left( \frac{u_2}{U_0} \right)^2 + \left( \frac{u_3}{U_0} \right)^2 \right) dV \quad (11.7)$$

This operation is carried out to relate velocity field and the body velocity. This operation has the advantage of solving only for a body with one unit of velocity. Now, there is no need to solve the equation for all velocities. The integral can be written as

$$\mathbb{M} = \int_V \left( \left( \frac{u_1}{U_0} \right)^2 + \left( \frac{u_2}{U_0} \right)^2 + \left( \frac{u_3}{U_0} \right)^2 \right) dV \quad (11.8)$$

The value of the integral Eq. (11.8) is independent of body velocity and every geometry and ordination can be evaluated. The kinetic energy of the fluid field is

$$E = \frac{\rho_\ell U_0^2}{2} \mathbb{M} \quad (11.9)$$

The derivative with respect to time of  $\mathbb{M}$  (or in other words, the added mass is fixed) is zero. The derivative of Eq. (11.9) utilizing the chain rule yields

$$\frac{dE}{dt} = \rho U_0 \mathbb{M} \frac{dU_0}{dt} \quad (11.10)$$

<sup>13</sup>The undersign spend a semester to study this material and another semester as a teaching assistance. Even with this experience, it is considered to be difficult topic.

Yet, the power need to supply the field to carry this change has to be

$$\text{power} = F U_0 \quad (11.11)$$

The energy conservation requires that power supplied have to be equal to the change of the kinetic energy of the field where here,  $F$  is the force and  $U$  is the velocity, hence

$$F U_0 = \frac{dE}{dt} = \rho_\ell U_0 \mathbb{M} \frac{dU}{dt} \quad (11.12)$$

Canceling the velocity on both sides of Eq. (11.12) Provides

$$F = \rho_\ell \mathbb{M} \frac{dU}{dt} \quad (11.13)$$

Eq. (11.13) can get the standard form  $F = m a$  when  $\rho_\ell \mathbb{M}$  is denoted the added mass. This added mass is not actual liquid mass but rather a representation of equivalent mass if it would be accelerated with the body. A entire fluid field is accelerated but not in the same neither uniform amount of amount. So, instead of doing the calculations for the entire field it is done for a simpler representation is utilized.

The simple case to calculate is a two dimensional cylinder with radius,  $R$ . In this case the radial velocity is given ?? and it is copied into here as

$$U_r = U_0 \cos \theta \left( 1 - \frac{a^2}{r^2} \right) \quad (11.14)$$

The tangential velocity is ??

$$U_\theta = U_0 \sin \theta \left( 1 + \frac{a^2}{r^2} \right) \quad (11.15)$$

Eq. (11.8) is independent of the coordinate system, so a cylinder coordinate system is used and  $U_r$  and  $U_\theta$  can be used instead the regular Cartesian coordinate that was used in the definition. Thus the added mass is

$$\mathbb{M} = \int_R^\infty \int_0^{2\pi} \left[ \left( \frac{U_\theta}{U_0} \right)^2 + \left( \frac{U_r}{U_0} \right)^2 \right] r d\theta dr = \pi R^2 \quad (11.16)$$

The added mass of cylinder is exactly the same of the volume as it displaced multiply the density of liquid (as oppose to the density of the cylinder). Thus, when light cylinder is moving in a heavy flow, the resistance will be mostly due to the acceleration of liquid.

For sphere the potential is given by spherical coordinates

$$\phi = U_0 \cos \theta \left( r + \frac{R^3}{2r^2} \right) \quad (11.17)$$

The radial velocity is then

$$U_r = U_0 \cos \theta \left( 1 - \frac{R^3}{r^3} \right) \quad (11.18)$$

while the tangential velocity ( $\theta$ ) is

$$U_\theta = -U_0 \sin \theta \left( 1 + \frac{R^3}{2r^3} \right) \quad (11.19)$$

Substituting Eq. (11.18) and Eq. (11.19) into the definition Eq. (11.16) yield

$$\mathbb{M} = \frac{2\pi}{3} R^3 \quad (11.20)$$

which is half of the volume of the sphere. Thus, the added mass is a strong function of the geometry and not affected by the body's velocity (it was eliminated). Yet, one can see some logic (intuition) in this value. The body continuously penetrates two dimensional surfaces. The cylinder cut the surface with infinite length in one direction while the sphere cut in only in a finite length. While this explanation is not scientific, it does provide some hints to the expecting value.

Suppose that body **A** and body **B** are connected by a thin rod the added masses of the bodies can be summed up (because the linearity of the problem). This idea is not totally accepted by all researchers. Yet, there is interest in bodies that are long slender which lead to slender body theory of integration of infinity thin slices (think long submarine for example, even ships). These coefficients (properties) are depend on the orientation of the body. Hence, they are function of all the angles (all the three angles). The observed current research work this days (as 2020) still assumes that the added masses are constant (for the moving coordinate). Hence, all these research works should be considered a progress point at best at this stage.

The term added mass has been misused by many and in following examples of such cases will be discussed.

#### Example 11.4: Poking Styrofoam

Level: Advance

The following video <https://www.youtube.com/watch?v=g5ihS9QFAP0> shows a person explaining or demonstrating the added mass concept. He was attempting to poke hole in a small square floating on the water. The small square was pushed into and no hole in square could be made. On the other hand, when experiment is done with the bigger square and the researcher is successful to poke a hole. The researcher claims that the reason for the different result is the added mass. Is the researcher is right? Explain.

#### Solution

The forces that act on the styrofoam are buoyancy, gravity in the opposite direction, and the force that apply to poke the hole,  $F_{poke}$ . Taking downward as a positive

direction provides

End of Ex. 11.4

$$\sum F = F_{poke} + \overbrace{m_{styrofoam} g}^{\text{gravity force}} - \overbrace{A \Delta x \rho_{\ell}}^{\text{buoyancy force}} \quad (11.4.a)$$

The right trick is to recognize that at equilibrium the gravity (force) is equals to buoyancy (force). Thus, the focus has to be on the deviation from equilibrium.

$$\sum F = F_{poke} + \overbrace{-A \Delta x \rho_{\ell}}^{\text{buoyancy force}} \quad (11.4.b)$$

In Eq. (11.4.b) the buoyancy refers to the change from equilibrium and it is not same as in Eq. (11.4.a). Locking the equation Eq. (11.4.b) it is recognized that  $F_{poke}$  about same in both cases. The only thing that is different is the area of styrofoam,  $A$ . Since styrofoam depth is limited amount of  $(\Delta)x$  as long as area  $A$  is large enough the poking makes a hole. Thus, in this analysis no added mass present because there is no movement and the claim of the scientist wrong and it is not issue of added mass. The depth of the styrofoam is pushed into liquid is

$$d = \frac{F_{poke}}{A \rho_{\ell}} \quad (11.4.c)$$

Where  $F_{poke}$  is the force required to poke a hole in that specific styrofoam.

### Example 11.5: Obtaining Energy from Truck

Level: Intermediate

A truck is driven on highway. If you want to reduce your energy consumption what you will do. What kind of action you should take? Go behind the truck? get away from the truck? Move in-front of the truck? Is this operation is related to added mass phenomenon?

#### Solution

This question has a practical application if you drive on large distance. The truck accelerates the air around it and the air has a velocity and acceleration in the direction of the truck. This velocity field is stronger as closer you get to the truck. (assume safety is not an issue). The field at the back of the truck is larger and hence more accessible. From the discussion about it is related to added mass.

## 11.6 Transfer Mechanisms and Transfer Properties

### 11.6.1 History of Transfer Properties

The basic common theory for this “transfer” effect is that a wave moves along the ship change  $\mathcal{GM}$  during the time ship passes the wave. This idea as far it can be judged was suggested by France and Shin (France, Levadou, Treacle, Paulling, Michel, and Moore

2003; Shin, Belenky, Paulling, Weems, and Lin 2004). The ship, according to this theory, has to be exposed to a waves with a frequency of twice the natural ship frequency (van Laarhoven 2009) which is defined as

$$\omega_{\phi} = \sqrt{\frac{\rho \ell g V_0 GM}{I_{xx} + M_{add}}} \quad (11.21)$$

where  $\omega_{\phi}$  is the natural frequency and  $M_{add}$  is the added mass of the ship<sup>14</sup> and the other properties are regular terms used in the book This equation can be recognized as the regular term from a simple pendulum. In addition, the wave length has to be at certain conditions (while they are not specified). In the said paper, it seem from the equation has a simple units mistake and the added mass should be replaced by the added moment of inertia (logically and dimensionally). (to be added as example to the dimensional analysis chapter) It is not clear from the model what causes the rolling beside that the rolling is possible (if the natural frequency was correct). This model disregards the ship shape, shifting rotation point, and other important characteristics. There are several problems with the assumptions that this model build on and they will be discussed later.

When engineers do not understand a certain topic, a committee (unknown 2008; Krata and Wawrzyński 2016) issue recommendation with total disregard to the common believe physics and realizing that something is going on (It is nice that someone notice it). For example, IMO issue in 2008 equation for estimate the frequency (time of the period)<sup>15</sup> as

$$\tau = \frac{2 c b}{GM_0} \quad (11.22)$$

where<sup>16</sup>

$$c = 0.373 + 0.023 \frac{b}{D} - 0.00043 s \quad (11.23)$$

where:

$c$  is coefficient describing ships transverse gyration radius,

$b$  ship width,  $s$  ships length at waterline,

$D$  denotes the ships draft, and

$GM_0$  initial transverse metacentric height.

It is interesting in this equation that equation (committee) recognized that the compartmental approach is erroneous. Yet, the physical sources of the energy transfer from one dimension to another approach (or to multi-dimensional movements) are unknown. Clearly this equation is only valid only in special cases at this stage the equation to be considered unknown.<sup>17</sup>

<sup>14</sup>Should be moment of the inertia

<sup>15</sup>One can only wonder why the 2 is not swallowed into coefficient  $c$ .

<sup>16</sup>The author would recommend that these kind of equations should be in dimensional analysis section. Keep in mind when the various coefficients given their dimensions also should be provided.

<sup>17</sup>This author speculated that about the affecting parameters. Yet, it will appear in later version.



### 11.6.2 Introduction

In general the resistance to movement has three different components, the body mass, added mass, and the transfer properties. In physics the first part is studied and is the fundamental of many branches of physics. The second part is presented in graduate class or by used by researchers mostly dealing the interaction of liquid–solid. The third part was discovered by the undersign and probably will be study by researchers and later move to graduate students. This idea is novel for which only minimum reviews have been done. The added mass/added moment of inertia are results of the movement uniform medium or at least continuous. Namely when the density on body surfaces might be different values but they are continuous (for example compressible flow). What happen when the media is not continuous either because it contains two or more phases. Furthermore, when the body moves through a shock or other discontinuities (large wave) there additional resistance (actually transfer of energy). These “resistances” occur only because the sharp change of the density.

The transfer properties are different than the added mass in several aspects. The transfer components property of the body and possibly appear only when there is a sharp change in liquid density for the rotation and the body is asymmetrical for linear movements. The transfer properties can be represented by a matrix of  $6 \times 6$ <sup>18</sup>. Notice that the issue of units, not all the elements have the same units. The transfer properties located on main diagonal are zero. The transfer properties are not resistance to the body movement but rather acting as to transfer energy (hence the name transfer properties) from one mode to another<sup>19</sup>. The transfer properties are mostly related to the velocity and not the acceleration.

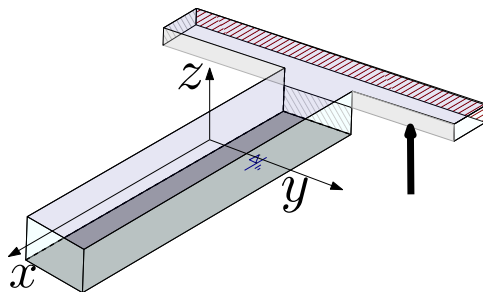


Fig. 11.6 – T shape floating to demonstrate the 3D effect The rolling creates yaw and Pitch. The transfer mechanism depends on the transfer properties. The transfer properties can be calculated for every geometrical shape and orientation<sup>20</sup>. The cause of the transfer

<sup>18</sup>There might be better representations. This undersign cowardly admits temporary defeat in this case.

<sup>19</sup>This statement is not entirely correct. It should be mentioned that part of the transfer has “efficiency” issues and not all the energy is transferred.

<sup>20</sup>The textbook was not intended to be a research paper. Yet, because lack of information, the book has sections which look like a research paper.

is the change in location of the buoyancy centroid (for the rotation motions)<sup>21</sup>. The only body shape that does not transfer from one mode to another is sphere. There is source of waves or other exciting forces/moments (beside the magnetic forces) that can cause transfer. Thus, the transfer of the motion from one mode to another is because symmetry (or asymmetry or lack of symmetry) of the body. Here, the transfer mechanisms are divided into two categories: one, from rotation in one direction to rotation in another direction, and two from rotation to linear motion (vis versa). The transfer from a linear motion to a linear motion is considered as the second category<sup>22</sup>.

To understand this transfer consider the T-shape body depicted in Fig. 11.6. The extend part of the body is just touching the liquid. In this case, the regular marine coordinate system is adapted for this discussion (if cannot avoided). If the body is rotating around the  $x$  coordinate (roll or around another parallel line), the buoyancy centroid changes the location when the body rotated. The change in buoyancy centroid move has a component in the  $x$  direction. Yet, in this case, as opposed to the extruded body, the buoyancy centroid moves to a new location with in a different plane of the original  $z$ - $y$  plane. In other words, the buoyancy centroid,  $\mathbf{B}$  has two components one in  $y$  direction (as before) and new one in the  $x$ . Assuming that the body was at equilibrium, these changes in  $\mathbf{B}$  centroid location creates a moment in the  $y$  axis (pitch) and also around  $z$  (yaw). This T-shape effects are created by added submerged volume, however similar effects can be created by removing part of the submerged body. The meaning of removing body is that a body when rotated exhibits a depression. Generally, if the body is not symmetrical round  $y$  any rotation around the  $x$  coordinate creates rotation  $y$  coordinate and  $z$  coordinate. The gravity centroid remains the same location on the body. Thus, there is a new moment that acting the new direction. The described effect is of roll causing pitch (and yaw). Yet, the same argument can describe pitch movement causing roll. In this discussion, the source of the rotation is irrelevant. Another source the rotation transfer is asymmetry in the movement or the orientation but this topic is to advance and is not discussed in this version.

The previous discussion dealt with the transfer from one rotation to another rotation due to the floating body asymmetry (in previous case the asymmetry was around

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<sup>21</sup>The movements inside the body caused by roll (for example) can also cause transfer energy like moving bodies or transfer of liquids.

<sup>22</sup>In discussion with one of the reader has pointed out that in the future rotation to linear will be considered the third category. Time will tell.

the  $yz$  plane). It can be expanded to movement of linear like heave to rotation. The main difference in this category is that the moment transferred into force. It must be mentioned that the energy during the transfer is constant (if no losses occurs) and the transfer reduces the oscillations or the movement because the energy is transferred. Additionally, the transfer does not occur linearly and it depend on the change of the buoyancy centroid. The forces in the  $x$  coordinate are equal and opposite to each other and hence cancel each other.

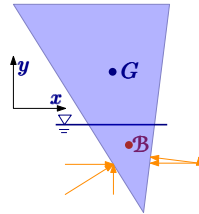


Fig. 11.7 – Extruded triangle to explain the movement transfer from Heave to Rolling.

However, the forces in the  $y$  direction are not equal and cause a moment which will result in the rolling rotation. That is, in this case the asymmetry in around  $y$  coordinate transfers energy from the heave to roll. If the floating body is not perfect symmetry it also might create a movement in other directions. Note that if the body was made from straight lines going down, it will not have any effect on the other linear direction motion.

The conceptual explanation has to get engineering expression. First, the transfer from roll to pitch requires to knowledge the change in  $B$  due to the roll. The calculation of the change in the  $x$  direction is done in a similar fashion to the change buoyancy centroid was ascertained for symmetrical body. These calculations are 3-D nature at least for the presentation. As this book (at least this version) is pioneer this aspect and the technique presented here to calculate is crude. Not to be bogged with the mathematics the functionally assumed to be known. Consider the body shown in Fig. 11.6. First, the change in the  $z - y$  plane of buoyancy is considered. The body is divided into many small slices of with thickness of  $dx$ . For each slice the change in buoyancy and it is denoted as  $dB$ . This value is a function of  $x$  and same time the area can be calculated it denoted as  $A(x)$  (in mathematical term it referred as the weight function). The change in the  $y$  direction is

$$\Delta B(\theta)_x = \frac{1}{V_0} \int_{V_0} \Delta B(x)A(x)dx \tag{11.24}$$

The change in the  $y$  direction is

$$\Delta B(\theta)_y = \frac{1}{V_0} \int_{V_0} y A(x)_{at\ new\ \theta} dx \tag{11.25}$$

It can be noticed that change in the  $x$  is not relevant to the  $y$  direction.

**Example 11.6: Heave motion**

**Level: Intermediate**

Calculate the change in the  $y$  direction of the body provided in the figure.

