INVARIANCE ANALYSIS AND SOME NEW EXACT ANALYTIC SOLUTIONS OF THE TIME-FRACTIONAL COUPLED DRINFELD-SOKOLOV-WILSON EQUATIONS

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Abstract. In this work, the fractional Lie symmetry method is used to find the exact solutions of the time-fractional coupled Drinfeld-Sokolov-Wilson equations with the Riemann-Liouville fractional derivative. Time-fractional coupled Drinfeld-Sokolov-Wilson equations are obtained by replacing the first-order time derivative to the fractional derivatives (FD) of order α in the classical Drinfeld-Sokolov-Wilson (DSW) model. Using the fractional Lie symmetry method, the Lie symmetry generators are obtained. With the help of symmetry generators, FCDSW equations are reduced into fractional ordinary differential equations (FODEs) with Erdélyi-Kober fractional differential operator. Also, we have obtained the exact solution of FCDSW equations and shown the effects of non-integer order derivative value on the solutions graphically. The effect of fractional order α on the behavior of solutions are studied graphically. Finally, new conservation laws are constructed along with the formal Lagrangian and fractional generalization of Noether operators. It is quite interesting the exact analytic solutions are obtained in explicit form.

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1. Introduction

Water waves in oceans is always a topic of great interest for researchers. However, the phenomena is very complex and no single model is available that may describe the fully non-linear dynamics of the waves. Many mathematical models have been proposed by researchers for simpler case where the pressure difference in vertical direction is negligible. It means the horizontal length scale (wavelength) is larger than the depth of the fluid. Further, the fluid is ir-rotational. In the recent few years, many modified models of the problem have been proposed by researchers by using Fractional Derivatives (FD) (a derivative of arbitrary order). However, the topic of fractional derivatives is quite old and reported first in 1695 by G.W. Leibniz, its application remains limited till last century. In the last decade, researchers have recognised the usefulness of FD to adequately model a problem than the traditional integer derivatives. Fractional order differential equations are the generalizations of classical integer-order differential equations. Many researchers are devoted to the interpretation, properties and applications of fractional order integrals and derivatives [?, ?, ?]. In the recent years, fractional differential equations (FDEs) have been studied frequently to model various physical problems in hydrology, visco-elasticity, mechanics, neurons, image processing, physics, control-theory, electrochemistry and finance [?, ?, ?, ?, ?, ?, ?]. Fractional differential equations (FDEs) attract considerable interest in the various fields of engineering and science. Many powerful and efficient methods have been

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developed to obtain the exact and numerical solutions of FDEs like (G'/G)-expansion method [?], functional variable method [?], modified trial equation method [?], exponential function method [?], sub equational method [?], homotopy perturbation method [?] and so on.

The Lie symmetry method was first introduced by Sophus Lie [?]. The main aim of this method is to obtain the infinitesimal generators which leaves the considered differential equation invariant in form. This method provides powerful structure in the working of differential equations. Adapting the Lie group analysis method and proposing the prolongation formulas for fractional derivatives, Gazizov et al. [?] studied the symmetry properties of fractional order differential equations with the help of Riemann-Liouville and Caputo fractional derivatives [?]. Since few attributes of FD are non-identical to the integer order derivatives, obtaining the Lie symmetries and the conservation laws for fractional partial differential equations (FPDEs) are of interest for researchers. Despite the importance of conservation laws in internal properties and, existence and uniqueness analysis of differential equations, the conservation laws for FDEs are not widely discussed. The real-world physical processes can be better modeled by FDEs rather by integer order differential equations. Fractional order representations possess long memory characteristics that makes the system behave in more realistic manner.

Most phenomena in Physics, Astrophysics and fluid dynamics are non-linear in nature. Among these non-linear phenomena, the dynamics of water waves in ocean is quite fascinating. Many models have been proposed for the shallow water waves involving mostly integer order derivatives. Fractional order derivative significantly affects the properties of the equation. In the recent few years, lot of studies have appeared on the "Fractional Derivatives (FD)" to model these problems more accurately. We have considered the time-fractional coupled Drinfeld-Sokolov-Wilson equations (FCDSW) [?, ?] which are used are used to describe the model of shallow water waves.

$$D_t^{\alpha} u + a w w_x = 0, \tag{1.1}$$

$$D_t^{\alpha} w + b w_{xxx} + c w u_x + d u w_x = 0. \tag{1.2}$$

Here α denotes the order of FD with $0 < \alpha \leq 1$ and a, b c and d are nonzero constants. For $\alpha = 1$, Eqs. (??)-(??) represent the classical DSW equations, which was first introduced by Drienfel'd and Sokolov [?, ?] and studied by Wilson [?].

In the recent years, many numerical and analytical solutions of FCDSW equations have been presented by the researchers due to its wide applications in the modeling of water waves. Yao et al. [?] have studied the bifurcations of travelling wave solutions of generalized DSW equations. Recently, Sahoo and Ray [?] have obtained the double-periodic solutions of FCDSW equations in the shallow water waves.

In this work, we have obtained the explicit solution expression of FCDSW equations. We have applied fractional Lie group method to obtain the symmetry properties and conservation laws for the FCDSW equations. Using the Lie symmetry transformations, the time fractional coupled DSW equations are reduced into fractional differential equations with Erdélyi-Kober operator. The fractional order differential equations are in Euler-Lagrange forms. Therefore, the conservation laws of these equations is obtained by using the Noether's theorem by Lie symmetries.

The primary contents of this paper are as follows: A brief introduction is given in detail about the fractional order differential equations in section 1. Section 2 contains preliminaries in which definition of fractional derivative and the basic idea of fractional Lie symmetry method are presented. In section 3, we have applied the proposed method to the FCDSW equations and determined the symmetry generators of fractional equations. Using these symmetry generators, FCDSW equations are reduced into the fractional ordinary differential equations with the help of Erdélyi-Kober fractional differential operator with Riemann fractional derivative. In section

4, exact solution of FCDSW equations is obtained and the nonlinearity property of the solution is studied with the help of two and three dimensional plots. In section 5, new conservation laws have been developed along with new conserved vectors using new conservation theorem and fractional generalization of Noether operators. In section 5, conclusion is presented about the whole study.

