Geometrical optics: Meridional and Skew Rays

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Abstract

In the framework of geometrical optics, let us study meridional and skew rays, those rays that we can find in optical fibres. The study is based on the book by Keigo Iizuka, entitled Engineering Optics.

Keywords: **Optics, Geometrical Optics, Fibers, Rays.**

Subject Areas: **Physics, Optics.**

1. Introduction

Geometrical optics, or ray optics, puts the emphasis on the light paths. A particular focus is in the calculation of the light path in inhomogeneous media and in the design of optical instruments [1]. Here we will study in particular two rays that we can find in optical fibers: the meridional ray and the skew ray. The study is based on the book by Keigo Iizuja, entitled Engineering Optics [1].

Before the specific study, let us remember some notions of ray tracing and some useful calculus.

2. Rays and ray tracing

In [2], the ray tracing is discussed. It is remembered that the geometrical optics is based on the concept of rays of light, assumed as straight lines in any homogeneous medium. The rays change their paths at surfaces separating two media having differing refractive indices. It is often needed to trace the path of a ray through an optical system, which generally contains a succession of refracting or reflecting surfaces separated by given distances along the optical axis. This axis is a line along which there is some degree of rotational symmetry in an optical system such as a camera lens, microscope or telescopic sight. For a system composed of simple lenses and mirrors, the axis passes through the center of curvature of each surface, and coincides with the axis of rotational symmetry. For an optical fiber, the optical axis is along the center of the fiber core, and is also known as the fiber axis.

In geometrical optics, a point source occupies no volume and radiates light in all directions. Point sources do not exist, however objects are simply viewed as a collection of point sources. Every object point is a point source, which is radiating in all directions. In general, and according to their directions, the rays of light, fall into three main classes: meridional, paraxial, and skew [2]. For an optical system possessing a rotational symmetry, meridional rays lie in the plane containing the lens axis and an object point lying to one side of the axis [2]. This plane is called the meridional plane. If the object point lies on the axis, all rays are necessarily meridional [2]. Let us stress that, in a frame of coordinates x, y, z, where z is the optical axis and y is perpendicular to this axis, a meridional ray is a ray confined in the plane y-z.

A paraxial ray is a ray which makes a small angle θ to the optical axis of the system, and lies close to the axis throughout the system. This ray allows three important approximations for calculation of the ray's path, namely $\sin\theta \approx \theta$, $\tan\theta \approx \theta$ and $\cos\theta \approx 1$. When the paraxial approximation is no longer valid, the concepts used in the Gaussian optics, the geometrical optics which uses this approximation, lose their meaning [3]. Then, “we can no longer use terms such as cardinal points, focal length, and magnification. Instead, we must go to exact ray tracing” [3]. As a consequence, we can have the meridional non-paraxial optics and the non-meridional non-paraxial optics.

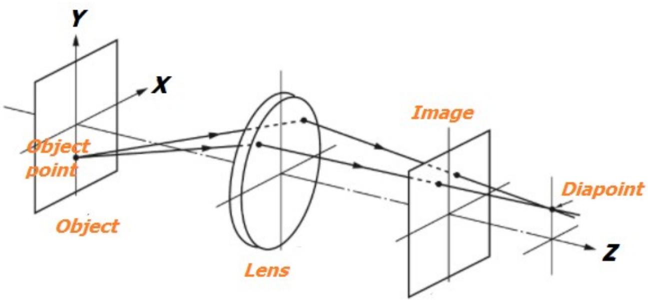
Sometimes, the meridional plane is mentioned as the “meridian” plane, and the meridional ray, the “meridian” ray.

“An object point off axis and the axis of a rotation-symmetric optical system form a plane called the meridian plane of the object point. A ray from the object point in the meridian plane is called a meridian ray; a ray not in the meridian plane is called a skew ray. A meridian ray obviously remains a meridian ray after refraction according to the refraction law, since the meridian plane is the plane of incidence, and a skew ray remains a skew ray. The optical system which we investigate co-ordinates to each object ray an image ray. The point where the image ray intersects the meridian plane is called the “diapoint” of the object.” [4]

Any ray, or bundle of rays, emerging from the object space will reappear as another (unique) ray, or bundle of rays, in the image space [5]. Mathematically, this is called a mapping of the object space into the image space by the optical system, the intervening medium thus determines the type of mapping or transformation [5].

Before, we have mentioned the paraxial rays. They form an important limiting class of rays that has many applications. The paraxial rays lie throughout their length so close to the optical axis that their aberrations are negligible [2]. “The ray tracing formulas for paraxial rays contain no trigonometric functions and are therefore well-suited to algebraic manipulation. A paraxial ray is really only a mathematical abstraction, … Skew rays, on the other hand, do not lie in the meridional plane, but they pass in front of or behind it and pierce the meridional plane at the diapoint. A skew ray never intersects the lens axis. Skew rays are much more difficult to trace than meridional rays” [2].