**Solution**

### 11.6.3 Transfer Linear Motion to Rotating Motion

The transfer mechanism from a linear motion to a rotational motion can be demonstrated by examining two motions. In this discussion the most common motions are presented which are heave (linear) and roll (rotational). For simplicity it is assumed that liquid level flat. The body translates to a lower point in the liquid as shown in Fig. 11.8. The rotation point changes when the body moves up or down. The old rotational point due to movement does not exist any more. Previously, the body experiences zero net force and while currently there is a net force. For the roll rotation, the rotation point is extremely important because it dictates the moment of inertia of the body (ship). Yet, as a first approximation, the rotation is assumed to be at the same center line as before but at the new liquid surface in the middle of the surface (moving from  $\mathcal{A} \rightarrow \mathcal{A}'$  which is exact description for extruded bodies, but more complicated for other bodies.). In this case, the governing equation of movement must include the variation of the moment of inertia due to height change.

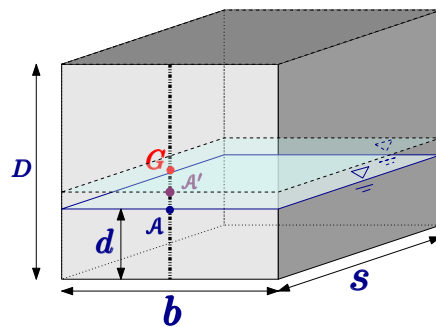


Fig. 11.8 – Coupling between the Heave and roll. Notice figure exhibit the old and new liquid levels.

The rotation location change with the  $y$  coordinate is discussed below. First effect of the change is the change in the body moment of inertia. Second effect is more important which is the stability changes and the body could cross the stability line on the stability dome. For the case of  $\rho_s/\rho_\ell < 0.5$ , when the body moves down it could enter inside the unstable zone. The reason for it that body effectively increases the ratio of the density. On the opposite case when  $\rho_s/\rho_\ell > 0.5$  when the body moves up it enter the inside the unstable zone. That is the body effectively decrease the density ratio. Hence, this mechanism in which the linear motion is transfer to rotation (rolling). The transfer mechanism is clear but at this point the transfer property is not well defined. It can be noticed that no acceleration is required for this energy transfer.

The rotation location change with the  $y$  coordinate is discussed below. First effect of the change is the change in the body moment of inertia. Second effect is more important which is the stability changes and the body could cross the stability line on the stability dome. For the case of  $\rho_s/\rho_\ell < 0.5$ , when the body moves down it could enter inside the unstable zone. The reason for it that body effectively increases the ratio of the density. On the opposite case when  $\rho_s/\rho_\ell > 0.5$  when the body moves up it enter the inside the unstable zone. That is the body effectively decrease the density ratio. Hence, this mechanism in which the linear motion is transfer to rotation (rolling). The transfer mechanism is clear but at this point the transfer property is not well defined. It can be noticed that no acceleration is required for this energy transfer.

The previous discussion dealt with the effect of the transfer from a linear motion to a rotation motion. The

When the body is turning the resistance (damping forces) is changing according the angle. In addition the added mass of the floating body changes. One of the reason for this change asymmetrical resistance on both side of the body. For example, consider the T shape body that discussed before when it makes rolling rotation. Beside the effect of the rolling on the pitch and the yaw that creates also heave. The heave is created because one side is liquid (water) and has larger resistance as compared to the other side. The results of this resistance to lift the ship. On the other part of cycle it has the

opposite effect.

#### 11.6.4 The Parallel Axes Theorem for Added Mass

### Meta

The change of the rotation origin affects only the added moment of inertia. The added moment of the floating body sometime give or calculated for specific location. However, the rotation can move from one location to another. One of the problems facing the individuals who calculating the floating body movement is the change of added moment of inertia from one axis to another. The floating body rotation axis can be significant. The reason to the change of the location can be different load of the ship for which the rotation point is at the liquid plane or other reasons.

In regular body, it was shown in section 5.3 on page 53 that it can used by a simple equation to transfer the property. The regular transfer of properties cannot be used here because the fluid does not have the rigid connection that were required from the solid body. The question, before lurching into calculating the moment of from scratch, can the old calculations be used.

The energy created by the body around the new axis is

$$E = \frac{\rho \ell}{2} \iiint_V U^2 dV \quad (11.26)$$

The velocity due to the rotation is  $\omega \sqrt{x'^2 + y'^2}$ . This velocity is expressed in the new coordinate system as  $\omega \sqrt{(x + \Delta x)^2 + (y + \Delta y)^2}$

The new axis is parallel to the old axis. It was shown that the integration of the velocity at the moving body surface.

### Meta End

#### 11.6.5 Experimental Observation

People have to find value of the added properties from experiential work. For example, (Aso, Kan, Doki, and Mori 1991; Brennen 1982; Molin, Remy, and Rippol 2007) have attempted to separate the various components like the fluid drags and Reynolds number effects. Notice that added mass should not be affected by the Reynolds number. There uniform method to carry these experiments and evaluated the values. This especially complicated for mid range of Reynolds numbers where actually the experiments were carried.<sup>23</sup>

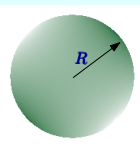
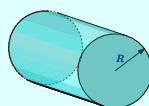
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<sup>23</sup>What a mess! These are the reasons we love fluid mechanics.

### 11.7 Added Mass and Transfer Properties

Selected Added mass properties. At this state there is no transfer properties worded out.

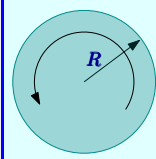
Table 11.1 – Moment of Inertia and Other Data

Shape Name	Geometry	Added Mass	Transfer	remarks
Sphere		$\frac{2}{3}\pi R^3$	0	Symmetrical
Cylinder		$\pi R^2$	0	Symmetrical

### 11.8 Added Moment of Inertia

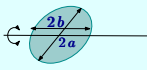
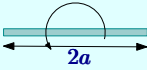
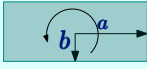
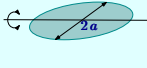
Selected Added moment of inertia or presented.

Table 11.2 – Moment of Inertia and Other Data

Shape Name	Geometry	$I$	Transfer	remarks
Circle		0	0	Symmetrical

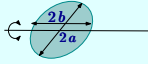
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Table 11.2 – Moment of inertia (continue)

Shape Name	Geometry	$I$	Transfer	remarks	
Ellipse		$\frac{\pi}{8} (a^2 - b^2)^2$	0	Symmetrical	
Flat Plate		$\frac{\pi}{8} a^4$	0	Symmetrical	
Rectangle		$\pi a^4 k$		0	Symmetrical
		b/a	k		
		0.0	0.125		
		0.1	0.147		
		0.2	0.150		
		0.5	0.150		
1.0	0.234				
Circle On Z		$\frac{16}{45} a^5$	0	Symmetrical	

Continued on next page

Table 11.2 – Moment of inertia (continue)

Shape Name	Geometry	$I$	Transfer	remarks	
Rectangle		$a^3 b^2 k$	0	Symmetrical	
		b/a			k
		0.1			0.8833
		0.2			0.7398
		0.3			0.6713
		0.4			0.6067
		0.5			0.5489
1.0	0.3556				

At first glance, this should be totally different however other claim that there at least weak connection. The solution leads to the infinite series which be computed. The results are shown the following figure.

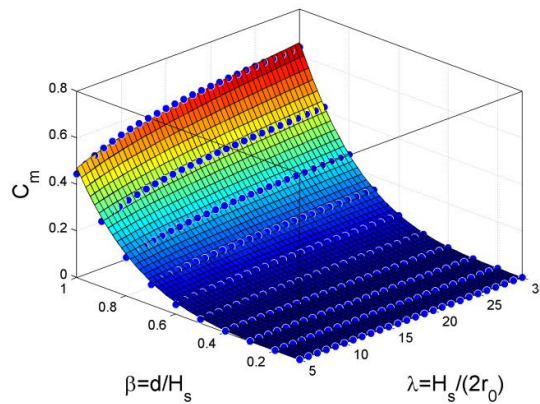


Fig. 11.9 – Added mass for vertical cylinder moving at liquid. The parameters shown in the figure  $\lambda$  is ratio of the diameter cylinder and of the total length, and  $\beta$  is ratio of the total length to wet length similar to density ratio.



Appendices

11.A Introduction

This appendix can be skipped by most people and it is mostly for crazy people like this undersign who do not have anything to do in his life. This section deals with a proof for boundary conditions and two with the parallel transport of inertia. This information was published before somewhere, however, this author failed to find it.

The potential function fluid is a function that when a gradient is applied to it exhibits the velocity such

$$v = \nabla\phi \tag{11.A.1}$$

Which also imply the Laplace equation ( $\nabla^2\phi = 0$ ) is valid, when the flow is irrotational flow. If a potential function is conservative (curl is zero) then relationship between the integral in area and border for single function can be written as

$$\int_C Pdx + Qdy = \iint_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \tag{11.A.2}$$

where  $C$  is the curve and round the surface and rotating the counter-clock. This can be extended to two functions. First utilizing differentiation Product Rule as

$$\iint_A \left( \frac{\partial\phi_1}{\partial x} \frac{\partial\phi_2}{\partial x} \right) dA = \iint_A \left( \phi_1 \frac{\partial\phi_1}{\partial x} - \phi_1 \frac{\partial^2\phi_2}{\partial x^2} \right) dA \tag{11.A.3}$$

first integral on the right hand side can be converted by utilizing Green's theory to be

$$\iint_A \left( \frac{\partial\phi_1}{\partial x} \frac{\partial\phi_2}{\partial x} \right) dA = \int_C \left( \phi_1 \frac{\partial\phi_1}{\partial n} \right) dA - \iint_A \left( \phi_1 \frac{\partial^2\phi_2}{\partial x^2} \right) dA \tag{11.A.4}$$

if  $\phi_1 = \phi_2 = \phi$  than Eq. (11.A.4) can be written as

$$\iint_A \left( \frac{\partial\phi}{\partial x} \right)^2 dA = \int_C \left( \phi \frac{\partial\phi}{\partial n} \right) dA - \iint_A \left( \phi \frac{\partial^2\phi}{\partial x^2} \right) dA \tag{11.A.5}$$

Three dimensional version (and notice  $C = S_{in} + S_{out}$ ) of Eq. (11.A.5) is

$$\iiint_V \left( \frac{\partial\phi}{\partial x} \right)^2 dV = \iint_{S_{in}+S_{out}} \left( \phi \frac{\partial\phi}{\partial n} \right) dA - \iiint_V \left( \phi \frac{\partial^2\phi}{\partial x^2} \right) dV \tag{11.A.6}$$

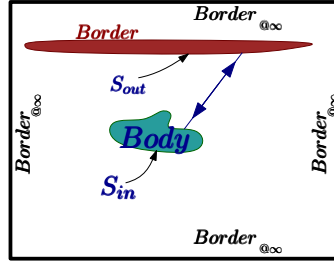


Fig. 11.A.1 – Integration paths of Added Mass.

$S_{in}$  denotes to the boundary on the body and  $S_{out}$  denotes to the boundary on other bodies or boundary of the container that contain the body. To get the total kinetic energy the other two components should be included and become

$$\iiint_V \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 \right] dV = \iint_{S_{in}+S_{out}} \left( \phi \frac{\partial \phi}{\partial n} \right) dA - \iiint_V \phi \overbrace{\left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right)}^{\sim 0} dV \quad (11.A.7)$$

The left hand side represent the kinetic energy in the domain bounded by the body and infinity. The right hand side represents the component that contribute to this integral. The first term on the right represents the perpendicular velocity at the bodies that is in the domain boundary. The second term vanishes because the Laplasian is zero inside the domain (continuity equation). What left on the right split into two as

$$\iint_{S_{in}+S_{out}} \left( \phi \frac{\partial \phi}{\partial n} \right) dA = \overbrace{\iint_{S_{in}} \left( \phi \frac{\partial \phi}{\partial n} \right) dA}^{\text{on the body}} + \iint_{S_{out} \text{ boundary}} \left( \phi \frac{\partial \phi}{\partial n} \right) dA + \iint_{S_{out \infty}} \overbrace{\left( \phi \frac{\partial \phi}{\partial n} \right) dA}^{\sim 0} \quad (11.A.8)$$

This integral suggests that the closer the boundary is to the body the larger is the added mass. Furthermore, it suggests that once the boundary contributions is determined then it can be used for other bodies. Yet, this conclusion is not widely accepted and controversial.

Anther point that it is controversial but should be mentioned is the combined potential

$$\phi = U_{0x} \phi_1 + U_{0y} \phi_2 + U_{0z} \phi_3 + \omega_{0x} \phi_4 + \omega_{0y} \phi_5 + \omega_{0z} \phi_6 \quad (11.A.9)$$

There are some limitations that statement like the origin of the rotation have to same etc. Nevertheless,if that is accepted then it can be substitute into Eq. (11.A.8) so that it provide 36 terms which looks like

$$m_{ij} = \iint_S \phi_i \frac{\partial \phi_j}{\partial n} dA \quad (11.A.10)$$

As the total energy change based on added properties is then

$$E = \frac{\rho}{2} \sum_{i=1}^6 \sum_{j=1}^6 m_{ij} U_i U_j \quad (11.A.11)$$

Comparing the energy of the terms opposite of the diagonal the  $m_{ij} m_{ji}$  is done by subtracting one from each other to see if they are the same value. Notice that energy was evaluated according to Eq. (11.A.7) as a surface integral. Thus,

$$\iint_S \phi_i \frac{\partial \phi_j}{\partial n} dA - \iint_S \phi_j \frac{\partial \phi_i}{\partial n} dA = \iint_S \left( \phi_i \frac{\partial \phi_j}{\partial n} - \phi_j \frac{\partial \phi_i}{\partial n} \right) dA \quad (11.A.12)$$

The right hand side can be split into  $S_{in}$  and  $S_{out}$  as

$$\iint_S \left( \phi_i \frac{\partial \phi_j}{\partial n} - \phi_j \frac{\partial \phi_i}{\partial n} \right) dA = \iint_{S_{in}} \left( \phi_i \frac{\partial \phi_j}{\partial n} - \phi_j \frac{\partial \phi_i}{\partial n} \right) dA + \iint_{S_{out}} \left( \phi_i \frac{\partial \phi_j}{\partial n} - \phi_j \frac{\partial \phi_i}{\partial n} \right) dA \quad (11.A.13)$$

~ 0

The tricky part is to recognize that according to the identity Eq. (11.A.4) which is true if only when the boundary around body is at infinity. That is,

$$\iint_{S_{in}} \left( \phi_i \frac{\partial \phi_j}{\partial n} - \phi_j \frac{\partial \phi_i}{\partial n} \right) dA = \iiint_V \left( \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} - \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \right) dV \quad (11.A.14)$$

In a marine situation the liquid domain is made of liquid and gas (air), hence the sharp change in density does not seem to create a problem. That is, the boundary conditions required for Eq. (11.A.4) is required to be at infinity. The potential functions that used are not continuous and there is a problem. It seems that requirement that the boundary should be at infinity not satisfied therefore  $m_{ij} \neq m_{ji}$ . The conditions for  $m_{ij} = m_{ji}$  are such boundary has to be at infinity. Hence the symmetry of the added mass seems to be lacking.

**Example 11.7: Reverse Flow**

**Level: Advance**

A body that is rotating around its gravity centroid has two possibilities of rotation. First rotation is counter-clockwise and second rotation is clockwise. In most situations this change does not matter because of symmetry. For example, when an extruded square is rotating around its center it does not matter the direction of the rotation. The question is about a body that clearly imitatively seen that the direction should matter shown in Fig. 11.A.2. Does it matter the direction in the calculation of the added moment of inertia?

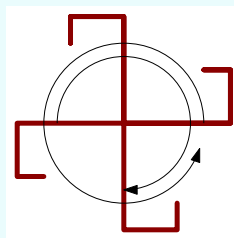


Fig. 11.A.2 – Direction of the rotation of arbitrary direction.

**Solution**

The answer to the question lay in the recognition what actually cause the added mass or the moment of inertia. The derivation about sum it in the concept that as long all the velocity components and rotation have the same origin as in this case the direction of the rotation is insignificant to the direction of the rotation. In any direction of the rotation the velocities on the surface will be same.

(?) claims no effect of the frequency as expected in theory.

# 12

## Rolling

### *12.1 Introduction*

While ships are built to be stable (for the most part), there are several situations where this requirement is not sufficient. Ships act as a giant pendulum but with several distinct characteristics (varying hinge point, varying moment of inertia, changing the damping elements, changing added mass, changing transfer properties etc ) which will be discussed in this chapter. Under certain conditions, these giant pendulums create a problem for the cargo on board. This issue leads to accidents in which the cargo (containers) is damaged. This problem could be due to the resonance of the ship with cargo (there could be other reasons). This chapter focus on this issue. There numerous investigations with estimate cost of over 10 millions dollars while cost of the actual damages was high as a billion dollars. Thus, this issue deserves a chapter examining the various factors of this problem and mechanisms in which they created. In this chapter, a discussion is presented on the general parameters and a dimensional analysis of the problem.

While the title of the chapter refers only to rolling rotation other aspects are also examined and discussed in this chapter.

#### **12.1.1 The Quasi-State Foundation**

It is essential to realize that quasi-state is necessity in order to be able to analysis this complicate situation. To make this point, an analogy to thermodynamics is provided. To understand thermodynamics, the material is categorized into equilibrium and none-equilibrium zones. That is, initially classes use and present the steam tables and treat the steam tables as Holy Grail for example, water boils at about 100°C at atmospheric

pressure. This ability to carry calculations, based on this rough estimate, simplifies the ability to obtain reasonable results. Yet, when precise values are required the effects of the heat flow rate, surface tension, and boiling surface must be taken into account<sup>1</sup>. The real reason for this separation is to create a helpful system so engineers can simplify the calculations. The same should be done for stability, and the quasi-equilibrium study should be done before the full dynamics is implemented.