2. Preliminaries

2.1. **Definition of Fractional derivative.** The Riemann-Liouville FD of a function $u(x,t)$, for order $\alpha > 0$, is defined as follows [?, ?, ?, ?]:

$$D_t^{\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\frac{\partial^m}{\partial t^m}\int_0^t (t-\sigma)^{m-\alpha-1}u(\sigma,x)d\sigma, & m-1 \leq \alpha \leq m, m \in N, \, t>0, \\ \frac{\partial^m}{\partial t^m}, & \alpha = m \in N, \end{cases} \tag{2.1}$$

where Γ denotes Euler's gamma function.

2.2. **Basic Idea of the Proposed Fractional Lie Symmetry Method.** We have considered the coupled time fractional non-linear PDEs with two independent variables given in the following form:

$$D_t^{\alpha}u = f(x,t,u,u_x,u_t,u_{xx},u_{xxx},w,w_x,w_t,w_{xx},w_{xxx},...), \tag{2.2}$$

$$D_t^{\alpha}w = g(x,t,u,u_x,u_t,u_{xx},u_{xxx},w,w_x,w_t,w_{xx},w_{xxx},...), \tag{2.3}$$

where $\alpha > 0$ and subscripts represent the partial derivatives.

Consider the following symmetry generator of one-parameter Lie group of transformations under which Eqs. (??) and (??) remain invariant

$$\begin{aligned} \tilde{x} &= x + \epsilon\xi(x,t,u,w) + O(\epsilon^2), \\ \tilde{t} &= t + \epsilon\tau(x,t,u,w) + O(\epsilon^2), \\ \tilde{u} &= u + \epsilon\eta(x,t,u,w) + O(\epsilon^2), \\ \tilde{w} &= w + \epsilon\nu(x,t,u,w) + O(\epsilon^2), \\ D_t^{\alpha}\tilde{u} &= D_t^{\alpha}u + \epsilon\eta^{\alpha,t}(x,t,u,w) + O(\epsilon^2), \\ D_t^{\alpha}\tilde{w} &= D_t^{\alpha}w + \epsilon\nu^{\alpha,t}(x,t,u,w) + O(\epsilon^2), \\ \frac{\partial\tilde{u}}{\partial\tilde{x}} &= \frac{\partial u}{\partial x} + \epsilon\eta^{x}(x,t,u,w) + O(\epsilon^2), \\ \frac{\partial\tilde{w}}{\partial\tilde{x}} &= \frac{\partial w}{\partial x} + \epsilon\nu^{x}(x,t,u,w) + O(\epsilon^2), \\ \frac{\partial^3\tilde{w}}{\partial\tilde{x}^3} &= \frac{\partial^3 w}{\partial x^3} + \epsilon\nu^{xxx}(x,t,u,w) + O(\epsilon^2), \end{aligned} \tag{2.4}$$

where ϵ is the group parameter and ξ, τ, η and ν are the infinitesimals of the transformations.

The infinitesimal generator X can be written in following form:

$$\mathbf{X} = \xi(x,t,u,w)\partial_x + \tau(x,t,u,w)\partial_t + \eta(x,t,u,w)\partial_u + \nu(x,t,u,w)\partial_w. \tag{2.5}$$

The kth order prolongation of the fractional vector field is given as

$$\begin{aligned} Pr^{(\alpha,k)}\mathbf{X} =& \mathbf{X} + \eta^{\alpha,t}\frac{\partial}{\partial u_t^{\alpha}} + \eta^{x}\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + ... + \eta^{xx...i_k}\frac{\partial}{\partial u_{xx...i_k}} \\ &+ \nu^{\alpha,t}\frac{\partial}{\partial w_t^{\alpha}} + \nu^{x}\frac{\partial}{\partial w_x} + \nu^{xx}\frac{\partial}{\partial w_{xx}} + ... + \nu^{xx...i_k}\frac{\partial}{\partial w_{xx...i_k}}, \quad k \geq 1. \end{aligned} \tag{2.6}$$

where the operators η^i and ν^i are extended infinitesimals [?] and $\eta^{\alpha,t}$, $\nu^{\alpha,t}$ are the fractional extended infinitesimals defined as follows:

$$\begin{aligned}
\eta^{\alpha,t} &= D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(u(D_t\tau)) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u),\\
\nu^{\alpha,t} &= D_t^\alpha(\nu) + \xi D_t^\alpha(w_x) - D_t^\alpha(\xi w_x) + D_t^\alpha(w(D_t\tau)) - D_t^{\alpha+1}(\tau w) + \tau D_t^{\alpha+1}(w),\\
\eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),\\
\nu^x &= D_x(\nu) - w_x D_x(\xi) - w_t D_x(\tau),\\
\nu^{xx} &= D_x(\nu^x) - w_{xx} D_x(\xi) - w_{xt} D_x(\tau),\\
\nu^{xxx} &= D_x(\nu^{xx}) - w_{xxx} D_x(\xi) - w_{xxt} D_x(\tau),
\end{aligned} \tag{2.7}$$

where D_x and D_t denote the total derivatives with respect to independent variables, defined as

$$\begin{aligned}
D_t &= \partial_t + u_t\partial_u + w_t\partial_w + u_{tt}\partial_{u_t} + w_{tt}\partial_{w_t} + u_{xt}\partial_{u_x} + w_{xt}\partial_{w_x}...,\\
D_x &= \partial_x + u_x\partial_u + w_x\partial_w + u_{xx}\partial_{u_x} + w_{xx}\partial_{w_x} + u_{tx}\partial_{u_t} + w_{tx}\partial_{w_t}....
\end{aligned} \tag{2.8}$$

