

On Robin's criterion for the Riemann Hypothesis

Frank Vega

the date of receipt and acceptance should be inserted later

Abstract Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^\gamma \times n \times \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 2022, Vega stated that the possible existence of the smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$ and $(\log n)^\beta < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$. Based on that result, we obtain a contradiction just assuming the existence of such possible smallest counterexample $n > 5040$ for the Robin inequality. By contraposition, we show that the Riemann hypothesis should be true.

Keywords Riemann hypothesis · Robin inequality · Sum-of-divisors function · Prime numbers · Counterexample

Mathematics Subject Classification (2010) 11M26 · 11A41 · 11A25

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n :

$$\sum_{d|n} d$$

F. Vega
CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
ORCID: 0000-0001-8210-4126
E-mail: vega.frank@gmail.com

where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n . Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robins(n) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and \log is the natural logarithm. The importance of this property is:

Theorem 1.1 Robins(n) holds for all natural numbers $n > 5040$ if and only if the Riemann hypothesis is true [3].

It is known that Robins(n) holds for many classes of numbers n . We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$.

Theorem 1.2 Robins(n) holds for all natural numbers $n > 5040$ that are square free [1].

Let $q_1 = 2, q_2 = 3, \dots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^m q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [1]. Now, we are able to use this recently result:

Theorem 1.3 The possible existence of the smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$ and $(\log n)^\beta < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$ [4].

Putting all together yields a proof for the Riemann hypothesis using the Theorem 1.3 as the principal argument.

2 Known Results

These are known results:

Lemma 2.1 For every $x > -1$ [2]:

$$\log(1+x) \geq \frac{x}{x+1}.$$

Lemma 2.2 For every real number x [2]:

$$e^x \geq 1+x.$$

Lemma 2.3 For every $x > -1$ [2]:

$$\frac{\log(1+x)}{x} \geq \frac{2}{x+2}.$$

3 A Central Lemma

The following is a key Lemma.

Lemma 3.1 *If the natural number $n > 5040$ is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$, then $\beta \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$ where $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.*

Proof If we apply the logarithm to the value of

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$$

then we obtain that

$$\sum_{i=1}^m \log\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}\right).$$

For some $1 \leq j \leq m$, we know that

$$\frac{q_j^{a_j+1}}{q_j^{a_j+1}-1} = 1 + \frac{1}{q_j^{a_j+1}-1}.$$

We use the Lemma 2.1 to show that

$$\begin{aligned} \log\left(1 + \frac{1}{q_j^{a_j+1}-1}\right) &\geq \frac{\frac{1}{q_j^{a_j+1}-1}}{\frac{1}{q_j^{a_j+1}-1} + 1} \\ &= \frac{1}{(q_j^{a_j+1}-1) \times \left(\frac{1}{q_j^{a_j+1}-1} + 1\right)} \\ &= \frac{1}{1 + (q_j^{a_j+1}-1)} \\ &= \frac{1}{q_j^{a_j+1}}. \end{aligned}$$

So,

$$\sum_{i=1}^m \log\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}\right) \geq \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$$

and thus,

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1} \geq e^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}}.$$

Using the Lemma 2.2, we have that

$$e^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$$

and therefore,

$$\beta \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}.$$

4 Main Insight

This is the main insight.

Lemma 4.1 *Suppose that $n > 5040$ is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$ and $q_m > e^{31.018189471}$. Then $(\log n)^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \geq 1.03352795481$.*

Proof If we apply the logarithm to the both sides of the inequality, then

$$\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times \log \log n \geq \log(1.03352795481).$$

Let's multiply the both sides of the inequality by e^γ ,

$$\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times e^\gamma \times \log \log n \geq e^\gamma \times \log(1.03352795481).$$

From the Theorem 1.2, we know that

$$\begin{aligned} e^\gamma \times \log \log n &\geq e^\gamma \times \log \log N_m \\ &> f(N_m) \\ &= \prod_{i=1}^m \left(1 + \frac{1}{q_i}\right) \end{aligned}$$

since $n > 5040$ is an Hardy-Ramanujan integer, $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m and thus, $n \geq N_m$ and N_m is square free. Hence, we would have that

$$\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times \prod_{i=1}^m \left(1 + \frac{1}{q_i}\right) \geq e^\gamma \times \log(1.03352795481).$$

If we apply the logarithm to the both sides again, then

$$\log \left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) + \sum_{i=1}^m \log \left(1 + \frac{1}{q_i}\right) \geq \log(e^\gamma \times \log(1.03352795481)).$$

We use the Lemma 2.3 to show that

$$\begin{aligned}
\log\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right) &= \log\left(1 + (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}})\right) \\
&\geq \frac{2 \times (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}})}{(-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}) + 2} \\
&= \frac{2 \times (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}})}{1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \\
&> 2 \times (-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}) \\
&= -2 + 2 \times \left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right)
\end{aligned}$$

since

$$-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} > -1.$$

For some $1 \leq j \leq m$, we know that

$$\begin{aligned}
\log\left(1 + \frac{1}{q_j}\right) &\geq \frac{\frac{1}{q_j}}{\frac{1}{q_j} + 1} \\
&= \frac{1}{q_j \times (\frac{1}{q_j} + 1)} \\
&= \frac{1}{1 + q_j}
\end{aligned}$$

according to the Lemma 2.1. However, we note that

$$-2 + 2 \times \left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right) + \sum_{i=1}^m \frac{1}{1 + q_i} > 0$$

when $q_m > e^{31.018189471}$. In addition, we have that

$$0 > \log(e^\gamma \times \log(1.03352795481))$$

and finally, the proof is complete.

5 Main Theorem

We conclude with the following statement:

Theorem 5.1 *The Riemann hypothesis is true.*

Proof Suppose that $n > 5040$ is the possible smallest number such that $\text{Robins}(n)$ does not hold. By the Theorem 1.3, we know that $q_m > e^{31.018189471}$ and $(\log n)^\beta < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$. From the Lemma 3.1, we know that

$$(\log n)^\beta \geq (\log n)^{\left(1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right)}$$

and therefore, we would have that

$$(\log n)^{\left(1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right)} < 1.03352795481 \times \log(N_m)$$

when $n > 5040$ is the possible smallest number such that $\text{Robins}(n)$ does not hold. Thus, we would obtain that

$$(\log n)^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} < 1.03352795481$$

since n must be an Hardy-Ramanujan integer and so, $\log n \geq \log N_m$. However, we know the previous inequality cannot be satisfied because of the Lemma 4.1. By contraposition, we show that the Riemann hypothesis is true, since we obtain a contradiction just assuming the possible smallest counterexample for the Robin inequality greater than 5040. Certainly, this is a direct consequence of the Theorem 1.1.

Acknowledgments

The author would like to thank his mother, maternal brother and his friend Sonia for their support.

References

1. Choie, Y., Lichiardopol, N., Moree, P., Solé, P.: On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux* **19**(2), 357–372 (2007). DOI 10.5802/jtnb.591
2. Kozma, L.: Useful Inequalities. http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf (2022). Accessed on 2022-02-02
3. Robin, G.: Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. pures appl* **63**(2), 187–213 (1984)
4. Vega, F.: Robin's criterion on divisibility. *The Ramanujan Journal* (2022). DOI 10.1007/s11139-022-00574-4. To appear in *The Ramanujan Journal*