The point where the rotation hinge appears is important to establish in any pendulum analysis. Thus, the movement in  $z$  direction, also known as heave, changes the location of the rotation point. A change in  $x$  and  $y$  also causes a change in the rotation point but to a smaller extent. In the analysis, these changes should be ignored and quasi-state should be assumed. Thus, in this chapter when the rotation point is needed for calculation, this assumption is utilized.

Obviously, the location of the rotation hinge of the pendulum is important for the ship movement. The value of the moment of inertia depends on this location. Yet, numerous works dealt with the dynamics situations (where the ship is not in equilibrium in  $z$  direction), yet, the rotation hinge is unknown (or worse the wrong rotation point is used like the metacenter or gravity as recommended in (Fossen 2011)). In dynamics situations, the hinge point deviates slightly from the quasi-equilibrium. Hence, the knowledge of this hinge point at quasi-equilibrium is essential and should be known before tackling the dynamics aspects. In this chapter, deals with quasi-equilibrium state and attempt to establish it. Furthermore, the deviations from the normal (perfectly oriented) such as trim and list are not dealt at this stage.<sup>2</sup>

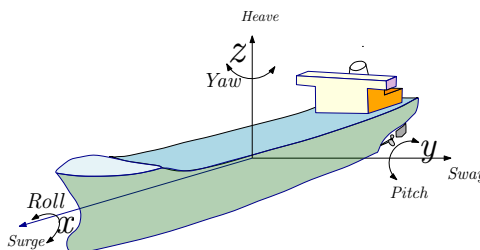


Fig. 12.1 – Six motions of ship with their names (Floating bodies).

What was established here is that there is **no** problem with the linear motions: heave, sway, surge. For the sway and surge (check Fig. 12.1), no ship will be capsizing because these motions (neglecting the transfer of energy mechanisms)<sup>3</sup>. In the extreme, the

<sup>1</sup>The purity of the water is deliberately does not appear in the list because they common and yet less important. For example, the change in salinity for common material like sea water can be a few degree while the smooth surface change the boiling temperature by about 30°C or more.

<sup>2</sup>These jargon terms should be translated. One can check the chapter on definitions in this book. Basically, List refers to a change from a symmetry like ship leans to a side instead being upright. Trim means the change depth at bow and stern locations.

<sup>3</sup>This remark should be limited by the suggestion offered by several individuals that tsunami can turn even large ship. Yet, this remark is true for "normal" waves. The heave motion can created situations where at the end of the motion body become temporary unstable and in some instance force roll and pitch rotations. From stability point of view, the roll will have to be first to occur.

heave (in theory) can cause drowning of the floating body if the magnitude of movement is such it exposed the opening to the liquid. However, if there is enough extra floating, most waves amplitude are not sufficient to capsizing which are not common (this author is not aware of a single case of such issue was for a commercial ship)<sup>4</sup>.

**Example 12.1: Heave motion**
**Level: Intermediate**

A large container ship (the Ever Giving) with the size of 60 [m] width and 400 [m] long is pushed 1[m] into the sea. Estimate the force that requires to create this movement. How much water the ship has to be covered to create this force. The ship height is 60 [m] and ship is about 50% submerged by the water. How large force is required to be to push this ship 25% down? What is the estimated acceleration of ship at this point?

**Solution**

The weight of the 1 cubic meter of water is about 1 tonne which for simplicity, this value is taken as standard for this example. The force is then simply evaluated as  $60 \times 400 = 24,000$  tonnes. To push the ship 1 [m] the ship has to be covered with 1[m] of water. Large ship do not covered with a meters of water and thus heave movement is limited. To have 25% movement it has to be covered by the same amount of water. In case of Ever Giving the entire ship has to be covered by 15 [m] of water (it is very fast wave of 45 [m] and very long 400 [m] long). The force is  $15 \times 60 \times 400$  tonnes. The question of the acceleration is important because it determines the time the body will be under the liquid which determines possibility of drowning. It can be shown that this situation (to be constructed example in chapter 13) it analogous to a block with density of half of the liquid at depth of 15[m]. The mass of the block is  $V \rho_s$  and the net force is  $V (\rho_\ell - \rho_s)$ . The acceleration is the ratio of these two quantities

$$a = \frac{V (\rho_\ell - \rho_s) g}{V \rho_s} = \left( \frac{\rho_\ell}{\rho_s} - 1 \right) g \quad (12.1.a)$$

which in this case is about  $g$ . Thus, to cause drowning the wave speed has to be in order magnitude or larger than  $\sqrt{2L/g}$ . Where  $L$  in this case is the ship length.

It was demonstrated that the heave is not dangerous up to the point where the ship openings is exposed to the liquid. The heave movement is mainly concern of the increase of the resistance to the flow (ship movement)<sup>5</sup>. The yaw rotation (again without the transfer mechanism) is not dangerous because the obvious observation (see Fig. 12.1). The pitch rotation can be caused by either by the transfer mechanisms or waves from the  $x$  or  $y$  (non symmetrical wave) directions. Of the three rotation motions, the rolling is the most important and dangerous to the ship. It not to say that there are no situations where a small boat along the pitch rotation can overturned (the author

<sup>4</sup>The author was supposed to be served on Mazada owned by Zim Israel Navigation Co. of Haifa on her last vogue 1981. The ship sank because admission of to much water due to leaking storage compartments. But, the ship did not capsized because of heave movement.

<sup>5</sup>The added mass acting as effective increase the ship size and hence increase the ship volume. This volume increase increase the resistance of the ship.

personally observed such case of capsizing). These situations simply are less common and a specialty case study. After examining numerous papers on the effects of rolling this author had a strong urge to utilize the standards fluid mechanics tools to write a governing equation(s). As usual, the examination previous work will be followed by a discussion on various terms of the governing equation with circumstances under which the equation can be simplified. Due to the mathematical difficulties, the previous work focus was on the mathematical aspects of the problems. That is, little care was given to the physical meaning of the terms and mainly focus was on solving the mathematics. For example, (Taylan 2004; Taylan 2000) considered the speed of ship and damping force as the mathematical functionality.

The rolling rotation has unique features that makes it very difficult to understand as compared to regular pendulum rotation motion. The fact is that until this point, no one even formulate the complete governing equation<sup>6</sup> and discuss what physical elements and how they interact. The "regular" rotation has forces/moments balanced around a fix point which may or may not have external inputs (sometimes referred as exciting input). Floating body among other things has several elements that are unique such shifting rotation point and shifting (added) mass and/moment of inertia. For example, a ship body that changes from movement in  $5^\circ$  relative ship  $x$  (the length of the ship) to straight movement can change about 50% of the added mass of the body. The second issue is transformation mechanism from one rotation to another which was addressed in section 11.6 on page 151. This author has not found a single source to which discuss the transfer mechanism from one rotation mode to another. As it was in chapter 11 the added mass is really a single component, that is zero when there is no acceleration (strange that is point is not accepted by many in fact when they write the governing equations.). Additionally, the transfer properties are truly a second order tensor<sup>7</sup>. One can wonder how one can write equation(s) when so much is ignored/unknown. While the solution of the commonly believe theory were conceptually wrong, their solution might or not be correct. As writing these statements, it has not been investigated<sup>8</sup>.

With so much unknown about ship movement and structure, this chapter starts to build from a simple case (rolling) and will be generalized later. The points that need to mention are: just recently it was discovered that the added mass is a single component. The transfer properties are actually are a new concept that was developed

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<sup>6</sup>Several individuals wrote to dispute this point. They pointed to a book edited by Lewis (1988). The book is example of the point that is made here. It contains the equations that assume rotation at arbitrary point and constant added mass or added moment of inertia. This is not include that the book got the added mass based on the common believe which violate the Newton's second law. Yet, when they were asked about how they calculate the ship moment of inertia and around what point the ship is rotated around, why were clueless. This point is correct regardless to individuals feelings. Yet, it demonstrate how entrenched fix rotation point concept etc, individuals even not aware how wrong they are!

<sup>7</sup>This author initially somehow thought that they are the same. It is strange to so many errors in one field.

<sup>8</sup>This remarks is because many people asked about it. This author has no intention to get into wave iteration with solid structure. This area is important but this author find other topics more suitable for him. Please do not press on this issue.



only in the last year. Additionally, the ship rotation hinge movement was found only less than a year ago. With all these new elements, attempts will be write the governing equation for a pure rolling movement. First, a description what has been done before is a must.

The basic common believe ideology for this “transfer” effect is that a wave moves along the ship change  $GM$  during the time ship passes the wave. This idea as far it can be judged was suggested by France and Shin (France, Levadou, Treakle, Paulling, Michel, and Moore 2003; Shin, Belenky, Paulling, Weems, and Lin 2004).The ship, according to this theory, has to be exposed to a waves with a frequency of twice the natural ship frequency (van Laarhoven 2009) which is defined as

$$\omega_{\phi} = \sqrt{\frac{\rho_{\ell} g V_0 GM}{I_{xx} + m_{add}}} \quad (12.1)$$

where  $\omega_{\phi}$  is the natural frequency and  $m_{add}$  is the added mass of the ship<sup>9</sup> and the other properties are regular terms used in this book. This equation can be recognized as the regular term from a simple pendulum. In addition, the wave length has to be at certain conditions (while they are not specified). In the said paper, it seem from the equation has a simple units mistake and the added mass should be replaced by the added moment of inertia (logically and dimensionally). (to be added as example to the dimensional analysis chapter) It is not clear from the model what causes the rolling beside that the rolling is possible (if the natural frequency was correct). This model disregards the ship shape, shifting rotation point, added mass variations, and other important characteristics. There are several problems with the assumptions that this model build on and they will be discussed later.

When engineers do not understand a certain topic, a committee (unknown 2008; Krata and Wawrzyński 2016) issue recommendation with total disregard to the physics<sup>10</sup>. For example, IMO issue in 2008 equation for estimate the frequency (time of the period)<sup>11</sup> as

$$\tau = \frac{2cb}{GM_0} \quad (12.2)$$

where<sup>12</sup>

$$c = 0.373 + 0.023 \frac{b}{D} - 0.00043 s \quad (12.3)$$

Where:

$c$  coefficient describing ships transverse gyration radius,  
 $b$  ship width,

<sup>9</sup>Should be moment of the inertia common error or erroneously naming.

<sup>10</sup>It is interesting to point that Krata, Przemysław get information from the experiments observations.

<sup>11</sup>One can only wonder why the 2 is not swallowed into coefficient  $c$ .

<sup>12</sup>The author would recommend that these kind of equations should be in a dimensional analysis section. Keep in mind, when the various coefficients given their dimensions also should be provided.

$s$	ships length at waterline,
$D$	the ships draft, and
$\mathbf{GM}_0$	initial transverse metacentric height.

It is interesting in this equation that the equation (committee) recognized that the compartmental approach is erroneous. Yet, the physical sources of the energy transfer from one dimension to another approach (or to multi-dimensional movements) are magical. Clearly this equation is valid only in special cases at this stage to be considered unknown.<sup>13</sup>

## 12.2 Historical Background

One of the latest work (Remola 2018) formulated the governing equation as

$$I_{xx} \frac{d^2\theta}{dt^2} + M_D \left( \theta, \frac{d\theta}{dt} \right) \frac{d\theta}{dt} + M_R \theta = M_W(t) \quad (12.1)$$

where  $M_D$  is the damping function which is depended on the angle and the angular velocity,  $M_R \cong \mathbf{GZ} \rho_\ell g V_0$  (where the arm is  $\mathbf{GZ}$  and  $\rho_\ell g V_0$  is weight). The (mass) moment of inertia,  $I_{xx}$ , is in the  $yz$  plane and is assumed to be calculated for the rotation around  $\mathcal{M}$  point (metacenter and the added property is assumed to be fixed and include the body moment of inertia). No research reported at what the angle for the added moment of inertia moment of inertia was calculated. In fact, there are researches papers pointing that the added moment of inertia of **circular** shape (or very close to it) has some value while theoretically it must be zero (maybe it was rotated around another point. Anyone who knows how this happened?). Various sources are assuming that the causing the rolling is the waves from different directions. In construction, this equation it was assumed to be without coupling to other motions in the other directions based on (Spyrou and Thompson 2000). Some other accepting this assumption like (Bulian, Francescutto, and Fucile 2009) which are based on “because a good balance is achieved between simplicity and accuracy.” One can wonder if indeed any estimate on the errors was made<sup>14</sup>.

The actual equation that most researchers are trying to solve is without the damping effect. An example of such work on the ship rolling is by (Jovanoski and Robinson 2009) pointing to quite well know phenomena to stability zones that dampingless equation created. Under the assumption of specific wave it is assumed (Allievi and Soudack 1990) that the governing can be written as

$$\frac{d^2\theta}{dt} + \frac{\overbrace{M_R}^{\omega_0^2}}{I_{xx}} \theta = W_A \cos(\omega_W t) \quad (12.2)$$

<sup>13</sup>This author speculated that about the affecting parameters. Yet, it will appear in later version.

<sup>14</sup>One would love to see this estimate.

where  $\omega_0$  natural frequency  $\omega_W$  disturbing wave frequency  $W_A$  is the normalized exciting amplitude.

**Example 12.2: The Natural Frequency**

**Level: Intermediate**

This question is inspired by the book (Rawson and Tupper 2001) it depicted here to demonstrate how much the field in a need get advanced with time. In this demonstration the effect changing  $\mathcal{B}$  (or  $\mathcal{GM}$ ) is not shown since it is only demonstration.

“A box with a length of 50[m], width 12[m], draught 3[m] above the water (freeboard) 6[m]. The height of the gravity centroid above the bottom is 4.5[m]. Assuming that the weight is uniformly distributed throughout the length and section of the vessel, and neglecting the effects of associated water, calculate the free periods of roll, pitch and heave in salt water.”

Converting removing some of jargon from the question. It also had to added the hidden assumptions like neglect the added moment of inertia, resistance liquid (damping force) etc.

**Solution**

The gravity centroid is given in the data and it is excessive information. The ship length should be used for pitch frequency. The buoyancy centroid is given by  $3[m]/2 = 1.5[m]$ . The value of  $\mathcal{GB} = 4.5 - 1.5 = 3[m]$ . The value of  $\mathcal{BM}$  can be calculated as  $I_{xx}/V_0 = (I_{xx}/s)/(A_0)$ . The moment of inertia is

$$\frac{I_{xx}}{s} = \frac{b^3}{12} = \frac{12^3}{12} = 144[m^3] \tag{12.2.a}$$

The area is

$$A_0 = b d = 12 \times 3 = 36[m^2] \tag{12.2.b}$$

Thus  $\mathcal{GM}$  is  $144/36 - 3 = 1[m]$ . Notice that using the table will provide the same result. The natural frequency is

$$\omega = \sqrt{\frac{M_R}{I_{xx}}} \tag{12.2.c}$$

It must be noticed that even though the notation of  $I_{xx}$  for both cases the moment of inertia is referring to the mass (not m but l) rotating around line for box is rotating. Moment of inertia box (cuboid) around its center line (that is a line parallel to  $x$  the goes center of the box is

$$I_{xx} = \frac{1}{12} \overbrace{\rho_l d b s}^{box\ mass} g (D^2 + b^2) \tag{12.2.d}$$

The actual rotation will be around point  $\mathcal{A}$  which is at the liquid surface. In this case, the density is such the change moment of inertia is  $\mathcal{GA} = 4.5 - 3 = 1.5[m]$  and the

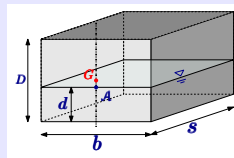


Fig. 12.1 – Cuboid For frequency analysis notice the change of moment of inertia.

continue Ex. 12.2

change value to be  $\Delta I_{xx} = 1.5^2 \text{ mass}$ . For comparison the ratio between  $I$  and  $\Delta I$  is

$$\frac{\Delta I_{xx}}{I_{xx}} = \frac{\rho_s GA^2}{\frac{\rho_s}{12} (b^2 + D^2)} = \frac{12 D^2 \left( \frac{1}{2} - \frac{\rho_s}{\rho_\ell} \right)}{D^2 \left( \left( \frac{b}{D} \right)^2 + 1 \right)} = \frac{12 \left( \frac{1}{2} - \frac{\rho_s}{\rho_\ell} \right)}{\left( \frac{b}{D} \right)^2 + 1} \quad (12.3)$$

The results depicted in Fig. 12.2. It clear from the figure that the comparison believe to assume that the density ratio is insignificant is contain large and significant error of factor 2 – 4 (hundreds percents). This error is less significant in very small range of  $\rho_s/\rho_\ell \sim 0.5$

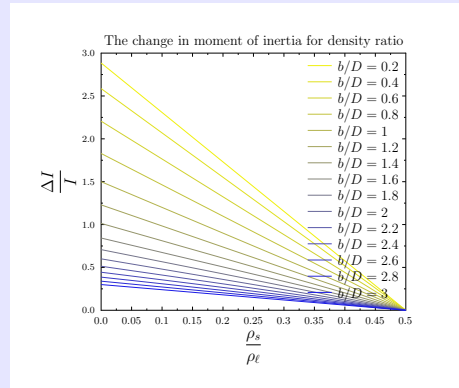


Fig. 12.2 – Effect the density ratio on the moment of inertia. Notice that presentation is in the range of 0–0.5 density ratio and the range of 0.5 to 1 should be symmetrical and thus it does not appear in the figure.