Now, we focus on the expressions for $\eta^{\alpha,t}$ and $\nu^{\alpha,t}$. The generalized Leibnitz rule is given by

$$D_t^\alpha(f(t)h(t)) = \sum_{m=0}^{\infty} \binom{\alpha}{m} D_t^m f(t) D_t^{\alpha-m} h(t), \tag{2.9}$$

where

$$\binom{\alpha}{m} = \frac{\Gamma(\alpha+1)}{\Gamma(m+1)\Gamma(\alpha+1-m)}.$$

Now, using Leibnitz's rule (**??**) in the expressions of $\eta^{\alpha,t}$ and $\nu^{\alpha,t}$, we have

$$\begin{aligned}
\eta^{\alpha,t} &= D_t^\alpha(\eta) - \alpha D_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{m=0}^{\infty}\binom{\alpha}{m} D_t^m(\xi) D_t^{\alpha-m} u_x - \sum_{m=0}^{\infty}\binom{\alpha}{m+1} D_t^{m+1}(\tau) D_t^{\alpha-m} u\\
\nu^{\alpha,t} &= D_t^\alpha(\nu) - \alpha D_t(\tau)\frac{\partial^\alpha w}{\partial t^\alpha} - \sum_{m=0}^{\infty}\binom{\alpha}{m} D_t^m(\xi) D_t^{\alpha-m} w_x - \sum_{m=0}^{\infty}\binom{\alpha}{m+1} D_t^{m+1}(\tau) D_t^{\alpha-m} w.
\end{aligned} \tag{2.10}$$

The chain rule for a composite function is as follows (see [?]):

$$\frac{d^\alpha f(g(t))}{dt^\alpha} = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-g(t)]^r \frac{d^k f(g)}{df^k}\frac{\partial^\alpha}{\partial t^\alpha}[(g(t))^{k-r}]. \tag{2.11}$$

Using Eqs. (**??**) and (**??**) in Eqs. (**??**) with $f(t) = 1$, we have

$$\begin{aligned}
D_t^\alpha(\eta) =& \partial_t^\alpha \eta + \Big(\eta_u \partial_t^\alpha u - u\partial_t^\alpha \eta_u\Big) + \Big(\eta_w \partial_t^\alpha w - w\partial_t^\alpha \eta_w\Big) + \sum_{m=1}^{\infty}\binom{\alpha}{m}\partial_t^m \eta_u D_t^{\alpha-m} u\\
&+ \sum_{m=1}^{\infty}\binom{\alpha}{m}\partial_t^m \eta_w D_t^{\alpha-m} w + \mu_1 + \mu_2,\\
D_t^\alpha(\nu) =& \partial_t^\alpha \nu + \Big(\nu_w \partial_t^\alpha w - w\partial_t^\alpha \nu_w\Big) + \Big(\nu_u \partial_t^\alpha u - u\partial_t^\alpha \nu_u\Big) + \sum_{m=1}^{\infty}\binom{\alpha}{m}\partial_t^m \nu_w D_t^{\alpha-m} w\\
&+ \sum_{m=1}^{\infty}\binom{\alpha}{m}\partial_t^m \nu_u D_t^{\alpha-m} u + \lambda_1 + \lambda_2,
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
\mu_1 &= \sum_{m=2}^{\infty}\sum_{n=2}^{m}\sum_{j=2}^{n}\sum_{r=0}^{j-1}\binom{\alpha}{m}\binom{m}{n}\binom{j}{r}\frac{1}{j!}\frac{t^{m-\alpha}}{\Gamma(m+1-\alpha)}(-u)^r\frac{\partial^n}{\partial t^n}(u^{j-r})\frac{\partial^{m-n+j}\eta}{\partial t^{m-n}\partial u^j},\\
\mu_2 &= \sum_{m=2}^{\infty}\sum_{n=2}^{m}\sum_{j=2}^{n}\sum_{r=0}^{j-1}\binom{\alpha}{m}\binom{m}{n}\binom{j}{r}\frac{1}{j!}\frac{t^{m-\alpha}}{\Gamma(m+1-\alpha)}(-w)^r\frac{\partial^n}{\partial t^n}(w^{j-r})\frac{\partial^{m-n+j}\eta}{\partial t^{m-n}\partial w^j},\\
\lambda_1 &= \sum_{m=2}^{\infty}\sum_{n=2}^{m}\sum_{j=2}^{n}\sum_{r=0}^{j-1}\binom{\alpha}{m}\binom{m}{n}\binom{j}{r}\frac{1}{j!}\frac{t^{m-\alpha}}{\Gamma(m+1-\alpha)}(-u)^r\frac{\partial^n}{\partial t^n}(u^{j-r})\frac{\partial^{m-n+j}\nu}{\partial t^{m-n}\partial u^j},\\
\lambda_2 &= \sum_{m=2}^{\infty}\sum_{n=2}^{m}\sum_{j=2}^{n}\sum_{r=0}^{j-1}\binom{\alpha}{m}\binom{m}{n}\binom{j}{r}\frac{1}{j!}\frac{t^{m-\alpha}}{\Gamma(m+1-\alpha)}(-w)^r\frac{\partial^n}{\partial t^n}(w^{j-r})\frac{\partial^{m-n+j}\nu}{\partial t^{m-n}\partial w^j}.
\end{aligned}\tag{2.13}$$

Thus, Eq. (**??**) yields

$$\begin{aligned}
\eta^{\alpha,t} =& \partial_t^\alpha\eta + \Big(\eta_u - \alpha D_t(\tau)\Big)\partial_t^\alpha u - u\partial_t^\alpha\eta_u + \Big(\eta_w\partial_t^\alpha w - w\partial_t^\alpha\eta_w\Big) + \mu_1 + \mu_2\\
&+ \sum_{m=1}^{\infty}\left[\binom{\alpha}{m}\partial_t^m\eta_u - \binom{\alpha}{m+1}D_t^{m+1}(\tau)\right]D_t^{\alpha-m}(u) + \sum_{m=1}^{\infty}\binom{\alpha}{m}\partial_t^m\eta_w D_t^{\alpha-m}w\\
&- \sum_{m=1}^{\infty}\binom{\alpha}{m}D_t^m(\xi)D_t^{\alpha-m}u_x,\\
\nu^{\alpha,t} =& \partial_t^\alpha\nu + \Big(\nu_w - \alpha D_t(\tau)\Big)\partial_t^\alpha w - w\partial_t^\alpha\nu_w + \Big(\nu_u\partial_t^\alpha u - u\partial_t^\alpha\nu_u\Big) + \lambda_1 + \lambda_2\\
&+ \sum_{m=1}^{\infty}\left[\binom{\alpha}{m}\partial_t^m\nu_w - \binom{\alpha}{m+1}D_t^{m+1}(\tau)\right]D_t^{\alpha-m}(w) + \sum_{m=1}^{\infty}\binom{\alpha}{m}\partial_t^m\nu_u D_t^{\alpha-m}u\\
&- \sum_{m=1}^{\infty}\binom{\alpha}{m}D_t^m(\xi)D_t^{\alpha-m}w_x.
\end{aligned}\tag{2.14}$$

