

On Robin's criterion for the Riemann Hypothesis

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Abstract Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^\gamma \times n \times \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 2022, Vega stated that the possible existence of the smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$ and $(\log n)^\beta < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$. Based on that result, we obtain a contradiction just assuming the existence of such possible smallest counterexample $n > 5040$ for the Robin inequality. By contraposition, we show that the Riemann hypothesis should be true.

Keywords Riemann hypothesis · Robin inequality · Sum-of-divisors function · Prime numbers · Counterexample

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1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n :

$$\sum_{d|n} d$$

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where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n . Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say Robins(n) holds provided

$$f(n) < e^\gamma \times \log \log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and \log is the natural logarithm. The importance of this property is:

Theorem 1.1 Robins(n) holds for all natural numbers $n > 5040$ if and only if the Riemann hypothesis is true [3].

It is known that Robins(n) holds for many classes of numbers n . We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$.

Theorem 1.2 Robins(n) holds for all natural numbers $n > 5040$ that are square free [1].

Let $q_1 = 2, q_2 = 3, \dots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^m q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [1]. Now, we are able to use this recently result:

Theorem 1.3 The possible existence of the smallest counterexample $n > 5040$ of the Robin inequality implies that $q_m > e^{31.018189471}$ and $(\log n)^\beta < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$ [4].

Putting all together yields a proof for the Riemann hypothesis using the Theorem 1.3 as the principal argument.

2 A Central Lemma

These are known results:

Lemma 2.1 For every $x > -1$ [2]:

$$x \geq \log(1+x) \geq \frac{x}{x+1}.$$

Lemma 2.2 For every real number x [2]:

$$e^x \geq 1+x.$$

The following is a key Lemma.

Lemma 2.3 If the natural number $n > 5040$ is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$, then $\beta \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$ where $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1}-1}$.

Proof If we apply the logarithm to the value of

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$$

then we obtain that

$$\sum_{i=1}^m \log\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}\right).$$

For some $1 \leq j \leq m$, we know that

$$\frac{q_j^{a_j+1}}{q_j^{a_j+1} - 1} = 1 + \frac{1}{q_j^{a_j+1} - 1}.$$

We use the Lemma 2.1 to show that

$$\begin{aligned} \log\left(1 + \frac{1}{q_j^{a_j+1} - 1}\right) &\geq \frac{\frac{1}{q_j^{a_j+1} - 1}}{\frac{1}{q_j^{a_j+1} - 1} + 1} \\ &= \frac{1}{(q_j^{a_j+1} - 1) \times \left(\frac{1}{q_j^{a_j+1} - 1} + 1\right)} \\ &= \frac{1}{1 + (q_j^{a_j+1} - 1)} \\ &= \frac{1}{q_j^{a_j+1}}. \end{aligned}$$

So,

$$\sum_{i=1}^m \log\left(\frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}\right) \geq \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$$

and thus,

$$\prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1} \geq e^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}}.$$

Using the Lemma 2.2, we have that

$$e^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}$$

and therefore,

$$\beta \geq 1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}.$$

3 Main Insight

This is the main insight.

Lemma 3.1 *Suppose that $n > 5040$ is an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$ and $q_m > e^{31.018189471}$. Then $(\log n)^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} \geq 1.03352795481$.*

Proof If we apply the logarithm to the both sides of the inequality, then

$$\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times \log \log n \geq \log(1.03352795481).$$

Let's multiply the both sides of the inequality by e^γ ,

$$\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times e^\gamma \times \log \log n \geq e^\gamma \times \log(1.03352795481).$$

From the Theorem 1.2, we know that

$$\begin{aligned} e^\gamma \times \log \log n &\geq e^\gamma \times \log \log N_m \\ &> f(N_m) \\ &= \prod_{i=1}^m \left(1 + \frac{1}{q_i}\right) \end{aligned}$$

since $n > 5040$ is an Hardy-Ramanujan integer, $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m and thus, $n \geq N_m$ and N_m is square free. Hence, we would have that

$$\left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) \times \prod_{i=1}^m \left(1 + \frac{1}{q_i}\right) \geq e^\gamma \times \log(1.03352795481).$$

If we apply the logarithm to the both sides again, then

$$\log \left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right) + \sum_{i=1}^m \log \left(1 + \frac{1}{q_i}\right) \geq \log(e^\gamma \times \log(1.03352795481)).$$

We use the Lemma 2.1 to show that

$$-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \geq \log \left(1 + \left(-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right) \right) = \log \left(\sum_{i=1}^m \frac{1}{q_i^{a_i+1}} \right)$$

since

$$-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} > -1.$$

For some $1 \leq j \leq m$, we know that

$$\begin{aligned} \log\left(1 + \frac{1}{q_j}\right) &\geq \frac{\frac{1}{q_j}}{\frac{1}{q_j} + 1} \\ &= \frac{1}{q_j \times \left(\frac{1}{q_j} + 1\right)} \\ &= \frac{1}{1 + q_j} \end{aligned}$$

according to the Lemma 2.1. However, we note that

$$-1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}} + \sum_{i=1}^m \frac{1}{1 + q_i} \gg 0$$

when $q_m > e^{31.018189471}$, where the symbol \gg means "much greater than". In addition, we have that

$$0 > \log(e^\gamma \times \log(1.03352795481))$$

and finally, the proof is complete.

4 Main Theorem

We conclude with the following statement:

Theorem 4.1 *The Riemann hypothesis is true.*

Proof Suppose that $n > 5040$ is the possible smallest number such that Robins(n) does not hold. By the Theorem 1.3, we know that $q_m > e^{31.018189471}$ and $(\log n)^\beta < 1.03352795481 \times \log(N_m)$, where $N_m = \prod_{i=1}^m q_i$ is the primorial number of order m , q_m is the largest prime divisor of n and $\beta = \prod_{i=1}^m \frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1}$ when n must be an Hardy-Ramanujan integer of the form $\prod_{i=1}^m q_i^{a_i}$. From the Lemma 2.3, we know that

$$(\log n)^\beta \geq (\log n)^{\left(1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right)}$$

and therefore, we would have that

$$(\log n)^{\left(1 + \sum_{i=1}^m \frac{1}{q_i^{a_i+1}}\right)} < 1.03352795481 \times \log(N_m)$$

when $n > 5040$ is the possible smallest number such that Robins(n) does not hold. Thus, we would obtain that

$$(\log n)^{\sum_{i=1}^m \frac{1}{q_i^{a_i+1}}} < 1.03352795481$$

since n must be an Hardy-Ramanujan integer and so, $\log n \geq \log N_m$. However, we know the previous inequality cannot be satisfied because of the Lemma 3.1. By contradiction, we show that the Riemann hypothesis is true, since we obtain a contradiction just assuming the possible smallest counterexample for the Robin inequality greater than 5040. Certainly, this is a direct consequence of the Theorem 1.1.

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References

1. Choie, Y., Lichiardopol, N., Moree, P., Solé, P.: On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux* **19**(2), 357–372 (2007). DOI 10.5802/jtnb.591
2. Kozma, L.: Useful Inequalities. http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf (2022). Accessed on 2022-02-02
3. Robin, G.: Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. pures appl* **63**(2), 187–213 (1984)
4. Vega, F.: Robin's criterion on divisibility. *The Ramanujan Journal* (2022). DOI 10.1007/s11139-022-00574-4. To appear in *The Ramanujan Journal*