Clearly, the original author of the question was not aware of this effect. Even if the added mass and other effects were ignored the error due to the change of moment of inertia change the results. The density ratio is  $3/(3 + 6) = 0.33$  and  $b/D = 12/(3 + 6) = 1.3$ . In this case, the error from the graph is  $\Delta II \sim 0.75$ . Or more precisely

$$\frac{12 \times (0.5 - 0.33)}{1.3^2 + 1} = 0.758$$

The weight of the ship can be ascertain by  $\rho g d s b$  and it is  $1.020 \times 9.8 \times 3 \times 50 \times 12 = 17992.8 [kN]$  The moment of inertia of the cuboid is

$$I_c = \frac{1 \times 9.8 \times 3 \times 12 \times 50(9^2 + 12^2)}{12} = 330,750 [\text{tonne m}]$$

where  $I_c$  in this case is the moment of inertia around the center of the cuboid. The actual value around the rotation point is  $330,750 * 1.758 = 581,458.5 [\text{tonne m}]$  Another way to calculate  $GM$  is obtained by using Eq. (9.1.c) which is copied to here as well<sup>15</sup>

$$GM = \frac{b^2}{12d} - \left( \frac{D}{2} - \frac{d}{2} \right) = \frac{12^2}{12 \times 3} - \frac{9}{2} - \frac{3}{2} = 1 [m]$$

End of Ex. 12.2

Now the calculation of the frequency.

$$\frac{M_R}{I} = \frac{\cancel{V_0} \rho \ell g \left[ \frac{b^2}{12d} - \left( \frac{D}{2} - \frac{d}{2} \right) \right]}{\frac{1}{12} \cancel{V_0} \rho \ell (D^2 + b^2)} = \sqrt{g f(\text{geometry})} \quad (12.2.e)$$

$$\omega = \sqrt{\frac{1 \times 9.}{a}}$$

Salient point has to be made for dampingless rotation, even in this situation the resonance can kick in. Just to make it clear consider the oversimplified equation

$$I_{xx} \frac{d^2\theta}{dt} + (\omega_0^2 - W_A \cos(\omega_W t)) \theta = 0 \quad (12.4)$$

$\omega_W$  is natural frequency to the longitudinal wave frequency, and  $\omega_0$  is the natural frequency of the ship  $W_A$  is a measure of the waves amplitude. This equation is referred in the literature is Mathieu equation. This equation has several range in which the situation become over excited and create reason for failure.

### 12.3 Physical Considerations

#### 12.3.1 The Effect of Added Properties

At the current stage, it is important to discuss the physics significance of the added properties. It was long recognized that the added properties are a very strong function of the geometry. In the “Basic of Fluid Mechanics” by this undersign (also see chapter 11) it was shown the assigned value to added properties are for the main axes are given in tables (currently under construction only two values). These values are only added mass coefficients and they are not the actual added mass. The added mass coefficients can be used to ascertain the actual value see example in added mass chapter.

The added moment of inertia depends on two factors, first, is the change of the wetted shape, and second, the change of the rotation point. The change from symmetrical like rectangle to non symmetrical like trapezoid is significant. These changes are on top the change of the ship moment of inertia (due to the change of the rotation point). This change can be calculated using somewhat similar technique as the slender body theory (by adding the change to the existing one). On the other hand, the some area (volume) has to remove, some complications in the development of this. It can noticed that if the cross area and change of

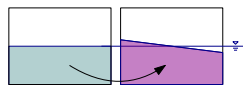


Fig. 12.1 – Change of Wetted Area and thus change of added moment of inertia. This change is a typical of any floating body.

<sup>15</sup>When discussing this question with a colleague she point out that the author is bit hypocrite not using dimensionless form. The undersign has to agree with her but he has to cater to his audience.

the hinge point, the added moment of inertia will remain the same as opposed to partially submerged body the movement creates a change in the added mass.

The current researchers do not deviate from a constant value. That is, the total moment of inertia expressed as  $I_{ship} + I_{add}$  when both are considered to be constant (Remola 2018). Where *add* denotes the added properties. While not explicitly stated in all the articles examined by this author, it is assumed that the rotation point was taken in either gravity centroid or metacenter both wrong points. It is well established that the added moment of inertia is constant for constant or fix geometrical shape (Landweber and Miloh 1980). In Fig. 12.2 shows that added moment of inertia calculated by using turbulence and other strange effects (that should not be part of the calculations). The most depressing of this research work is that there is no information on the important parameters like the location of the rotation point so even model body moment of inertia can be estimated (so much work so little usefulness). Yet, even with lack of real information, unintentionally the information that the added moment of inertial is depend on the angle visible from the figure. The results of this work clearly are not credible and the actual value cannot be ascertain but the existence of the effect is visible.

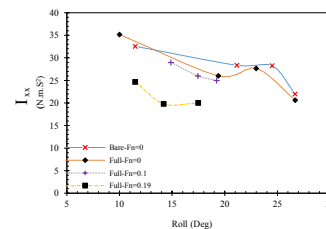


Fig. 12.2 – Added moment of inertia After Kianejad et al. Notice that the left subtitle was change from original to the standard in the book.

The second issue that affects the variations of the added moment of inertia is the change of the immersed geometry. For example, a square body starts as rectangular (depends on the density ratio) and changes to extruded trapezoid or extruded triangle depends on the angle). Hence, at every angle the body has different immersed geometry and therefore different added moment of inertia. The calculated Fig. 12.2 by Kianejad et al (Kianejad, Enshaei, and Ranmuthugala 2017) shows more or less constant value (one of the turbulent added mass work). If one accept this calculations that a constant value can be used<sup>16</sup>. If one insist on a constant value (reduction of accuracy idea) a

<sup>16</sup>This author did not work on the specific code that done by researchers. Yet, one cannot wonder if the equations used in the code are appropriate as they include viscosity and turbulence none which should be used in added properties. Thus, it possible that this added moment of inertia is defined differently than the common added moment of inertia. Additionally, the paper abstracted the derivations and does not specify what is really going on and hence it hard to ascertain the validity of the work. Several other papers by other research groups suggested this approach and yet none of them even provide the location of the rotation point for which it was calculated. In this stage, this undersigned takes this work to mean none standard added moment of inertia which some how include the dissipation terms and should be really assumed to in the fictional realm.

better representation is in nature of<sup>17</sup> the added moment of inertia.

$$\overline{I_{add}} = 0.5 (I_{add|_{min}} + I_{add|_{max}}) \quad (12.1)$$

Where  $I_{add|_{min}}$  is the minimum added moment of inertia and  $I_{add|_{max}}$  is estimate of the maximum In general this term should be written as

$$I_{add} = I_{add|_0} + f(\theta) \quad (12.2)$$

The value of  $I_{add|_0}$  is at zero angle and  $f(\theta)$  can be positive or negative to maximum value of  $I_{add|_0}$ . Kianejad et al calculated the value of the specific ship of the added moment of inertia (Kianejad, Enshaei, and Ranmuthugala 2017).<sup>18</sup> The work by Kianejad et al was based on work by Korotkin and others (Korotkin 2008) which original definition of the added mass is based the surrounding material for ideal flow (that is the viscosity and turbulence must be ignored as these not part of the added properties.). Between these two works there were additional steps or works that and the research groups who was using these techniques. At writing this section it is not clear who introduce this strange definition of the added moment of inertia into "research" stream. The fact that there the added moment of inertia includes additional effects makes it none standard added properties. Due to the fact that these values are not based on known standards they should be ignored and hope that they should recalculated and converted to regular standard in the future.

Perhaps the biggest issue in understanding rolling is the change of location of the rotation point or hinge. As opposed to commonly believed, the rotation point is not at the metacenter but at the surface of the liquid (for quasi-equilibrium calculations and extruded liked bodies). For two-dimensional shape, it is at the middle of the liquid line. Additional rotation point of ship is the gravity center advocated by Fossen (2011). There not even an attempt for justification for the wrong point not even to a minimum consideration for the location..

### Example 12.3: The Rotation location

Level: Advanced

One misunderstood point in rotation of ship especially for rolling motion is the center of rotation. This point has profound effect on the frequency and other parameters. The liquid surface (wetted area) changes during the rolling motion. Demonstrate that the rotation point is the middle of the liquid surface for two dimensional shape.

#### Solution

As first approximation the ship is assumed to have a constant mass. This assumption is reasonable for the case where the ship shape does not have a large abrupt shape and the acceleration of the movement is relatively small. For practical purposes, a change of the mass up to 5% is still within zone for this analysis (much more complicate

<sup>17</sup>The source of the added moment of inertia is dragging of liquid due to the body geometry. Perfect cylindrical shape do no drag any mass on rotation because the fiction is not considered and thus the added moment of inertia has no added moment of inertia (yet it has added mass).

<sup>18</sup>The researchers could not answer the questions this author present to them.

**End of Ex. 12.3**

analysis results from stringent requirements).

When the ship mass is constant (quasi steady state) then the instantaneous displacement (assuming the heave movement is insignificant) the displacement volume is constant under these two assumptions. If the ship has continuous liquid surface (that is, the ship is not catamaran or trimaran) the rotation is at some point on the liquid surface. As long the ship shape is continuous, the subtracted area has to be equal added area. These two areas are, at the limit, straight triangles. The two triangles have the same area if the height and the base are the same in the two triangles. This condition occurs when the heights is the same which means that the rotation must occur at the middle the liquid surface.

The rotation point moves to a new location when the body/ship is turning to a new angle because the middle of the liquid surface is at a new location which depends on the geometry. Almost all the shapes of ships are such that create a change in the hinge line or hinge point. The only exception to this perfectly extruded bodies. A cylinder or cuboid are an example of extruded body were the rotation location does not change. This effect has to be taken into account when calculating moment of inertia of body etc. The significance of this effect can be demonstrated by triangle up versus triangle down (which sometime stable in one way but not other) . Thus even the experimental attempts to measure it when ignoring the shifting of the hinge line (Balcer 2004) are not fruitful. Note, even the correct initial hinge line was erroneously identified at the metacenter (with the exception of an individual Lautrup (Lautrup 2011)<sup>19</sup>. The literature survey shows that not much work was done on these topics. For example Markeev (2017) study a moving mass on the pendulum in 2017 which is some what related work by Legaza (Legeza 2020) which dealt with movable hinge for building. While some works have been done on these two topics, these topics are not well understood and a theatrical base is still missing.

**Example 12.4: Square Rotation Point Movement****Level: Intermediate**

A square is rotating  $5^\circ$ . Assume that density ratio is  $\rho_s/\rho_\ell = 0.25$  Where is the location of the rotation point at that angle? How much the moment of the inertia change by percent?

**Solution**

<sup>19</sup>It fact what missed by this author because later Lautrup made contradicting statement which confused this undersign.



End of Ex. 12.4

The geometry is depicted in Fig. 12.3. The trapezoid area has to be the same as before because the displaced volume (area) remains the same. The angle is given and according to the above requirement the center must stay at the same physical location and hence the two triangles have the same area and cancel each other. Note the actual physical rotation point moves (rotates). The body is rotating around the middle of the liquid surface (of the body).

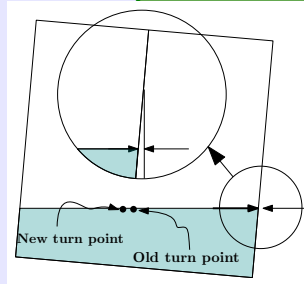


Fig. 12.3 – Change of the rotation point geometry. For this geometry these points are the same.

Example 12.5: Triangle Rotation Point Movement

Level: Intermediate

A triangle with the tip in side the liquid is turned about  $5^\circ$ . Calculate the physical change of the rotation point. Assume that the triangle is isosceles with angle  $2\alpha$  and the density ratio is 0.25 with side length  $a$ .

Solution

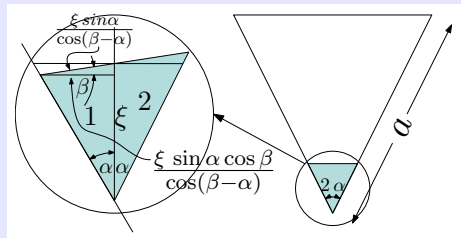


Fig. 12.4 – Triangle rotation point movement explanation.

As in Ex. 12.4 assumption is that the area (actually the volume) is constant. The volume of the triangle is

$$A = a^2 \cos \alpha \sin \alpha = \frac{a^2 \sin 2\alpha}{2} \tag{12.5.a}$$

where  $a$  is, in this case, the side of the triangle. Notice that two triangles are considered. The submerged area is quarter of the total area  $V_0 = 0.25 V$  or  $A_0 = 0.25 A$ . Hence, the submerged area is

$$A = \overbrace{\frac{1}{4}}^{\rho_s / \rho_\ell} \frac{a^2 \sin 2\alpha}{2} \tag{12.5.b}$$

continue Ex. 12.5

The height of the submerged area can be evaluated as it was done before for many examples as

$$\frac{\rho_s/\rho_\ell}{4} \xi^2 \sin \alpha = A_1 + A_2 \quad (12.5.c)$$

where  $A_1$  is the area of the left side triangle and  $A_2$  is the area of the right side triangle. The dimensions of the new shape can be obtained by calculating the two zones (triangles 1 & 2). It can be noticed that triangle 1 is smaller than triangle 2 as well as their top side. The left triangle (based on the observation at Fig. 12.4) is then

$$A_1 = \frac{\xi^2}{2} \frac{\sin(\alpha)}{\cos(\theta + \alpha)} \quad (12.5.d)$$

The second area can be obtained utilizing sine law for the entire triangle to obtain the top side value. Notice, the cos is because the angle is  $\pi/2 - \dots$ . With this information the other two sides can be evaluated. The area of triangle 2 is half the base times the height hence

$$A_2 = \frac{\xi^2}{2} \frac{\sin \alpha \cos \beta}{\cos(\alpha + \beta)} \quad (12.5.e)$$

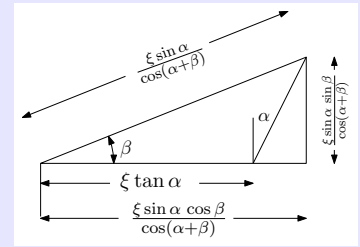


Fig. 12.5 – The upper of the right hand triangle.

The total area is the sum of the two triangles and it is

$$A = \frac{\xi^2}{2} \frac{\sin(\alpha)}{\cos(\theta + \alpha)} + \frac{\xi^2}{2} \frac{\sin \alpha \cos \beta}{\cos(\alpha + \beta)} \quad (12.5.f)$$

Eq. (12.5.f) can be simplified as

$$A = \frac{\xi^2 \sin \alpha}{2 \cos(\alpha + \beta)} (1 + \cos \beta) \quad (12.5.g)$$

This area has to be to the area initial area Eq. (12.5.b) thus

$$\frac{1}{4} \frac{a^2 \sin 2\alpha}{\sin \alpha} = \frac{\xi^2 \sin \alpha}{2 \cos(\alpha + \beta)} (1 + \cos \beta) \quad (12.5.h)$$

After rearrangement Eq. (12.5.h) can be written as

$$\left(\frac{\xi}{a}\right)^2 = \frac{\rho_\ell/\rho_s}{4} \frac{\sin 2\alpha \cos(\alpha + \beta)}{\sin \alpha (1 + \cos \beta)} \quad (12.5.i)$$

or after simplification Eq. (12.5.i) becomes

$$\frac{\xi}{a} = \sqrt{\frac{1}{2} \frac{\cos \alpha \cos(\alpha + \beta)}{1 + \cos \beta}} \quad (12.5.j)$$

End of Ex. 12.5

This found value  $\xi$  denoted as  $\xi^*$  and is used to determine the upper part of triangles (new surface) and the difference between  $S_2 - S_1$  is the change of the rotation location.

$$\frac{\Delta S}{a} = \frac{S_2 - S_1}{a} = \frac{\xi^* \sin \alpha}{\cos(\alpha + \beta)} - \frac{\xi^* \sin \alpha}{\cos(\beta - \alpha)} \quad (12.5.k)$$

The change is in the direction of the surface and the  $x$  component or the  $y$  component can be evaluated.

With these two angles the a graph can be drawn.

### 12.3.2 The change of density effect

The change of the density due to either the temperature or the salinity or any other reason changes the buoyancy of the floating body. Thus, this change in the location of the rotation point changes the moment of inertia relative to the rotation point. The relative change of the density can be in order a few percent points (very common) to 20% or 30%. Yet, even in the case of 3% which can in turn the change the moment of inertia by 7% (hint  $I \sim x^2$ ). This change is large enough to be noticed in the calculations. This book deals with other bodies and hence in the cases where density has a large and significant change the natural frequency also change significantly.

## 12.4 Rolling Governing Equations

In this section the rolling from the simplest to the most general are discussed. The presentation starts with simple case and progress to a complicate situation.