The infinitesimal generator X must satisfy the invariance conditions [**?**] for Eqs. (**??**) and (**??**), which are given as follows:

$$Pr^{(\alpha,\,k)}X(\Delta u)|_{\Delta u=0} = 0, \quad \text{and} \quad Pr^{(\alpha,\,k)}X(\Delta w)|_{\Delta w=0} = 0, \tag{2.15}$$

where $\Delta u = D_t^\alpha u - f$ and $\Delta w = D_t^\alpha w - g$.

3. **FCDSW equations**

3.1. **Lie symmetries.** The time fractional coupled DSW equations are as follows:

$$D_t^\alpha u + aww_x = 0, \tag{3.1}$$

$$D_t^\alpha w + bw_{xxx} + cwu_x + duw_x = 0, \tag{3.2}$$

where $D_t^\alpha(u)$ and $D_t^\alpha(w)$ are Riemann-Liouville FDs of order α with respect to t. Applying prolongation of fractional vector field on Eqs. (**??**) and (**??**), we obtain

$$\eta^{\alpha,t} + aw\nu^x + a\nu w_x = 0, \tag{3.3}$$

$$\nu^{\alpha,t} + cw\eta^x + c\nu u_x + dw\nu^x + d\eta w_x + b\nu^{xxx} = 0. \tag{3.4}$$

Now, using the Eqs. (**??**) and (**??**) in the Eqs. (**??**) and (**??**), we get the following set of infinitesimals for FCDSW equations:

$$\xi = \frac{1}{3}\alpha c_3 x + c_1, \quad \tau = c_3 t + c_2, \quad \eta = -\frac{2}{3}\alpha c_3 u \quad \text{and} \quad \nu = -\frac{2}{3}\alpha c_3 w, \tag{3.5}$$

where c_1, c_2 and c_3 are the arbitrary constants.

Since the lower limit of the integral (**??**) is fixed, therefore for preservation of its structure under the transformations (**??**), the condition $\tau(x,t,u,w)|_{t=0} = 0$ should hold. Therefore, c_2 must be zero (i.e. $\tau = c_3 t$).

So, the symmetry generators to form a Lie algebra of Eqs. (**??**) and (**??**) are found as:

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \tag{3.6}$$

$$\mathbf{X}_3 = \frac{1}{3}\alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}\alpha u\frac{\partial}{\partial u} - \frac{2}{3}\alpha w\frac{\partial}{\partial w}. \tag{3.7}$$

Theorem 3.1. *A solution $u = z_1(x,t)$ and $w = z_2(x,t)$, is invariant solutions of Eq. (**??**) and (**??**) iff*
*(i) $u = z_1(x,t)$ and $w = z_2(x,t)$ satisfy the FPDE (**??**) and (**??**), respectively.*
and
(ii) $u = z_1(x,t)$ and $w = z_2(x,t)$ are the invariant surfaces, i.e.
$\boldsymbol{X} z_1 = 0 \iff \left(\xi(x,t,u,w)\frac{\partial}{\partial_x} + \tau(x,t,u,w)\frac{\partial}{\partial_t} + \eta(x,t,u,w)\frac{\partial}{\partial_u} + \nu(x,t,u,w)\frac{\partial}{\partial_w}\right)z_1 = 0,$ *and*
$\boldsymbol{X} z_2 = 0 \iff \left(\xi(x,t,u,w)\frac{\partial}{\partial_x} + \tau(x,t,u,w)\frac{\partial}{\partial_t} + \eta(x,t,u,w)\frac{\partial}{\partial_u} + \nu(x,t,u,w)\frac{\partial}{\partial_w}\right)z_2 = 0.$

3.2. **Symmetry reduction of FCDSW equations.** In this section, we obtain the reduced equations for (**??**) and (**??**) by imposing the Lie symmetries. For the vector field $\mathbf{X}_3$, the characteristic equations will be

$$\frac{dx}{\alpha x} = \frac{dt}{3t} = \frac{du}{-2\alpha u} = \frac{dw}{-2\alpha w}. \tag{3.8}$$

After solving the Eq. (**??**), we obtain the following similarity variables:

$$z = xt^{-\frac{\alpha}{3}}, \quad u = f(z)t^{-\frac{2\alpha}{3}}, \quad w = g(z)t^{-\frac{2\alpha}{3}}. \tag{3.9}$$

Theorem 3.2. *The transformations (**??**) reduce the Eqs. (**??**) and (**??**) in the fractional non-linear ordinary equations given as follows:*

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} f\right)(z) + agg_z = 0, \tag{3.10}$$

and

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} g\right)(z) + cgf_z + dfg_z + bg_{zzz} = 0, \tag{3.11}$$

*with the Erdélyi-Kober fractional differential operator $P_\beta^{\tau,\alpha}$ [**?**, **?**] defined as*

$$\left(P_\beta^{\tau,\alpha} f\right) := \prod_{j=0}^{m-1}\left(\tau + j - \frac{1}{\beta}z\frac{d}{dz}\right)\left(K_\beta^{\tau+\alpha,m-\alpha} f\right)(z), \tag{3.12}$$

and

$$\left(P_\beta^{\tau,\alpha} g\right) := \prod_{j=0}^{m-1}\left(\tau + j - \frac{1}{\beta}z\frac{d}{dz}\right)\left(K_\beta^{\tau+\alpha,m-\alpha} g\right)(z), \tag{3.13}$$

where

$$m = \begin{cases} [\alpha]+1, & \alpha \notin N, \\ \alpha, & \alpha \in N, \end{cases} \tag{3.14}$$

$$\left(K_{\beta}^{\tau+\alpha,\, m-\alpha} f\right)(z) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (u-1)^{\alpha-1} u^{-(\tau+\alpha)} f\left(z u^{\frac{1}{\beta}}\right) du, & \alpha > 0, \\ f(z), & \alpha = 0, \end{cases} \tag{3.15}$$

and

$$\left(K_{\beta}^{\tau+\alpha,\, m-\alpha} g\right)(z) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (w-1)^{\alpha-1} w^{-(\tau+\alpha)} g\left(z w^{\frac{1}{\beta}}\right) dw, & \alpha > 0, \\ g(z), & \alpha = 0 \end{cases} \tag{3.16}$$

are the Erdélyi-Kober fractional integral operators [**?**].

Proof: When $m-1 < \alpha < m, m = 1, 2, 3, ...,$ from Riemann-Liouville FD, we have

$$D_t^{\alpha} u(x,t) = \frac{\partial^m}{\partial t^m} \left[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} s^{\frac{-2\alpha}{3}} f\left(x s^{\frac{-\alpha}{3}}\right) ds \right]. \tag{3.17}$$

Let $p = \frac{t}{s}$, then $ds = -\frac{t}{p^2} dp$.

So Eq. (**??**) can be expressed as

$$\begin{aligned} D_t^{\alpha} u(x,t) &= \frac{\partial^m}{\partial t^m} \left[\frac{t^{m-\frac{5\alpha}{3}}}{\Gamma(m-\alpha)} \int_1^{\infty} (p-1)^{m-\alpha-1} p^{m+1-\alpha-\frac{2\alpha}{3}} f\left(z p^{\frac{\alpha}{3}}\right) dp \right] \\ &= \frac{\partial^m}{\partial t^m} \left[t^{m-\frac{5\alpha}{3}} \left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},\, m-\alpha} f \right)(z) \right] \\ &= \frac{\partial^{m-1}}{\partial t^{m-1}} \left[\frac{\partial}{\partial t} \left(t^{m-\frac{5\alpha}{3}} \left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},\, m-\alpha} f \right)(z) \right) \right]. \end{aligned} \tag{3.18}$$

For $z = x t^{\frac{-\alpha}{3}}$ and a function $\phi(z) \in C^1(0,\infty)$, we get

$$t \frac{d}{dt} \phi(z) = t z_t \phi'(z) = t x \left(-\frac{\alpha}{3}\right) t^{\frac{-\alpha}{3}-1} \phi'(z) = -\frac{\alpha}{3} z \frac{d}{dz} \phi(z). \tag{3.19}$$

From relation (**??**), Eq. (**??**) can be re-written as follows:

$$D_t^{\alpha} u(x,t) = \frac{\partial^{m-1}}{\partial t^{m-1}} \left[\left(t^{m-1-\frac{5\alpha}{3}} \left(\left(m - \frac{5\alpha}{3} - \frac{\alpha}{3} z \frac{d}{dz} \right) \left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},\, m-\alpha} f \right)(z) \right) \right) \right]. \tag{3.20}$$

Repeating the same process $m-1$ times, we obtain

$$D_t^{\alpha} u(x,t) = t^{-\frac{5\alpha}{3}} \prod_{j=0}^{m-1} \left(1 - \frac{5\alpha}{3} + j - \frac{\alpha}{3} z \frac{d}{dz} \left(K_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3},\, m-\alpha} f \right)(z) \right). \tag{3.21}$$

Using Eq. (**??**) in Eq. (**??**), we obtain

$$D_t^{\alpha} u(x,t) = t^{-\frac{5\alpha}{3}} \left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\, \alpha} f \right)(z). \tag{3.22}$$

Therefore, Eq. (**??**) can be written into a non-linear FODE as follows:

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\, \alpha} f \right)(z) + a g g_z = 0. \tag{3.23}$$

For reducing Eq. (**??**), let $m-1<\alpha<m$, $m=1,2,3,\ldots$. Then, from the definition of Riemann-Liouville FD and using the similar procedure as above, we have

$$D_t^{\alpha} w(x,t) = t^{\frac{-5\alpha}{3}}\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} g\right)(z). \tag{3.24}$$

Therefore, Eq. (**??**) can be written into a non-linear FODE as follows:

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{5\alpha}{3},\alpha} g\right)(z) + cgf_z + dfg_z + bg_{zzz} = 0. \tag{3.25}$$

4. Exact analytic solution of the FCDSW equations

From the similarity analysis, we have $u = f(z)t^{a_1}$ and $w = g(z)t^{a_2}$, where $a_1 = a_2 = \frac{-2\alpha}{3}$ and $z = xt^{-b}, (b = \frac{\alpha}{3})$. In order to determine the exact explicit solution expression of FCDSW equations, let us first introduce $f(z) = A_1 z^{k_1}$ and $g(z) = A_2 z^{k_2}$, The parameters A_1, A_2, k_1 and k_2 are the real valued constants. As mentioned in the work of Costa et al. [**?**] and Bira et al. [**?**], we have

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t (t-r)^{-\beta} r^{a_1} f(xr^{-b})dr. \tag{4.1}$$

Putting $\tau = \frac{r}{t}$ in the above equation, we obtain

$$\begin{aligned}\frac{\partial^\beta u}{\partial t^\beta} &= \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^1 (1-\tau)^{-\beta} t^{a_1-\beta+1}\tau^{a_1} f(z\tau^{-b})d\tau \\ &= \frac{\partial}{\partial t}\left[t^{a_1-\beta+1}\left(F_\beta^{a_1,b} f\right)(z)\right],\end{aligned} \tag{4.2}$$

where

$$\left(F_\beta^{a_1,b} f\right)(z) = \frac{1}{\Gamma(1-\beta)}\int_0^1 (1-\tau)^{-\beta}\tau^{a_1} f(z\tau^{-b})d\tau.$$

After some manipulation, Eq.(**??**) can be rewritten as

$$\frac{\partial^\beta u}{\partial t^\beta} = t^{a_1-\beta}\left[\left(1-\beta+a_1-bz\frac{d}{dz}\right)\left(F_\beta^{a_1,b} f\right)(z)\right]. \tag{4.3}$$