### 12.5 The Simplest Case

The extruded symmetrical long body is the most simply to write the governing equation. This is the case where no external (exciting) force is present and no transfer mechanism exist. The limitation of the analysis here that situation has to realistic. This requirement is that the damping has to considered and accounted for. A limit dimensional analysis suggests that the size of the added moment of inertia has to be accounted for. In this stage, the simple free body diagram is used. It is assumed that the natural situation is that at zero degree, that is no list. The internal forces (moments) are  $I_{body} + I_{add}d\theta^2/dt^2$ . For small angle, the added moment of inertia can be written as

$$I_{added}(\theta) = I_{added_0} + \theta g_I \quad (12.1)$$

where  $g_I$  is the coefficient accounting to the change of the geometry of the wetted area. The same change of the wetted geometry change should affect the damping force mostly because the change of surface area (the line of contact). However, first glance dimensional analysis shows that this change is small but become more substantial for larger angles (this assumption the major source of limitation of this analysis to be realistic.). Hence, damping is assumed to be constant. Armed with these elements

provides base to a governing equation (this equation is not really suitable for checking physical situations).

$$(I_{body} + I_{added_0} + \theta g_I) \frac{d^2\theta}{dt^2} + M_D \frac{d\theta}{dt} + M_R\theta = 0 \quad (12.2)$$

The value of  $g_I$  can be positive or negative and the effects are examined in following.

## Appendices

### 12.A Double Pendulum: Simplifications

In this section a discussion on the double pendulum which is similar too. Essentially ship is not a regular pendulum but it is a what is known as double pendulum with point mass. These points do not represent the ship nor the circular motion of the point 1. A gradual progression will be followed since material is written for a textbook. That is, first introduction to a circular double pendulum with point masses later a discussion about the circular double pendulum with mass distribution (moment of inertia instead of mass) will be introduced.

Later the next stage is to deal with the circular movement to linear where  $x_1$  is related to  $\theta_1$ .

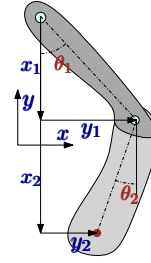


Fig. 12.A.1 – Simple double circular pendulum with point mass.

#### 12.A.1 Double Pendulum with Point Masses

The location of point 1 is

$$\begin{aligned} x_1 &= L_1 \sin \theta_1 \\ y_1 &= -L_1 \cos \theta_1 \end{aligned} \quad (12.A.1)$$

The location of point 2 is

$$\begin{aligned} x_2 &= x_1 + L_2 \sin \theta_2 \\ y_2 &= y_1 - L_2 \cos \theta_2 \end{aligned} \quad (12.A.2)$$

Taking the derivative to obtain the velocity

$$\begin{aligned} x'_1 &= \theta'_1 L_1 \cos \theta_1 \\ y'_1 &= \theta'_1 L_1 \sin \theta_1 \end{aligned} \quad (12.A.3)$$

In the same way for the second point

$$\begin{aligned}x_2' &= x_1' + \theta_2' L_2 \cos \theta_2 \\y_2' &= y_1' + \theta_2' L_2 \sin \theta_2\end{aligned}\quad (12.A.4)$$

The second derivative is

$$\begin{aligned}x_1'' &= \theta_1'^2 L_1 \sin \theta_1 + \theta_1'' L_1 \cos \theta_1 \\y_1'' &= \theta_1'^2 L_1 \cos \theta_1 + \theta_1'' L_1 \sin \theta_1\end{aligned}\quad (12.A.5)$$

the second derivative is for the second point

$$\begin{aligned}x_2'' &= x_1'' \theta_2'^2 L_2 \sin \theta_2 + \theta_2'' L_2 \cos \theta_2 \\y_2'' &= y_1'' + \theta_2'^2 L_2 \cos \theta_2 + \theta_2'' L_2 \sin \theta_2\end{aligned}\quad (12.A.6)$$

where  $T$  denotes the tension and  $m$  denotes the mass and standard stationary equation can be written. The forces that act on mass 1 and mass 2 are the tension in the rods and the gravity.

$$\begin{aligned}m_1 x_1'' &= -T_1 \sin \theta_1 + T_2 \sin \theta_2 \\m_1 y_1'' &= T_1 \cos \theta_1 T_2 \cos \theta_2 - m_1 g\end{aligned}\quad (12.A.7)$$

and for the second point

$$m_2 x_2'' = T_2 \sin \theta_2 \quad (12.A.8)$$

$$m_2 y_2'' = T_2 \cos \theta_2 - m_2 g \quad (12.A.9)$$

Basically there are 4 equations and 4 unknowns ( $T_1, T_2, \theta_1$ , and  $\theta_2$ ). Notice that  $x_1, x_2, y_1$ , and  $y_2$  are a simple function of  $L$  and  $\theta$  with the right indexes. After the elimination or conversion of "dummy" variables ( $T_1, T_2$ ) results in two equations with unknowns which can be solved by numerical methods like Runge Kutta method. No recommendations on the specific methods is provided.

The results are strongly depend on the initial conditions. There several conclusions can be drawn from the results which are very relevant to floating body stability. The chief conclusion is that standard natural frequency does not exist. The results are more chaotic that one expect. For small angle the behavior is close to a simple pendulum. Even with going through more complicate examination one can see that situation get more complex (this with including the change of moment of inertia and other factors).

## 12.A.2 Double Pendulum with Circular Motion with Mass Distribution

This section is out reach for many of the readers because it involve advance physics and concept like Lagrangian. These concepts are used because the complication of writing the governing equations. The usage of Lagrangian provides the governing equations

in a standard form. Equations presented in a standard form and thus element the need to usage of vectors. With the Lagrangian methods there is no need to solve for accelerations or a need to solve for inertial velocities. So simple demonstration of usage of Lagrangian for simple pendulum is presented. The Lagrangian function is defined as the difference between the total kinetic energy,  $T$  and total potential energy  $U$  of system.

$$L = T - U \quad (12.A.10)$$

For simple pendulum with point mass with arm length,  $\ell$  the potential energy is relative the lowest point and for small angle. The velocity is

$$v = \ell \dot{\theta} \quad (12.A.11)$$

The total kinetic energy is

$$T = \frac{m v^2}{2} = \frac{m \ell^2 \dot{\theta}^2}{2} \quad (12.A.12)$$

The potential energy,  $U$ , in case of simple pendulum, depends only on the height. For simplicity the potential energy is assigned  $U(\theta = 0) = 0$  Thus the potential energy can be written as

$$U = y m g = \ell (1 - \cos \theta) m g \quad (12.A.13)$$

With these two terms the Lagrangian is explicitly can written as

$$L = T - U = \frac{g m (\ell \dot{\theta})^2}{2} - \ell (1 - \cos \theta) m g \quad (12.A.14)$$

For the next part The governing equation is obtained utilizing for zero non-conservative forces by

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad (12.A.15)$$

substituting  $L$  into Eq. (12.A.15) yields

$$m \ell^2 \theta'' + m g \ell \sin \theta = 0 \longrightarrow \theta'' + \frac{g}{\ell} \sin \theta = 0 \quad (12.A.16)$$

which is the standard equation for simple pendulum.

It can be noticed that coordinate system change some what. That the distances are measured to the center of the leg which has mass and moment of inertia. It is assumed that the moment of inertia is given. The distance  $x_1$  is this only to half of the leg and some can be said about the  $y_1$ . The distances can be express as

$$x_1 = \frac{\ell}{2} \sin \theta_1 \quad (12.A.17)$$

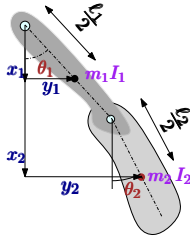


Fig. 12.A.2 – Double pendulum linear motion in circular motion with disperse mass.

$$y_1 = -\frac{\ell}{2} \cos \theta_1 \quad (12.A.18)$$

It is assumed that rod are evenly distribute otherwise new dimension can be introduced.

$$\begin{aligned} x_2 &= \ell_1 \sin \theta_1 + \ell_2 \frac{\sin \theta_2}{2} \\ y_2 &= -\ell_1 \cos \theta_1 - \ell_2 \frac{\cos \theta_2}{2} \end{aligned} \quad (12.A.19)$$

The kinetic energy can be expressed as

$$T = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + \frac{I_1 \theta_1'^2}{2} + \frac{I_2 \theta_2'^2}{2} \quad (12.A.20)$$

The potential energy is

$$U = g y_1 m_1 + g y_2 m_2 \quad (12.A.21)$$

Thus the Lagrangian is

$$L = T - U = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + \frac{I_1 \theta_1'^2}{2} + \frac{I_2 \theta_2'^2}{2} - m_1 g y_1 - m_2 g y_2. \quad (12.A.22)$$

Assuming no external forces the governing equations are obtained from

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial \theta_1'} - \frac{\partial L}{\partial \theta_1} &= 0 \\ \frac{d}{dt} \frac{\partial L}{\partial \theta_2'} - \frac{\partial L}{\partial \theta_2} &= 0 \end{aligned} \quad (12.A.23)$$

At this stage the continuation can be found the literature as it establish issue. The main point was to establish the method and demonstrate the ship rotation similar to double pendulum but with a twist in which the relationship between the hinge to physics is determined from ship determined.

### 12.A.3 Double Pendulum with Linear Motion

The relation between  $y_1$  (see Fig. 12.A.3) and  $\theta_1$  is determined from the cross section (for 2D simpler and it more complex for 3D.). This relationship is independent of the angular velocity and simply related to angle itself  $\theta_1$  and related based on geometrical parameters. This relationship obtained by following the instructions in chapter. Given this relationship only one body as control mass is needed.

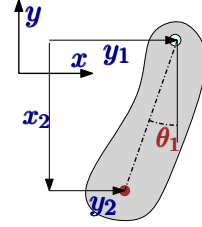


Fig. 12.A.3 – Double pendulum linear motion with disperse mass.

### 12.B Floating in Heave Movement

This exercise to demonstrate a proper way of handling the problems of floating bodies. This example with theoretical example in which a cylinder undergoes oscillations in the heave direction. This cylinder is unstable and will not keep moving the heave direction. Yet, the simplicity provide one with opportunity to examine the added mass's effect. In the literature this example is discussed present in numerous places. A floating body that is oscillated on liquid. The Lagrangian method is used instead the direct application of the Newton's second law because in that case there is no need to keep track of the forces directions. The Lagrangian is defined as

$$L \equiv T - U \quad (12.B.1)$$

The minus sign is code into the definition (a plus sign would simply give the total energy). The  $T$  represent the kinetic energy and  $U$  is the potential energy. The kinetic energy in this case is  $T = m \dot{x}^2/2$ . The potential energy in this case  $U = x g (m_{body} + m_a)$  where  $x$  is the location when the body is in equilibrium. Thus the Lagrangian is

$$L = \frac{[m_{body} + m_a(x)] \dot{x}^2}{2} - x g [m_{body} + m_a(x)] \quad (12.B.2)$$

Note the added mass,  $m_a$  is a function of the height (or depth) of the body immersion size. There are no conclusive functions that describe this functionality, however it is widely accepted that the added mass increases with the depth. For example In the excellent paper (Da Lozzo, Auricchio, and Calvi 2012)<sup>20</sup> the discussion was on a cylindrical pole where the liquid (sea water) level changing due to various conditions (wave, tide, etc). In the floating body is the solid part moving up and down and it is not cross flow (a bit different) At first glance, this should be totally different however

<sup>20</sup>This paper is the best paper written in ship stability or added mass that this author has seem, make sense, organized and logical. The only thing missing is the nomenclature. This kind of work has to be praised.



other claim that there at least weak connection. The solution leads to the infinite series which be computed. The results are shown the following figure.

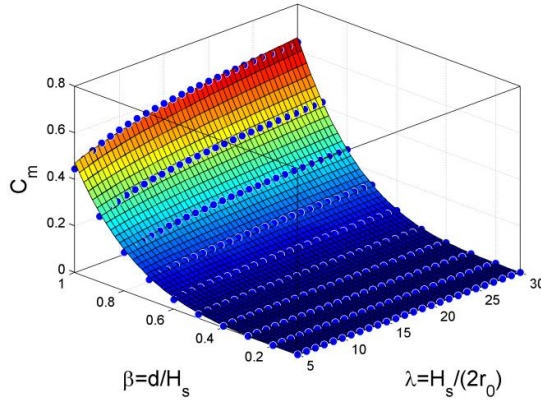


Fig. 12.B.1 – Added mass for vertical cylinder moving in and out liquid. The parameters shown in the figure  $\lambda$  is ratio of the diameter cylinder and of the total length, and  $\beta$  is ratio of the total length to wet length similar to density ratio.

As first approximation this functionality can be expressed as a linear.

$$m_a = m_{a0} + j x \tag{12.B.3}$$

where  $j$  is the linear coefficient

The standard approach equation is

$$\frac{d}{dt} \left( \frac{dL}{dx} \right) = \frac{dL}{dx} \tag{12.B.4}$$

In this case

$$\frac{d}{dt} (m_{body} + m_a(x) \dot{x}) = g (m_{body} + m_a(x)) + \overbrace{2x j g}^{correction} \tag{12.B.5}$$

Equation Eq. (12.B.5) can be solve analytically by separating and carrying the integral numerically.<sup>21</sup> This book is really focus on the mathematics. In any case, this equation can easily solve numerically. This equation is second order equation which is reduced to the familiar harmonic equation. It can be noticed that the equation is asymmetrical in nature in sense that add mass is acting damping force.

<sup>21</sup>The did not bottle with this integral which seem to be analytically possible.

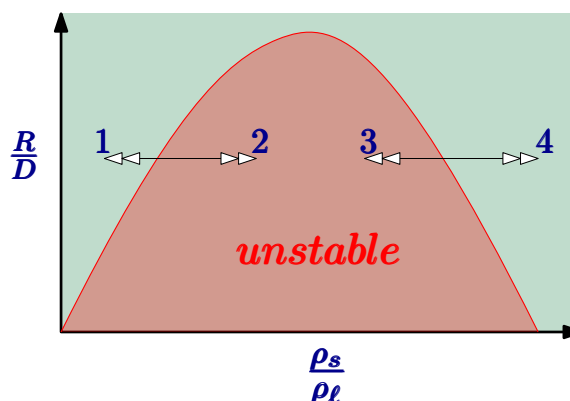


Fig. 12.B.2 – Stability chart for body that undergoes heave oscillations. Depending on the initial density ratio, the floating body can be unstable from both sides. In other words, if body is heavy it can be unstable when it goes down. On the other hand, when the body is light, if pushed too much into the liquid it becomes unstable.

In addition, it is important to discuss the limitations of the solution. When the floating body goes from the lower point to the upper point in heave movement it can be viewed as a process of change of density. That is, in the case of a body at greater depth it can be viewed as a heavier body or a body with a larger density. Effectively, the body through the movement changes the density and thus the density ratio. As shown in Chapter 8, the body can move in and out of the unstable zone. Once the body enters the unstable zone, the rolling movement “kick in” and the body starts to rotate. Yet if the rotation is very minimal, and the body returns to the stable zone, the rolling movement is curtailed and the body returns to its original position. On the other hand, if the movement is significant, the body will undergo rolling or pitch motions as well.

## 12.C Pendulum With Moving Hinge Point

### 12.D Change of Center of Rotation

At first the discussion deals with 2D cases as they are simpler compared to 3D. The center of rotation is at the middle of the liquid line (for a two-dimensional body). However, the change of the center of rotation is the average of addition to the right and subtraction to the left, assuming counter-clockwise rotation. In this derivation, it is assumed that the body is extruded, that is, it is only 2-D and the angle is very small. Later it will derive for measurable angles. In these derivations, the body is not assumed to be symmetrical. The total change is made of the change to the right and to the left. The change to the right is

$$\Delta x_r = \frac{dA_r}{dh} \quad (12.D.1)$$

The  $\Delta x_r$  represents the change to the right. If the body is symmetrical, that total change is  $\Delta x = \Delta x_r = \Delta x_l$ . The change to the left is  $\Delta x_l$  is

$$\Delta x_l = -\frac{dA_l}{dh} \quad (12.D.2)$$

The  $\Delta x_l$  represents the change of the contact point of surface to the left. The total change is then

$$\Delta x_{total} = \frac{dA_r}{dh} - \frac{dA_l}{dh} \quad (12.D.3)$$

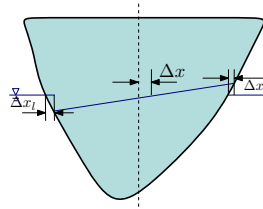


Fig. 12.D.1 - The derivative of the change of the rotation points.



# 13

## Dimensional Analysis

This chapter is dedicated to my adviser, Dr. E.R.G. Eckert.

Genick Bar-Meir

### *13.1 Introductory Remarks*

The reasons that this chapter is included in this book to eliminate the tables for various situation but to have dimensionless presentation so tables should be included in the equations them self. For example, in Metacenter method the parameter GM is not given in dimensionless form. What is the meaning of GM 1[m] while it say the ship is stable and does not indicate if it is very stable or not. One has to get additional information. The indication that this question of stability is has relatively small of parameters which indicate that it has to none-dimensionalized.