In the similar manner, we can find $\frac{\partial^\beta w}{\partial t^\beta}$ as follows:

$$\frac{\partial^\beta w}{\partial t^\beta} = t^{a_2-\beta}\left[\left(1-\beta+a_2-bz\frac{d}{dz}\right)\left(F_\beta^{a_2,b} g\right)(z)\right]. \tag{4.4}$$

Substituting Eqs.(**??**) and (**??**) in Eqs.(**??**) and (**??**), we obtain

$$\begin{aligned}&\left[\left(1-\beta+a_1-bz\frac{d}{dz}\right)\left(F_\beta^{a_1,b} f\right)(z)\right] + agg_z = 0, \\ &\left[\left(1-\beta+a_2-bz\frac{d}{dz}\right)\left(F_\beta^{a_2,b} g\right)(z)\right] + bg_{zzz} + cgf_z + dfg_z = 0.\end{aligned} \tag{4.5}$$

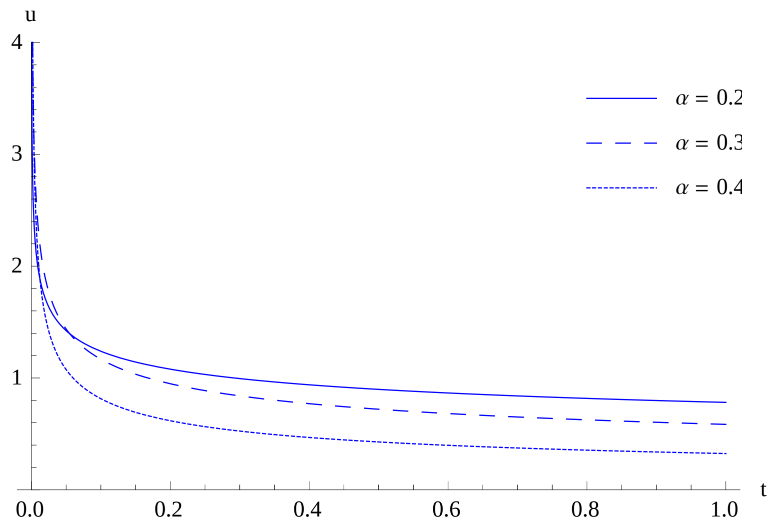
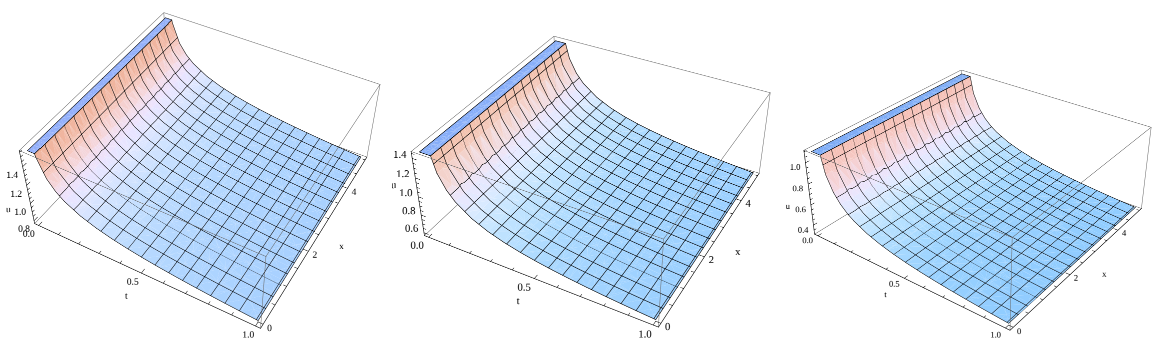


Fig. 1 2D profiles of the solution $u(x,t)$ for $0 < \alpha < 1$ and fixed x.



(a) (b) (c)

Fig. 2 3D profiles of the solution $u(x,t)$ for (a) $\alpha = 0.2$ (b) $\alpha = 0.3$ (c) $\alpha = 0.4$.

After calculating the operators $F_{\beta}^{a_1,\,b}(z^{k_1})$ and $F_{\beta}^{a_2,\,b}(z^{k_2})$ as in the work of Costa et al. [**?**], we obtain

$$\begin{aligned} F_{\beta}^{a_1,\,b}(z^{k_1}) &= \frac{1}{\Gamma(1-\beta)} \int_0^1 (1-\tau)^{-\beta} \tau^{a_1} z^{k_1} \tau^{-bk_1} d\tau \\ &= \frac{\Gamma(1+a_1-bk_1)}{\Gamma(2+a_1-bk_1-\beta)} z^{k_1}, \\ F_{\beta}^{a_2,\,b}(z^{k_2}) &= \frac{\Gamma(1+a_2-bk_2)}{\Gamma(2+a_2-bk_2-\beta)} z^{k_2}. \end{aligned} \tag{4.6}$$

Using (**??**) in Eqs. (**??**) we obtain

$$\begin{aligned} &\frac{\Gamma(1+a_1-bk_1)}{\Gamma(1+a_1-bk_1-\beta)} A_1 z^{k_1} + ak_2 A_2^2 z^{2k_2-1} = 0, \\ &\frac{\Gamma(1+a_2-bk_2)}{\Gamma(1+a_2-bk_2-\beta)} A_2 z^{k_2} + bk_2 A_2 (k_2-1)(k_2-2) z^{k_2-3} + ck_1 A_1 A_2 z^{k_1+k_2-1} + dk_2 A_1 A_2 z^{k_1+k_2-1}. \end{aligned} \tag{4.7}$$

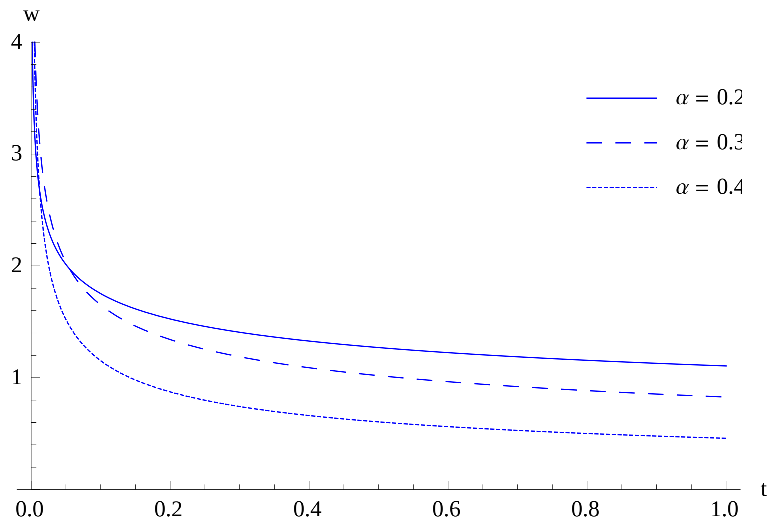
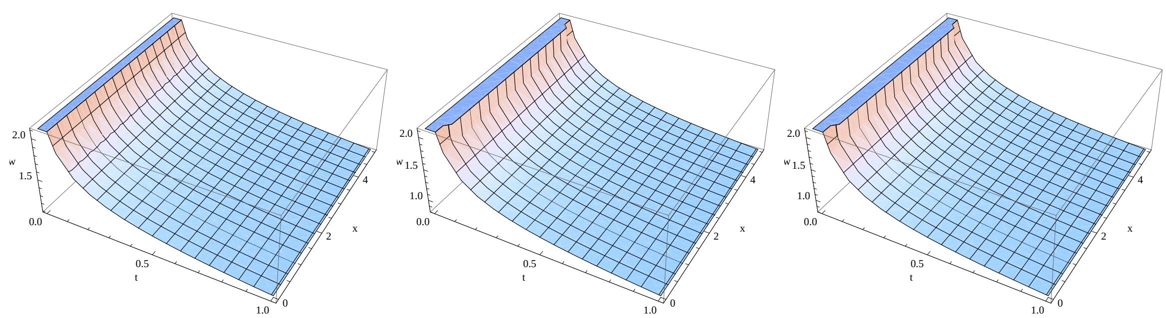


Fig. 3 2D profiles of the solution $w(x,t)$ for $0 < \alpha < 1$ and fixed x.



(a) (b) (c)

Fig. 4 3D profiles of the solution solution $w(x,t)$ for (a) $\alpha = 0.2$ (b) $\alpha = 0.3$ (c) $\alpha = 0.4$.

Substituting $\beta = \alpha$, $a_1 = a_2 = -\frac{2\alpha}{3}$, $b = \frac{\alpha}{3}$ in (**??**) and for $k_1 = k_2 = 1$, we obtain

$$A_1 = -\frac{1}{(c+d)}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \quad \text{and} \quad A_2 = \frac{1}{\sqrt{\{a(c+d)\}}}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}. \tag{4.8}$$

Hence the solution of original coupled FPDEs can be written as

$$u(x,t) = -\frac{1}{(c+d)}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\frac{x}{t^{\alpha}}, \tag{4.9}$$

and

$$w(x,t) = \frac{1}{\sqrt{\{a(c+d)\}}}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\frac{x}{t^{\alpha}}. \tag{4.10}$$

The above solution is called a dipole solution or singular solution. From Figures 1-4, one can observe that a change in the value of α affects the soliton behaviour in a fundamental way [**?**], which results that FD can be used to change the shape of waves without changing the nonlinearity and dissipative effect in the medium.

5. **Conservation laws**

In this part, the conserved vectors for the FCDSW equations using the new conservation theorem are determined. The conservation laws [?, ?] for the FCDSW equations have been also obtained.

The conservation laws for Eqs. (??) and (??) are defined as a vector field $T = (T^1, T^2)$, where $T^1 = T^1(x, t, u, w, ...)$ and $T^2 = T^2(x, t, u, w, ...)$ are called conserved vectors for Eqs. (??) and (??), if T^1 and T^2 satisfy the conservation theorem given as follows:

$$[D_t T^1 + D_x T^2]_{(??),(??)} = 0. \tag{5.1}$$

The formal Lagrangian of Eqs. (??) and (??) is obtained as

$$L = \gamma_1(x,t)(D_t^\alpha u + aww_x) + \gamma_2(x,t)(D_t^\alpha w + cu_x w + duw_x + bw_{xxx}). \tag{5.2}$$

Here ω_1 and γ_1 are the functions of x and t.

The Euler Lagrangian operators are defined as follows:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + ... + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha u}, \tag{5.3}$$

and

$$\frac{\delta}{\delta w} = \frac{\partial}{\partial w} - D_x \frac{\partial}{\partial w_x} + D_x^2 \frac{\partial}{\partial w_{xx}} - D_x^3 \frac{\partial}{\partial w_{xxx}} + ... + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha w}, \tag{5.4}$$

where $(D_t^\alpha)*$ is the adjoint operator of fractional differential operator D_t^α, given as follows:

$$(D_t^\alpha)^* = (-1)^m I_s^{m-\alpha} D_t^m,$$

where $I_s^{m-\alpha}$ is the right-hand-sided fractional integral operator of order $m - \alpha$, which is defined as

$$I_s^{m-\alpha} f(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_t^s \frac{f(x,p)}{(p-t)^{\alpha+1-m}} dp,$$

where $m = [\alpha] + 1$.

So, the adjoint equations can be written as

$$\frac{\delta L}{\delta u} = 0, \quad \text{and} \quad \frac{\delta L}{\delta w} = 0. \tag{5.5}$$

The component of conserved vectors are obtained by applying Noether operators to the Lagrangian. The fractional Noether operator for t-component can be written by the following formula [?, ?, ?]:

$$\begin{aligned} T^1 =& \tau \tilde{I} + \sum_{k=0}^{m-1} (-1)^k D_t^{\alpha-1-k}(W_1) D_t^k \frac{\partial L}{\partial D_t^\alpha u} - (-1)^m I\left(W_1, D_t^m \frac{\partial L}{\partial D_t^\alpha u}\right) \\ &+ \sum_{k=0}^{m-1} (-1)^k D_t^{\alpha-1-k}(W_2) D_t^k \frac{\partial L}{\partial D_t^\alpha w} - (-1)^m I\left(W_2, D_t^m \frac{\partial L}{\partial D_t^\alpha w}\right). \end{aligned} \tag{5.6}$$

Here,

$$I(f,g) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \int_t^T \frac{f(\tau,x)g(\mu,x)}{(\mu-\tau)^{\alpha+1-m}} d\mu d\tau. \tag{5.7}$$

Here $\tilde{I}$ denotes the identity operator, and $W_1 = \eta - \tau u_t - \xi u_x$ and $W_2 = \nu - \tau w_t - \xi w_x$ denote the Lie characteristic functions.