What is dimensionless analysis? Dimensional analysis refers to techniques dealing with units or conversion to a unitless system. The definition of dimensional analysis is not consistent in the literature which span over various fields (mostly in the thermo-fluid field) (Buckingham 1914; Buckingham 1915b; Buckingham 1915a). Possible topics that dimensional analysis deals with are consistency of the units, change order of magnitude, applying from the old and known to unknown (see the Book of Ecclesiastes), and creation of group parameters without any dimensions. In this chapter, the focus is on the creation the creation of dimensionless groups. These techniques gave birth to dimensional parameters which have a great scientific importance (Görtler 1975; Lang-

haar 1980). Since the 1940s<sup>1</sup>, the dimensional analysis is taught and written in all fluid, heat and mass transfer textbooks. The approach or the technique used in some of these books is referred to as Buckingham- $\pi$ -theory. The  $\pi$ -theory was coined by Buckingham. However, there is another technique which is referred to in the literature as the Nusselt's method. Both these methods attempt to reduce the number of parameters which affect the problem and reduce the labor in solving the problem. The key in these techniques lays in the fact of consistency of the dimensions of any possible governing equation(s) and the fact that some dimensions are reoccurring. The Buckingham- $\pi$  goes further and no equations are solved and even no knowledge about these equations is required. In Buckingham's technique only the dimensions or the properties of the problem at hand are analyzed. This author is aware of only a single class of cases where Buckingham's method is useful and or can solve the problem namely the pendulum class problem (and similar).

The dimensional analysis was independently developed by Nusselt and improved by his students/co-workers (Schmidt, Eckert) in which the governing equations are used as well. Thus, more information is put into the problem and thus a better understanding on the dimensionless parameters is extracted. The advantage or disadvantage of these similar methods depend on the point of view. The Buckingham- $\pi$  technique is simpler while Nusselt's technique produces a better result. Sometime, the simplicity of Buckingham's technique yields insufficient knowledge or simply becomes useless. When no governing equations are found, Buckingham's method has usefulness. It can be argued that these situations really do not exist in the Thermo-Fluid field. Nusselt's technique is more cumbersome but more precise and provide more useful information. Both techniques are discussed in this Chapter. The advantage of the Nusselt's technique are: a) compact presentation, b) knowledge what parameters affect the problem, c) easier to extent the solution to more general situations. In very complex problems both techniques suffer from inability to provide a significant information on the effective parameters such multi-phase flow etc.

It has to be recognized that the dimensional analysis provides answer to what group of parameters affecting the problem and not the answer to the problem. In fact, there are fields in thermo-fluid where dimensional analysis, is recognized as useless. For example, the area of multiphase flows there is no solution based on dimensionless parameters (with the exception of the rough solution of Martinelli). In the Buckingham's approach it merely suggests the number of dimensional parameters based on a guess of all parameters affecting the problem. Nusselt's technique provides the form of these dimensionless

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<sup>1</sup>The history of dimensional analysis is complex. Several scientists used this concept before Buckingham and Nusselt (see below history section). Their work culminated at the point of publishing the paper Buckingham's paper and independently constructed by Nusselt. It is interesting to point out that there are several dimensionless numbers that bear Nusselt and his students name, Nusselt number, Schmidt number, Eckert number. There is no known dimensionless number which bears Buckingham name. Buckingham's technique is discussed and studied in Fluid Mechanics while almost completely ignored by Heat and Mass Transfer researchers and their classes. Furthermore, in many advance fluid mechanics classes Nusselt's technique is used and Buckingham's technique is abandoned. Perhaps this fact can be attributed to tremendous influence Nusselt and his students had on the heat transfer field. Even, this author can be accused for being bias as the Eckert's last student. However, this author observed that Nusselt's technique is much more effective as it will demonstrated later.

parameters, and the relative relationship of these parameters.

### 13.1.1 Brief History

The idea of experimentation with a different, rather than the actual, dimension was suggested by several individuals independently. Some attribute it to Newton (1686) who coined the phrase of “great Principle of Similitude.” Later, Maxwell a Scottish Physicist played a major role in establishing the basic units of mass, length, and time as building blocks of all other units. Another example, John Smeaton (8 June 1724–28 October 1792) was an English civil and mechanical engineer who study relation between propeller/wind mill and similar devices to the pressure and velocity of the driving forces. Jean B. J. Fourier (1768-1830) first attempted to formulate the dimensional analysis theory. This idea was extend by William Froude (1810-1871) by relating the modeling of open channel flow and actual body but more importantly the relationship between drag of models to actual ships. While the majority of the contributions were done by thermo–fluid guys the concept of the equivalent or similar propagated to other fields. Aiméem Vaschy, a German Mathematical Physicist (1857–1899), suggested using similarity in electrical engineering and suggested the Norton circuit equivalence theorems. Rayleigh probably was the first one who used dimensional analysis (1872) to obtain the relationships between the physical quantities (see the question why the sky is blue story).

Osborne Reynolds (1842–1912) was the first to derive and use dimensionless parameters to analyze experimental data. Riabouchunsky<sup>2</sup> proposed of relating temperature by molecules velocity and thus creating dimensionless group with the byproduct of compact solution (solution presented in a compact and simple form).

Buckingham culminated the dimensional analysis and similitude and presented it in a more systematic form. In the about the same time (1915, Wilhelm Nusselt (November 25, 1882 – September 1, 1957), a German engineer, developed the dimensional analysis (proposed the principal parameters) of heat transfer without knowledge about previous work of Buckingham.

### 13.1.2 Theory Behind Dimensional Analysis

In chemistry it was recognized that there are fundamental elements that all the material is made from (the atoms). That is, all the molecules are made from a combination of different atoms. Similarly to this concept, it was recognized that in many physical systems there are basic fundamental units which can describe all the other dimensions or units in the system. For example, isothermal single component systems (which does not undergo phase change, temperature change and observed no magnetic or electrical effect) can be described by just basic four physical units. The units or dimensions are, time, length, mass, quantity of substance (mole). For example, the dimension or the units of force can be constructed utilizing Newton’s second law i.e. mass times acceleration  $\rightarrow m a = M L/t^2$ . Increase of degree of freedom, allowing this system

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<sup>2</sup>Riabouchunsky, Nature Vol 99 p. 591, 1915

to be non-isothermal will increase only by one additional dimension of temperature,  $\theta$ . These five fundamental units are commonly the building blocks for most of the discussion in fluid mechanics (see Table of basic units 14.1).

Table 13.1 – Basic Units of Two Common Systems

Standard System			Old System		
Name	Letter	Units	Name	Letter	Units
Mass	$M$	[kg]	Force	$F$	[N]
Length	$L$	[m]	Length	$L$	[m]
Time	$t$	[sec]	Time	$t$	[sec]
Temperature	$\theta$	[°C]	Temperature	$T$	[°C]
Additional Basic Units for Magnetohydrodynamics					
Electric Current	$A$	[A]mpere	Electric Current	$A$	[A]mpere
Luminous Intensity	$cd$	[cd] candle	Luminous Intensity	$cd$	[cd] candle
Chemical Reactions					
Quantity of substance	$\mathfrak{M}$	mol	Quantity of substance	$\mathfrak{M}$	mol

The choice of these basic units is not unique and several books and researchers suggest a different choice of fundamental units. One common selection is substituting the mass with the force in the previous selection ( $F, t, L, \text{mol}, \text{Temperature}$ ). This author is not aware of any discussion on the benefits of one method over the other method. Yet, there are situations in which first method is better than the second one while in other situations, it can be the reverse. Other selections are possible but not common and, at the moment, will not be discussed here.

**Example 13.1: What are the units of**
**Level: Easy**

What are the units of force when the basic units are: mass, length, time, temperature ( $M, L, t, \theta$ )? What are the units of mass when the basic units are: force, length, time, temperature ( $F, L, t, T$ )? Notice the different notation for the temperature in the two systems of basic units. This notation has no significance but for historical reasons remained in use.

Solution



**End of Ex. 13.1**

These two systems are related as the questions are the reversed of each other. The connection between the mass and force can be obtained from the simplified Newton's second law  $F = m a$  where  $F$  is the force,  $m$  is the mass, and  $a$  is the acceleration. Thus, the units of force are

$$F = \frac{M L}{t^2} \quad (13..a)$$

For the second method the unit of mass are obtain from Equation (13..a) as

$$M = \frac{F t^2}{L} \quad (13..b)$$

The number of fundamental or basic dimensions determines the number of the combinations which affect the physical<sup>3</sup> situations. The dimensions or units which affect the problem at hand can be reduced because these dimensions are repeating or reoccurring. The Buckingham method is based on the fact that all equations must be consistent with their units. That is the left hand side and the right hand side have to have the same units. Because they have the same units the equations can be divided to create unitless equations. This idea alludes to the fact that these unitless parameters can be found without any knowledge of the governing equations. Thus, the arrangement of the effecting parameters in unitless groups yields the affecting parameters. These unitless parameters are the dimensional parameters. The following trivial example demonstrates the consistency of units

**Example 13.2: Newton's law dimensions****Level: Easy**

Newton's equation has two terms that related to force  $F = m a + \dot{m} U$ . Where  $F$  is force,  $m$  is the mass,  $a$  is the acceleration and dot above  $\dot{m}$  indicating the mass derivative with respect to time. In particular case, this equation get a form of

$$F = m a + 7 \quad (13..c)$$

where 7 represent the second term. What are the requirement on equation (13..c)?

**Solution**

Clearly, the units of  $[F]$ ,  $m a$  and 7 have to be same. The units of force are  $[N]$  which is defined by first term of the right hand side. The same units force has to be applied to 7 thus it must

<sup>3</sup>The dimensional analysis also applied in economics and other areas and the statement should reflect this fact. However, this book is focused on engineering topics and other fields are not discussed.

End of Ex. 13.2

be in  $[N]$ .

Suppose that there is a relationship between a quantity  $a$  under the question and several others parameters which either determined from experiments or theoretical consideration which is of the form

$$D = f(a_1, a_2, \dots, a_i, \dots, a_n) \quad (13.1)$$

where  $D$  is dependent parameters and  $a_1, a_2, \dots, a_i, \dots, a_n$  are have independent dimensions. From these independent parameters  $a_1, a_2, \dots, a_i$  have independent dimensions (have basic dimensions). This mean that all the dimensions of the parameters  $a_{i+1}, \dots, a_n$  can be written as combination of the the independent parameters  $a_1, a_2, \dots, a_i$ . In that case it is possible to write that every parameter in the later set can written as dimensionless

$$\frac{a_{i+1}}{a_1^{p_1}, a_2^{p_2}, \dots, a_i^{p_i}} = \text{dimensionless} \quad (13.2)$$

The “non–basic” parameter would be dimensionless when divided by appropriately and selectively chosen set of constants  $p_1, p_2, \dots, p_i$ .

### Example 13.3: Clamping force

Level: Easy

In a experiment, the clamping force is measured. It was found that the clamping force depends on the length of experimental setup, velocity of the upper part, mass of the part, height of the experimental setup, and leverage the force is applied. Chose the basic units and dependent parameters. Show that one of the dependent parameters can be normalized.

#### Solution

The example suggest that the following relationship can be written.

$$F = f(L, U, H, \tau, m) \quad (13.d)$$

The basic units in this case are in this case or length, mass, and time. No other basic unit is need to represent the problem. Either  $L$ ,  $H$ , or  $\tau$  can represent the length. The mass will be represented by mass while the velocity has to be represented by the velocity (or some combination of the velocity). Hence a one possible choice for the basic dimension is  $L$ ,  $m$ , and  $U$ . Any of the other Lengths can be represented by simple division by the  $L$ . For example

$$\text{Normalize parameter} = \frac{H}{L} \quad (13.e)$$

**End of Ex. 13.3**

Or the force also can be normalized as

$$\text{Another Normalize parameter} = \frac{F}{mU^2 L^{-1}} \quad (13..f)$$

The acceleration can be any part of acceleration component such as centrifugal acceleration. Hence, the force is mass times the acceleration.

The relationship (13.1) can be written in the light of the above explanation as

$$\frac{D}{a_1^{p_1}, a_2^{p_2}, \dots, a_i^{p_i}} = F \left( \frac{a_{i+1}}{a_1^{p_{i+1,1}}, a_2^{p_{i+1,2}}, \dots, a_i^{p_{i+1,i}}}, \dots, \frac{a_n}{a_n^{p_{n,1}}, a_n^{p_{n,2}}, \dots, a_n^{p_{n,i}}} \right) \quad (13.3)$$

where the indexes of the power  $p$  on the right hand side are single digit and the double digits on the on the right hand side. While this “proof” shows the basic of the Buckingham’s method it actually provides merely the minimum number of the dimension parameters. In fact, this method entrenched into the field while in most cases provides incomplete results. The fundamental reason for the erroneous results is because the fundamental assumption of equation (13.1). This method provides a crude tool of understanding.

### 13.1.3 Dimensional Parameters Application for Experimental Study

The solutions for any situations which are controlled by the same governing equations with same boundary conditions regardless of the origin the equation. The solutions are similar or identical regardless to the origin of the field no matter if the field is physical, or economical, or biological. The Buckingham’s technique implicitly suggested that since the governing equations (in fluid mechanics) are essentially are the same, just knowing the parameters is enough to identify the problem. This idea alludes to connections between similar parameters to similar solution. The non-dimensionalization i.e. operation of reducing the number affecting parameters, has a useful by-product, the analogy in other words, the solution by experiments or other cases. The analogy or similitude refers to understanding one phenomenon from the study of another phenomenon. This technique is employed in many fluid mechanics situations. For example, study of compressible flow (a flow where the density change plays a significant part) can be achieved by study of surface of open channel flow. The compressible flow is also similar to traffic on the highway. Thus for similar governing equations if the solution exists for one case it is a solution to both cases.

The analogy can be used to conduct experiment in a cheaper way and/or a safer way. Experiments in different scale than actual dimensions can be conducted for cases where the actual dimensions are difficult to handle. For example, study of large air planes can done on small models. On the other situations, larger models are used to study small or fast situations. This author believes that at the present the Buckingham method has extremely limited use for the real world and yet this method is presented in the classes on fluid mechanics. Thus, many examples

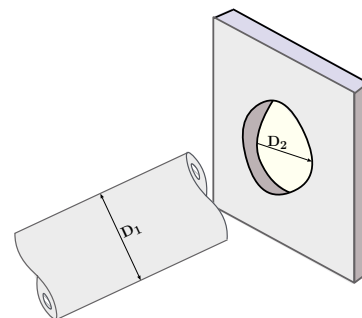


Fig. 13.1 – Fitting rod into a hole.

on the use of this method will be presented in this book. On the other hand, Nusselt's method has a larger practical use in the real world and therefore will be presented for those who need dimensional analysis for the real world. Dimensional analysis is useful also for those who are dealing with the numerical research/calculation. This method supplement knowledge when some parameters should be taken into account and why.

Fitting a rod into a circular hole (see Figure 13.1) is an example how dimensional analysis can be used. To solve this problem, it is required to know two parameters; 1) the rod diameter and 2) the diameter of the hole. Actually, it is required to have only one parameter, the ratio of the rod diameter to the hole diameter. The ratio is a dimensionless number and with this number one can tell that for a ratio larger than one, the rod will not enter the hole; and a ratio smaller than one, the rod is too small. Only when the ratio is equal to one, the rod is said to be fit. This presentation allows one to draw or present the situation by using only one coordinate, the radius ratio. Furthermore, if one wants to deal with tolerances, the dimensional analysis can easily be extended to say that when the ratio is equal from 0.99 to 1.0 the rod is fitting, and etc. If one were to use the two diameters description, further significant information will be needed. In the preceding simplistic example, the advantages are minimal. In many real problems this approach can remove cluttered views and put the problem into focus. Throughout this book the reader will notice that the systems/equations in many cases are converted to a dimensionless form to augment understanding.

#### 13.1.4 The Pendulum Class Problem

The only known problem that dimensional analysis can solve (to some degree) is the pendulum class problem. In this section several examples of the pendulum type problem are presented. The first example is the classic Pendulum problem.

**Example 13.4: Dimensionless Pendulum****Level: Easy**

Derive the relationship for the gravity [ $g$ ], frequency [ $\omega$ ] and length of pendulum [ $\ell$ ]. Assume that no other parameter including the mass affects the problem. That is, the relationship can be expressed as

$$\omega = f(\ell, g) \quad (13..g)$$

Notice in this problem, the real knowledge is provided, however in the real world, this knowledge is not necessarily given or known. Here it is provided because the real solution is already known from standard physics classes.<sup>4</sup>

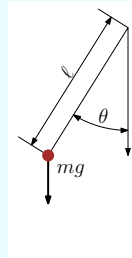


Fig. 13.2 – Figure for example (13.4).

**Solution**

The solution technique is based on the assumption that the indexical form is the appropriate form to solve the problem. The Indexical form

$$\omega = C_1 \times \ell^a g^b \quad (13..h)$$

The solution functional complexity is limited to the basic combination which has to be in some form of multiplication of  $\ell$  and  $g$  in some power. In other words, the multiplication of  $\ell g$  have to be in the same units of the frequency units. Furthermore, assuming, for example, that a trigonometric function relates  $\ell$  and  $g$  and frequency. For example, if a sin function is used, then the functionality looks like  $\omega = \sin(\ell g)$ . From the units point of view, the result of operation not match i.e. ( $sec \neq \sin(sec)$ ). For that reason the form in equation (13..h) is selected. To satisfy equation (13..h) the units of every term are examined and summarized the following table.

Table 13.2 – Units of the Pendulum Parameters

Parameter	Units	Parameter	Units	Parameter	Units
$\omega$	$t^{-1}$	$\ell$	$L^1$	$g$	$L^1 t^{-2}$

End of Ex. 13.4

Thus substituting of the Table 13.2 in equation (13..h) results in

$$t^{-1} = C_1 (L^1)^a (L^1 t^{-2})^b \implies L^{a+b} t^{-2b} \quad (13..i)$$

after further rearrangement by multiply the left hand side by  $L^0$  results in

$$L^0 t^{-1} = C L^{a+b} t^{-2b} \quad (13..j)$$

In order to satisfy equation (13..j), the following must exist

$$0 = a + b \quad \text{and} \quad -1 = \frac{-2}{b} \quad (13..k)$$

The solution of the equations (13..k) is  $a = -1/2$  and  $b = -1/2$ .