The other conserved vector T^2 for x-component is represented as

$$\begin{aligned}T^2 =&\xi\tilde{I} + W_1\left[\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}} + (D_x)^2\frac{\partial L}{\partial u_{xxx}} - ...\right] + W_2\left[\frac{\partial L}{\partial w_x} - D_x\frac{\partial L}{\partial w_{xx}} + (D_x)^2\frac{\partial L}{\partial w_{xxx}} - ...\right]\\ &+ D_x(W_1)\left[\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xxx}} + ...\right] + D_x(W_2)\left[\frac{\partial L}{\partial w_{xx}} - D_x\frac{\partial L}{\partial w_{xxx}} + ...\right]\\ &+ (D_x)^2(W_1)\left[\frac{\partial L}{\partial u_{xxx}} - ...\right] + (D_x)^2(W_2)\left[\frac{\partial L}{\partial w_{xxx}} - ...\right] + ... \end{aligned}\tag{5.8}$$

Now, the Lie characteristic functions for the vector $\mathbf{X}_3$ are obtained as

$$W_1 = -\frac{\alpha}{3}xu_x - tu_t - \frac{2\alpha}{3}u,\tag{5.9}$$

$$W_2 = -\frac{\alpha}{3}xw_x - tw_t - \frac{2\alpha}{3}w.\tag{5.10}$$

Now, substituting the value of the Lagrangian (**??**) in Eq. (**??**) and (**??**) and using the values of W_1 and W_2 from Eqs. (**??**) and (**??**), we have obtained the t-component of the conserved vector for $\mathbf{X}_3$ as follows:

$$\begin{aligned}T^1 &= \tau\tilde{I} + D_t^{\alpha-1}(W_1)D_t^0\frac{\partial L}{\partial D_t^\alpha u} + I\left(W_1, D_t\frac{\partial L}{\partial D_t^\alpha u}\right)\\ &+ D_t^{\alpha-1}(W_2)D_t^0\frac{\partial L}{\partial D_t^\alpha w} + I\left(W_2, D_t\frac{\partial L}{\partial D_t^\alpha w}\right)\\ &= \gamma_1 D_t^{\alpha-1}\left(-\frac{\alpha}{3}xu_x - tu_t - \frac{2\alpha}{3}u\right) + I\left[\left(-\frac{\alpha}{3}xu_x - tu_t - \frac{2\alpha}{3}u\right), (\gamma_1)_t\right]\\ &+ \gamma_2 D_t^{\alpha-1}\left(-\frac{\alpha}{3}xw_x - tw_t - \frac{2\alpha}{3}w\right) + I\left[\left(-\frac{\alpha}{3}xw_x - tw_t - \frac{2\alpha}{3}w\right), (\gamma_2)_t\right].\end{aligned}\tag{5.11}$$

Also, the x-component of the conserved vector for $\mathbf{X}_3$ is obtained in the following form:

$$\begin{aligned}T^2 &= \xi\tilde{I} + W_1\left[\frac{\partial L}{\partial u_x}\right] + W_2\left[\frac{\partial L}{\partial w_x} + (D_x)^2\frac{\partial L}{\partial w_{xxx}}\right] + D_x(W_2)\left[-D_x\frac{\partial L}{\partial w_{xxx}}\right]\\ &+ (D_x)^2(W_2)\left[\frac{\partial L}{\partial w_{xxx}}\right]\\ &= b(\gamma_2)_x\left(\alpha w_x + \frac{\alpha}{3}xw_{xx} + tw_{xt}\right) - b(\gamma_2)_{xx}\left(\frac{\alpha}{3}xw_x + tw_t + \frac{2\alpha}{3}w\right) - \gamma_1 a\left(\frac{\alpha}{3}xww_x + tww_t + \frac{2\alpha}{3}w^2\right)\\ &+ \gamma_2\left[du\left(-\frac{\alpha}{3}xw_x - \frac{2\alpha}{3}w - tw_t\right) + cw\left(-\frac{\alpha}{3}xu_x - tu_t - \frac{2\alpha}{3}u\right) + b\left(\frac{4\alpha}{3}w_{xx} + tw_{xxt} + \frac{\alpha}{3}xw_{xxx}\right)\right].\end{aligned}\tag{5.12}$$

6. **Conclusion**

In this article, we have applied the fractional Lie symmetry group-theoretic method to solve the time FCDSW partial differential equations. Firstly, we have determined the Lie point symmetries for FCDSW equations. Using the Lie symmetries, the time fractional coupled system of equations is transformed into a system of FODEs with the help of fractional Erdélyi-Kober differential operator. Using the symmetry analysis, we have obtained the exact solution of the FCDSW equations in explicit form. The effects of fractional order α on the solution's behaviour are shown graphically. From the figures, one can observe that a small change in the value α affects the soliton behaviour and the shape of wave, without changing the nonlinearity in the medium. With the help of Noether operators and new conservation theorem, the new conserved vectors are obtained successfully along with formal Lagrangian, which are used in the study of global behaviour and the stability of solutions of FCDSW equations. One can use conservation laws for mathematical analysis to develop appropriate numerical methods and for stability, uniqueness and existence analysis.

In this work, we have discussed only one vector field for both the systems for symmetry reductions; however, for remaining vector fields, the symmetry reductions can also be discussed. We have avoided the discussion of remaining vector fields due to the lack of physical importance of their results. There are some possible extensions of this study, e.g. symmetry analysis for space-time fractional systems of non-linear PDEs with or without variable coefficients. Some of the extensive work will be discussed in the future work.

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Conflict of Interest

The author declares that they have no conflict of interest.

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