What was found in this example is the form of the solution's equation and frequency. Yet, the functionality e.g. the value of the constant was not found. The constant can be obtained from experiment for plotting  $\omega$  as the abscissa and  $\sqrt{\ell/g}$  as ordinate.

According to some books and researchers, this part is the importance of the dimensional analysis. It can be noticed that the initial guess merely and actually determine the results. If, however, the mass is added to considerations, a different result will be obtained. If the guess is relevant and correct then the functional relationship can be obtained by experiments.

## 13.2 Buckingham- $\pi$ -Theorem

All the physical phenomena that is under the investigation have  $n$  physical effecting parameters such that

$$F_1(q_1, q_2, q_3, \dots, q_n) = 0 \quad (13.1)$$

where  $q_i$  is the "i" parameter effecting the problem. For example, study of the pressure difference created due to a flow in a pipe is a function of several parameters such

$$\Delta P = f(L, D, \mu, \rho, U) \quad (13.2)$$

In this example, the chosen parameters are not necessarily the most important parameters. For example, the viscosity,  $\mu$  can be replaced by dynamic viscosity,  $\nu$ . The choice is made normally as the result of experience and it can be observed that  $\nu$  is a function of  $\mu$  and  $\rho$ . Finding the important parameters is based on "good fortune" or perhaps intuition. In that case, a new function can be defined as

$$F(\Delta P, L, D, \mu, \rho, U) = 0 \quad (13.3)$$

<sup>4</sup>The reader can check if the mass is assumed to affect the problem then, the result is different.

Again as stated before, the study of every individual parameter will create incredible amount of data. However, Buckingham's<sup>5</sup> methods suggested to reduce the number of parameters. If independent parameters of same physical situation is  $m$  thus in general it can be written as

$$F_2(\pi_1, \pi_2, \pi_3, \dots, \pi_m) = 0 \quad (13.4)$$

If there are  $n$  variables in a problem and these variables contain  $m$  primary dimensions (for example M, L, T), then the equation relating all the variables will have  $(n-m)$  dimensionless groups.

There are 2 conditions on the dimensionless parameters:

1. Each of the fundamental dimensions must appear in at least one of the  $m$  variables
2. It must not be possible to form a dimensionless group from one of the variables within a recurring set. A recurring set is a group of variables forming a dimensionless group.

In the case of the pressure difference in the pipe (Equation (13.3)) there are 6 variables or  $n = 6$ . The number of the fundamental dimensions is 3 that is  $m = 3$  ([M], [L], [t]) The choice of fundamental or basic units is arbitrary in that any construction of these units is possible. For example, another combination of the basic units is time, force, mass is a proper choice. According to Buckingham's theorem the number of dimensionless groups is  $n - m = 6 - 3 = 3$ . It can be written that one dimensionless parameters is a function of two other parameter such as

$$\pi_1 = f(\pi_2, \pi_3) \quad (13.5)$$

If indeed such a relationship exists, then, the number of parameters that control the problem is reduced and the number of experiments that need to be carried is considerably smaller. Note, the  $\pi$ -theorem does not specify how the parameters should be selected nor what combination is preferred.

### 13.2.1 Construction of the Dimensionless Parameters

In the construction of these parameters it must be realized that every dimensionless parameters has to be independent. The meaning of independent is that one dimensionless parameter is not a multiply or a division of another dimensional parameter. In the above example there are three dimensionless parameters which required of at least one of the physical parameter per each dimensionless parameter. Additionally, to make these dimensionless parameters independent they cannot be multiply or division of each other.

For the pipe problem above,  $\ell$  and  $D$  have the same dimension and therefore both cannot be chosen as they have the same dimension. One possible combination

<sup>5</sup>E. Buckingham, "Model Experiments and the Forms of Empirical Equations," Transactions of the American Society of Mechanical Engineers, Vol. 37, 1915.

is of  $D$ ,  $U$  and  $\rho$  are chosen as the recurring set. The dimensions of these physical variables are:  $D = [L^1]$ , velocity of  $U = [L t^{-1}]$  and density as  $\rho = [M L^{-3}]$ . Thus, the first term  $D$  can provide the length,  $[L]$ , the second term,  $U$ , can provide the time  $[t]$ , and the third term,  $\rho$  can provide the mass  $[M]$ . The fundamental units,  $L$ ,  $t$ , and  $M$  are length, time and mass respectively. The fundamental units can be written in terms of the physical units. The first term  $L$  is the described by  $D$  with the units of  $[L]$ . The time,  $[t]$ , can be expressed by  $D/U$ . The mass,  $[M]$ , can be expressed by  $\rho D^3$ . Now the dimensionless groups can be constructed by looking at the remaining physical parameters,  $\Delta P$ ,  $D$  and  $\mu$ . The pressure difference,  $\Delta P$ , has dimensions of  $[M L^{-1} t^{-2}]$  Therefore,  $\Delta P M^{-1} L t^2$  is a dimensionless quantity and these values were calculated just above this line. Thus, the first dimensionless group is

$$\pi_1 = \frac{\overbrace{\Delta P}^{[M L^{-1} t^{-2}]}}{\overbrace{\rho}^{[M^{-1}]} \overbrace{D^3}^{[L]}} \frac{\overbrace{D}^{[L]}}{\overbrace{U^2}^{[t^2]}} = \frac{\overbrace{\Delta P}^{unitless}}{\rho U^2} \quad (13.6)$$

The second dimensionless group (using  $D$ ) is

$$\pi_2 = \frac{\overbrace{D}^{[L]}}{\overbrace{\ell^{-1}}^{[L^{-1}]}} = \frac{D}{L} \quad (13.7)$$

The third dimensionless group (using  $\mu$  dimension of  $[M L^1 t^{-1}]$ ) and therefore dimensionless is

$$\pi_3 = \mu \frac{\overbrace{1}^{[M^{-1}]}}{\overbrace{D^3}^{[L]}} \frac{\overbrace{D}^{[L]}}{\overbrace{U}^{[t]}} = \frac{\mu}{D U \rho} \quad (13.8)$$

This analysis is not unique and there can be several other possibilities for selecting dimensionless parameters which are “legitimately” correct for this approach.

There are roughly three categories of methods for obtaining the dimensionless parameters. The first one solving it in one shot. This method is simple and useful for a small number of parameters. Yet this method becomes complicated for large number of parameters. The second method, some referred to as the building blocks method, is described above. The third method is by using dimensional matrix which is used mostly by mathematicians and is less useful for engineering purposes.

The second and third methods require to identification of the building blocks. These building blocks are used to construct the dimensionless parameters. There are several requirements on these building blocks which were discussed on page 199. The main point that the building block unit has to contain at least the basic or fundamental unit. This requirement is logical since it is a building block. The last method is mostly used by mathematicians which leads and connects to linear algebra. The fact that this method used is the hall mark that the material was written by mathematician. Here, this material will be introduced for completeness sake with examples and several terms associated with this technique.



### 13.2.2 Similarity and Similitude

One of dimensional analysis is the key point is the concept that the solution can be obtained by conducting experiments on similar but not identical systems. The analysis here suggests and demonstrates<sup>6</sup> that the solution is based on several dimensionless numbers. Hence, constructing experiments of the situation where the same dimensionless parameters obtains could, in theory, yield a solution to problem at hand. Thus, knowing what are dimensionless parameters should provide the knowledge of constructing the experiments.

In this section deals with these similarities which in the literature some refer as analogy or similitude. It is hard to obtain complete similarity. Hence, there is discussion how similar the model is to the prototype. It is common to differentiate between three kinds of similarities: geometric, kinetics, and dynamic. This characterization started because historical reasons and it, some times, has merit especially when applying Buckingham's method. In Nusselt's method this differentiation is less important.

#### Geometric Similarity

One of the logical part of dimensional analysis is how the experiences should be similar to actual body they are supposed to represent. This logical conclusion is an addition and this author is not aware of any proof to this requirement based on Buckingham's methods. Ironically, this conclusion is based on Nusselt's method which calls for the same dimensionless boundary conditions. Again, Nusselt's method, sometimes or even often, requires similarity because the requirements to the boundary conditions. Here<sup>7</sup> this postulated idea is adapted.

Under this idea the prototype area has to be square of the actual model or

$$\frac{A_p}{A_m} = \left( \frac{\ell_{1prototype}}{\ell_{1model}} \right)^2 = \left( \frac{\ell_{2p}}{\ell_{2m}} \right)^2 \quad (13.9)$$

where  $\ell_1$  and  $\ell_2$  are the typical dimensions in two different directions and subscript  $p$  refers to the prototype and  $m$  to the model. Under the same argument the volumes change with the cubes of lengths.

In some situations, the model faces inability to match two or more dimensionless parameters. In that case, the solution is to sacrifice the geometric similarity to minimize the undesirable effects. For example, river modeling requires to distort vertical scales to eliminate the influence of surface tension or bed roughness or sedimentation.

#### Kinematic Similarity

The perfect kinetics similarity is obtained when there are geometrical similarity and the motions of the fluid above the objects are the same. If this similarity is not possible, then the desire to achieve a motion "picture" which is characterized by ratios

<sup>6</sup>This statement is too strong. It has to be recognized that the results are as good as the guessing which in most cases is poor.

<sup>7</sup>Because this book intend to help students to pass their exams, this book present what most instructors required. It well established that this over-strict requirement and under Nusselt's method it can be overcome.

of corresponding velocities and accelerations is the same throughout the actual flow field. It is common in the literature, to discuss the situations where the model and prototype are similar but the velocities are different by a different scaling factor.

The geometrical similarity aside the shapes and counters of the object it also can requires surface roughness and erosion of surfaces of mobile surfaces or sedimentation of particles surface tensions. These impose demands require a minimum on the friction velocity. In some cases the minimum velocity can be  $U_{min} = \sqrt{\tau_w/\rho}$ . For example, there is no way achieve low Reynolds number with thin film flow.

### Dynamics Similarity

The dynamic similarity has many confusing and conflicting definitions in the literature. Here this term refers to similarity of the forces. It follows, based on Newton's second law, that this requires similarity in the accelerations and masses between the model and prototype.

#### Example 13.5: Nusselt's technique

Level: Easy

What are the dimensional parameters that control the stability of floating bodies using Nusselt's technique assuming the body are with uniform density.

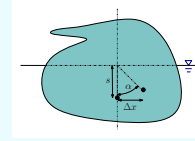


Fig. 13.1 – Floating body showing alpha and other dimensions.

#### Solution

The governing equation for the stability of floating body is

$$\theta = \alpha \quad (13.10)$$

where  $\theta$  is the arbitrary turning angle and  $\alpha$  is the angle resulting from change of the centroid of submerge volume due to the change in  $\theta$ . The relationship between the various geometrical parameters is determined connection according to Eq. (13.10). That relationship requires look at the component of triangle at Fig. 13.1. The base of the triangle is determined by

$$\Delta x = x_n - x_0 = \frac{V}{V_0} (x_a - x_r) \quad (13.1)$$

End of Ex. 13.5

and  $x_a$  and  $x_r$  is related to ratio

$$x_a = \frac{\int x dV}{\int dV} = \frac{\int x(x \tan \theta) d(x) dx}{V} = \frac{\tan \theta \int x^2 \overbrace{d(x) dx}^{dA}}{V} \quad (13..m)$$

or using the definition of moment of inertia Eq. (13..m) can be transferred into

$$x_a = \frac{\tan \theta I_{xx}}{V} \quad (13..n)$$

After the opposite side calculation, adjacent side ( $\mathbf{GB}$  center of buoyancy ( $\mathbf{B}$ ) center of gravity ) thus ( $\mathbf{G-B}$ ). The weight can be estimated as  $m = \rho_\ell A \mathbf{B} = \rho_s A \mathbf{G}$ . When  $A$  is typical cross section, thus

$$\mathbf{GB} = \mathbf{G} \left( 1 - \frac{\rho_s}{\rho_\ell} \right) \quad (13..o)$$

Combining equations (13..n) and (13..o) results in

$$\alpha = \frac{\frac{\tan \theta I_{xx}}{V}}{\mathbf{G} \left( 1 - \frac{\rho_s}{\rho_\ell} \right)} \quad (13..p)$$

Observation of Eq. (13..p) so dimensional group  $\frac{\rho_s}{\rho_\ell}$ , and  $\frac{I_{xx}}{\mathbf{G}V}$ . While the analysis was clumsy and rough it provides dimensionless parameters while Buckingham's method fails dramatically.

### 13.3 Summary

The two dimensional analysis methods or approaches were presented in this chapter. Buckingham's  $\pi$  technique is a quick "fix approach" which allow rough estimates and relationship between model and prototype. Nusselt's approach provides an heavy duty approach to examine what dimensionless parameters effect the problem. It can be shown that these two techniques in some situations provide almost similar solution. In other cases, these technique proves different and even conflicting results. The dimensional analysis technique provides a way to simplify models (solving the governing equation by experimental means) and to predict effecting parameters.



# 14

## Experiments for Stability

### *14.1 Introduction*

This chapter intended for students who taking a stability class or fluid mechanics beside professionals in marine industry. This chapter is intended to be supplemental to the academic material but in other situations it could be a stand alone material. While stability has mainly theoretical subjects. Yet, there are elements that should provide the student with a hands-on experience. For this reason, several experiments are designed to be performed by the instructor or by the students to provide the “missing” intuition. The floating body stability is not intuitive as the stability of placing body on solid surface and requires demonstration for various aspects. Even professionals could benefits from these experiments due to the fact that many of these aspects are totally innovative.

The first experiment deals with the stability diagram of floating bodies. Every geometry of floating body has its own stability diagram and diagram demonstrates three zones: stable, unstable, and neutral situations. The second experiment deals again with stability diagram but for different geometries. The third experiment the measurements of  $GM$  of the cylinder are carried out. The natural frequency of the floating body is measured for constant cross. The change of the rotation location is estimated. The principle axes experiment in which general shape of the “preferred” axes of rotation is examined.

The every experiment will be written as a separated document (that is, start from a new page) and appear in this collection.

## 14.2 Stability Points of Various Bodies

### 14.2.1 Concept

Every body has a stability points (angles or positions) either on solid surface or in liquid (while floating) for which the body is stable. The number of these points and their angle depends on the geometry of the body. The intention of this experiment is to build intuition in regards to the stability points and demonstrate the unpredictability of these angles. This discussion is for the uniform density bodies.

### 14.2.2 Description

The existences of stability points is a fundamental concept in the stability analysis which could be carried out by high school students , university students and by professionals. The experiment is designed to demonstrate the existence of clear points (angles) in which the body is stable around. At stability points, the body is stable while at other angles the body is unstable. These points are different from the angles when the body is at rest on solid surface. The neutral stability is also demonstrated in this experiment.

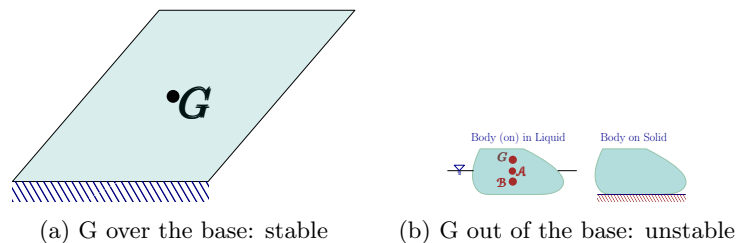


Fig. 14.1 – Body on solid surface stable and unstable.

These bodies shown in the Fig. 14.1 have different stability points on liquid. The body depicted in Fig. 14.1a has four point on solid and floating on liquid. While the body depicted in Fig. 14.1a has two points but on liquid these point are at different angles.

### 14.2.3 Apparatus

The experiment is based on observation and not measurement and is relatively simple to carry out. Yet, there are several issues to be pointed out, placing body on solid surface compared to placing the body in liquid, estimating the number stable points (again should be compared to solid), comparison change of one parameter of the geometry, and change of density ratio. A container (preferred transparent) with an ability to contain enough liquid in which the floating bodies can to be examined. Several bodies: extruded squire, extruded circular, extruded rectangular (with different ratios). It also should include a body similar to the one depicted Fig. 14.1b and one similar to one depicted in Fig. 14.1a. The same bodies should be made out two different materials (light and heavier like a styrofoam, and wood).

**14.2.4 Procedure**

State what are your predicting angles prior the experiments and tabulate the prediction in the following tables.

Table 14.1 – Angle of Stable points Styrofoam

Shape	Solid		Liquid	
	Prediction	Actual	Prediction	Actual
square				
rec ratio 1:1.1				
rec ratio 1:1.2				
rec ratio 1:1.3				
rec ratio 1:1.4				
rec ratio 1:1.5				
rec ratio 1:1.6				
rec ratio 1:2				
cylinder				
ball				

Table 14.2 – Angle of Stable points Oak

Shape	Solid		Liquid	
	Prediction	Actual	Prediction	Actual
square				
rec ratio 1:1.1				
rec ratio 1:1.2				
rec ratio 1:1.3				
rec ratio 1:1.4				
Continued on next page				

Table 14.2 – Angle of stable point Oak (continue)

Shape	Prediction	Actual	Prediction	Actual
rec ratio 1:1.5				
rec ratio 1:1.6				
rec ratio 1:2				
cylinder				
ball				

Place the bodies on solid surface and observed the results. Place the cylinder in different angle and observed if the cylinder change or stay in the placed angle. Repeat the same for the ball.

### 14.2.5 Data Analysis

Calculate the density of the oak and styrofoam. Using the information build a stability diagram for the rectangular shape.

### 14.2.6 Additional Questions

1. Is possible to build the same diagram for triangle?
2. Is possible to build this diagram for other shape? what are the requirements for this diagram on the geometry?



## 14.3 Cylinder Stability Diagram

### 14.3.1 Concept

This laboratory experiment is an exercise in stability. It is designed to demonstrate the stability of floating bodies and to familiarize the student with the concepts of buoyancy, neutral and stable and unstable situations, and stability dome. Every floating body has a stability diagram which is depicting stability for certain angle. The vertical position (angle) is often the most alluring for many applications. For example, rectangle in a vertical position is useful for marine applications.

### 14.3.2 Description

The stability of the floating body for specific angle has what known as the stability dome. The floating body buoyancy centroid,  $\mathcal{B}$ , depends on the body shape and on its orientation.

A floating body has several stable positions (angles). The vertical position mimic to some extent the shape of ship. A position can be stable under a certain density ratio range. The stability dome is introduced via the change of the density ratio. In other words, the experiment is designed to investigate the change density effect on the stability. The change of the density effect is obtained by adding sand (or liquid) to the cylinder. The additional material add weight and hence the effective density  $c$  changes. The schematic of the experiment is shown diagram Fig. 14.1.

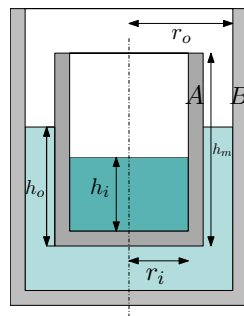


Fig. 14.1 – Schematic the stability dome Apparatus.

Note that the gravity centroid changes also during filing process and it is accounted by introducing the effective density ratio. The effective density concept is emulate body with a constant density.

This experimental is verification of the theory presented in this textbook. The dome shown in stability map represents the limits of floating body stability. The effective density of the floating body ( $\rho_s$ ) is controlled by the amount liquid in the cylinder. A discussion about the effective density is below. Thus, the density ratio can be varied by filling the cylindrical container with liquid or sand.

The core issue of the floating body stability is the locations of gravity centroid and buoyancy centroid. The gravity is fixed in case of uniformed density body. In

this case the gravity depends on the liquid level in the inner container (A). The effective density ratio (or the simulated solid body density) is changed by amount liquid poured into cylinder. As opposed to the fix case the buoyancy centroid can be calculated from the middle of the height of the inner cylinder (regardless of the thickness of the container.)

### 14.3.3 Apparatus

There are three components of the experimental set-up shown in Fig. 14.1:

1. Large vertically-standing cylinder containing liquid (fresh water) denoted in figure as B
2. Small cylinder with a detachable cap at one end denoted in figure as A
3. weighing scale

Insert the inner container to the large container empty. Fill the inner container with small incremental amount of liquid (see Fig. 14.1). Measure the height of the filling the liquid and records the mount. Check if the inner container is stable.

## 14.4 The procedure

1. Measure the weight the inner container and indicate it on the report
2. Fill the exterior container with liquid (fresh water or oil, etc)
3. Inserted the inner container into the larger container
4. Measure the thickness inner container wall
5. Measure and record the height  $h_o$ . This height is the initial height,  $h_{o0}$ .
6. Check if the cylinder is stable and record this information
7. Add small amount of liquid into the inner container.
8. Messure  $h_i$  and  $h_o$
9. Go item 6 if the inner cylinder is not full
10. If the inner cylinder is full, it is the end of the process.

### 14.4.1 Data Analysis

The data analysis is made from two sections: the initial and the process parts. The initial part deals with calibrations.

**Initial**

1. Calculate the density of the liquid using the weight
2. Use the value of  $h_{o0}$  to calculate and verify the weight of the empty inner cylinder,  $w_{i0}$ .

$$w_{i0} = h_2 \pi r_1^2 \rho_\ell \tag{14.1}$$

3. Calculate the initial gravity location of the empty inner container, (assuming thin walls)

$$G_0 = \frac{h_m^2 \pi r_o t}{\pi r_o^2 t + 2 h_m \pi r_o t} = \frac{h_m^2}{r_o + 2 h_m} \tag{14.2}$$

4. Calculate the buoyancy centroid using  $B_0 = \frac{h_o}{2}$

**Process**

Repeat the following steps for each additional liquid amount:

5. Calculate the buoyancy centroid as  $B = h_o/2$
6. Calculate the effective gravity centroid by

$$G_{eff} = \frac{w_{i0} G_0 + 0.5 h_i^2 \pi r_i^2 g \rho_\ell}{w_{i0} + h_i \pi r_i^2 g \rho_\ell} \tag{14.3}$$

7. Compare  $G_{eff}$  and  $h_B$ . If  $G_{eff} > B$  the experiment is not valid (it is not addressed the case of  $G$  over/above  $B$ ) for uniform density. This data point should be ignored. Continue the next data point.
8. The density ratio can be calculated by

$$\rho_s = \frac{2 h_G}{h_o} \rho_\ell \tag{14.4}$$

9. Draw stability dome for the inner container.

Table 14.1 – Table for the calculating various value

<i>index</i>	$h_o$	$h_i$	$B$	$G_{eff}$	$\rho_s$
1					

Continued on next page

Table 14.1 – Table for the calculating various values (continue)

<i>index</i>	$h_0$	$h_i$	$\mathcal{B}$	$G_{eff}$	$\rho_s$
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					

### 14.4.2 Additional Questions

Questions to the students:

1. Under what conditions the fact that the liquid (or other moving surfaces) is not relevant to the stability analysis?
2. Show that Eq. (14.3) the density of the liquid is irrelevant. Simplify the equation.
3. Explain how Eq. (14.3) was constructed.
4. What is the physical limit of this experiment. Why the experiment is limited?

## 14.5 $GM$ experiments

### 14.5.1 Concept

Simply stating, a body is stable if a small rotation is applied and body return to its original state. In the old method is done by measuring the  $GM$  as a marker of the stability. The body is stable when  $GM$  is positive. Note the  $GM$  is monotonic but not a linear function, that is,  $GM=2[m]$  is more than  $1[m]$  but not double in the “stability.”

### 14.5.2 Description

In this experiment, the stability marker,  $GM$ , is obtained for specific geometry. In doing so the concepts of stability, and metacenter are introduced. This concept was discussed in chapter 9 and here it is verified. Due to fact this material could be used as a stand alone this description is added.

The buoyancy centroid depends on submerged geometry. Assuming quasi steady state, that is the body in equilibrium with the liquid in the  $z$  direction and, forces act on the body gravity and buoyancy are equal. However, the only deviation from the quasi steady state is in the roll (change in angle of the body in the  $yz$  plane as defined in chapter 2).

Buoyancy depends on the body shape and orientation (angle). When the body is tilted in a small amount the buoyancy centroid moves as well. Thus, the buoyancy centroid movement is such that  $\mathcal{B}$  moves

to two possible zones with a border (the border is not shown explicitly) as shown in Fig. 14.1. The right side is stable (blue color) and the unstable (brown color). The border is made by a circular body which is neutral stability (under the  $G$  point). The point of intersection of the lines of vertical line from the new buoyancy centroid with the previous vertical line that goes through the  $\mathcal{B}$  is referred as the metacenter ( $\mathcal{M}$ ) and the distance between the center of gravity ( $G$ ) and  $\mathcal{M}$ , is referred as the metacentric height ( $GM$ , see Fig. 14.1).

The expression for the metacentric height  $GM$  is

$$GM = \frac{I_{xx}}{V_0} - GB \quad (14.1)$$

where  $I_{xx}$  is the floating body moment of inertia at the liquid line,  $GB$  is the distance between the gravity centroid and the old buoyancy centroid and  $V_0$  is volume of the displaced liquid. For stability the metacentric height  $GM$  must be positive. Stability (restoring force) increases with increasing  $GM$ .

In this experiment, the mission to estimate the metacentric height for several floating bodies.

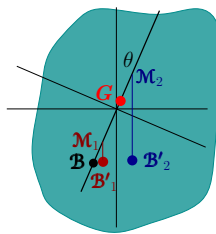


Fig. 14.1 – Genral change of  $\mathcal{B}$  with two possibilities.

### 14.5.3 Rig

There are three components of the experimental set-up:

1. A large pot (preferred in a cylinder shape) containing liquid (the liquid can be salt water, fresh water, oil, etc.),
2. scale,
3. a additional sand or liquid,
4. relatively smaller pot to be fitted into the first pot (thus it should be made from cylindrical shape or, at least, both can be any shape that fitted into each other proffered to be in a form that easy to calculate (or pre-calibrated) the volume), and,
5. and a metric balance to weigh the cylinders.

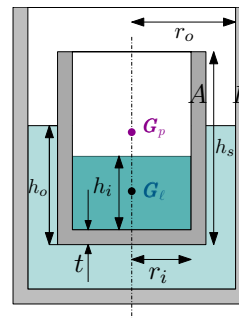


Fig. 14.2 – Apparatus for  $GM$ .

### 14.5.4 Procedure

1. Measure the weigh of the small pot make it  $w_s$
2. Place the small empty cylinder into the large vertical cylinder filled with liquid and observe stability (whether or not the cylinder is stable).
3. Add a small amount of sand/liquid into the small cylinder and describe if the small cylinder is unstable. Measure the distance of the small cylinder is submerged,  $h_o$ , and then measure the height of liquid/sand,  $h_i$ .
4. Continue adding sand until the small cylinder remains vertical, i.e., stable. Measure the submerged amount of cylinder. (This is the stability boundary)
5. Continue the process until the cylinder become stable again. Mark that point as the second stability boundary on the  $GM$  diagram or the stability diagram.
6. Remove the small cylinder and measure its weight. Also measure the height of sand and record your results on the data sheet.

### 14.5.5 Measurements

According to procedures described above, measure the following quantities:

Table 14.1 – Measurement table

Measurement Table					
Case	Mass (gm)	$h[cm]$	$h_i[cm]$		
Unstable					
Stable					

### 14.5.6 Data Analysis

Fig. 14.2 shows apparatus that used for this experiment which a cylinder (pot) inside another cylinder (pot). The inside cylinder is filled by material. The cylinder floats in a liquid with density of  $\rho_\ell$ , as it marked in schematic. The buoyancy centroid is at  $\mathcal{B} = h_o/2$ . The center of the actual gravity centroid,  $\mathcal{G}$ , is given by

$$\mathcal{G} = \frac{w_i (h_i/2 + t) + w_A h_s}{w_i + w_A} \tag{14.2}$$

The metacenter is given (assuming that it is cylinder) by

$$GM = \frac{\pi r_i^4}{\pi r_i^2} - (\mathcal{G} - \mathcal{B}) \tag{14.3}$$

Table 14.2 – Trapezoid properties and equations

The calculated values							
$h_o$	$h_i$	$\mathcal{B}$	$\mathcal{G}$	$GM$			

Continued on next page

Table 14.2 – Basic Cone properties (continue)

The calculated values							
$h_o$	$h_i$	$\mathcal{B}$	$G$	$GM$			

### 14.5.7 Additional Questions

1. In the vertical position, how many stable positions are there?
2. Is there any other layout or angles of the cylinder beside the vertical position that body is in stable positions? (hint, check the photos in this book) When the cylinder is layout in the liquid as shown in Fig. 14.3 are considered as neutral stable (that is, consider the symmetrical shape and its orientation in  $x$  coordinate as defined for ships.). What are the conditions that these cylinders for this to be true (hint, see the discussion about it in book). Is the cap at the cylinder end affects this neutral symmetry.
3. Consider a hollow circular cylinder of homogeneous material of specific gravity,  $s = 0.5$ , length,  $L$ , and diameter,  $D$ , stable at the position shown in Fig. 3. Find the formulas for the immersion of the cylinder  $h$  at this position.

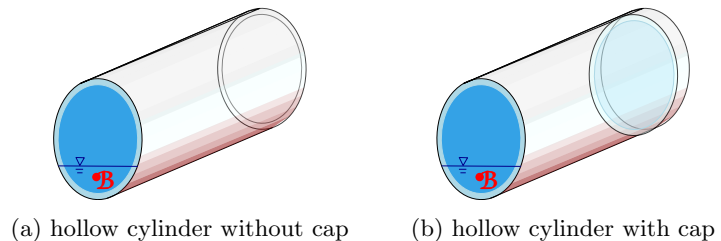


Fig. 14.3 – cylinder laying on liquid with and without cap.

4. Consider a hollow cylinder made with two plates of thickness  $t = \xi r$ , and the density of the solid is  $a \rho_\ell$ . Assume the cylinder is filled with air which mass can be ignored. what the range of  $\xi$  that this cylinder will float?
5. A hollow cylinder is with wall of uniform thickness,  $t$  is floating vertically in liquid. The ratio of the height and ratios of cylinder are  $L$  and  $r$  respectively. Determine the ranges of  $r/L$  for which the positions of the vertical cylinder is stable. The density of the liquid is  $\rho_\ell$ .



### 14.5.8 References

## Meta

Dr. Robert Ettema 57:020 Mechanics of Fluids & Transfer Processes Laboratory  
Experiment #2

ERIC SAVORY

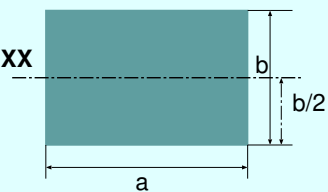
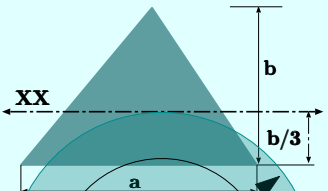
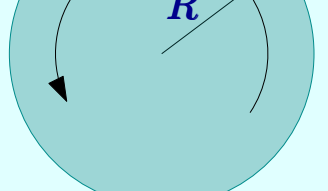
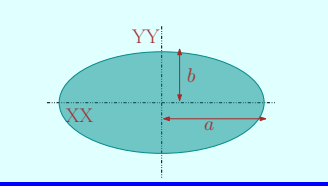
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## Meta End



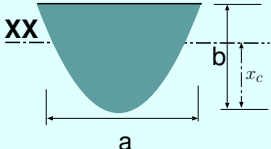
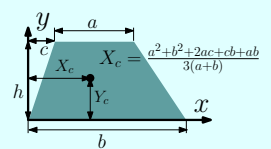
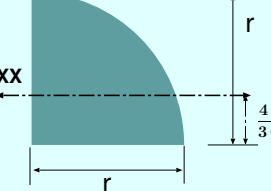
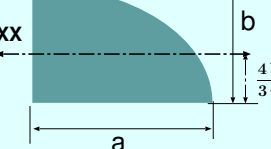
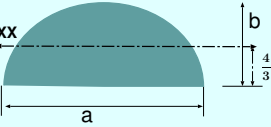
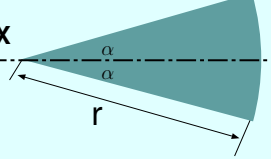
# Data

Table 1 – Moment of Inertia and Other Data

Shape Name	Geometry	Centroid	Area	$I$
Rectangle		$\frac{b}{2} ; \frac{a}{2}$	$ab$	$\frac{ab^3}{12}$
Triangle		$\frac{a}{3}$	$\frac{ab}{3}$	$\frac{ab^3}{36}$
Circle		$\frac{b}{2}$	$\frac{\pi b^2}{4}$	$\frac{\pi b^4}{64}$
Ellipse		$\frac{a}{2} \frac{b}{2}$	$\frac{\pi ab}{4}$	$\frac{ab^3}{64}$

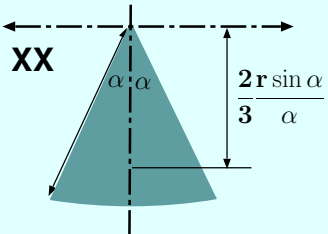
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Table 1 – Moment of inertia (continue)

Name	Geometry	Centroid	Area	<i>I</i>
$y = \alpha x^2$ <b>Parabola</b>		$\frac{3\alpha b}{15\alpha - 5}$	$\frac{6\alpha - 2}{3} \times \left(\frac{b}{\alpha}\right)^{\frac{3}{2}}$	$\frac{\sqrt{b}(20b^3 - 14b^2)}{35\sqrt{\alpha}}$
<b>Trapezoid</b>		$Y_c = \frac{h(2a+b)}{3(a+b)}$	$\frac{h(a+b)}{2}$	$\frac{h^3(3a+b)}{12}$
<b>Quadrant of Circle</b>		$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$	$r^4 \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)$
<b>Ellipsoidal Quadrant</b>		$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$	$ab^3 \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)$
<b>Half of Elliptic</b>		$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$	$ab^3 \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)$
<b>Circular Sector</b>		$0$	$2\alpha r^2$	$\frac{r^4}{4} \left(\alpha - \frac{1}{2} \sin 2\alpha\right)$

Continued on next page

Table 1 – Moment of inertia (continue)

Name	Geometry	Centroid	Area	$I$
Circular Sector		$\frac{2}{3} \frac{r \sin \alpha}{\alpha}$	$2\alpha r^2$	$I_{x'x'} = \frac{r^4}{4} \left( \alpha + \frac{1}{2} \sin 2\alpha \right)$



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