“If the object point lies on the lens axis, we trace only axial rays. However, for an extra-axial object point there are two kinds of rays to be traced, namely meridional rays, which lie in the meridional plane, shown in the familiar ray diagram of a system, and skew rays, which lie in front of or behind the meridional plane and do not intersect the axis anywhere. Each skew ray pierces the meridional plane at the object point and also at another point in the image space known as the diapoint of the ray [2]. The paths of two typical skew rays are shown diagrammatically" [2], in the schematic view given below.

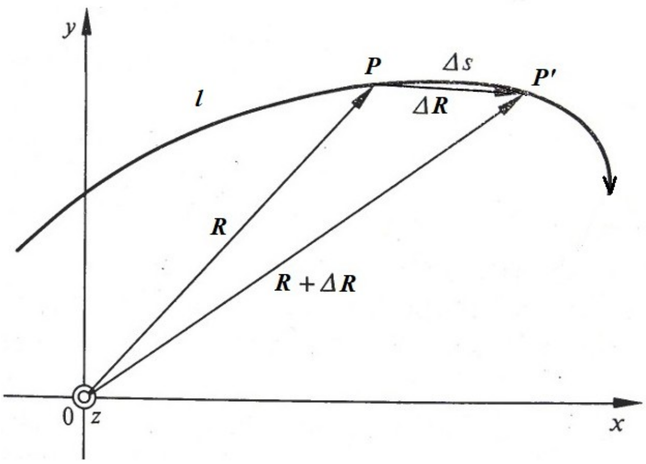


Path of two typical skew rays, from Ref. [2]. Note that the rays depicted in the Figure are symmetric with respect to the Y axis.

"Axial rays and meridional rays can be traced by relatively simple trigonometric formulas, or even graphically if very low precision is adequate. Skew rays, on the other hand, are much more difficult to trace, the procedure being discussed in Chapter 8", of the Ref. [2].

3. Mathematics: tangent unit vector and curvature

Some mathematical expressions are necessary for the description of the path of light.



Let us consider the following figure, where the position vector $\boldsymbol{R}$ is given.

The unit tangent vector is given by:

$$\hat{\mathbf{s}} = \frac{d\boldsymbol{R}}{ds}$$

In this formula, the curvilinear coordinate s is used. This is the arc length on the curve, given from a fixed origin on it.

Using rectangular coordinates: $\boldsymbol{R}=\hat{\boldsymbol{i}}\,x+\hat{\boldsymbol{j}}\,y+\hat{\boldsymbol{k}}\,z$, the unit tangent vector is:

$$\hat{\mathbf{s}}=\hat{\boldsymbol{i}}\frac{dx}{ds}+\hat{\boldsymbol{j}}\frac{dy}{ds}+\hat{\boldsymbol{k}}\frac{dz}{ds}$$

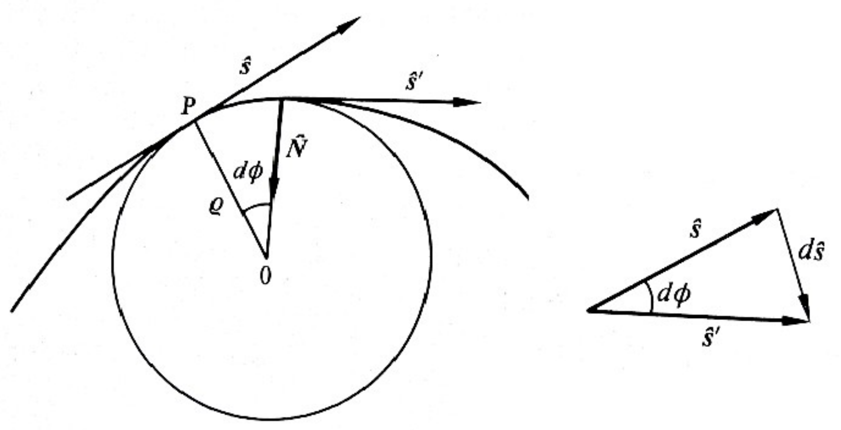
In cylindrical coordinate: $\boldsymbol{R}=\hat{\boldsymbol{r}}\,r+\hat{\boldsymbol{k}}\,z$, $\hat{\boldsymbol{r}}=\hat{\boldsymbol{i}}\cos\phi+\hat{\boldsymbol{j}}\sin\phi$, $\hat{\boldsymbol{\phi}}=-\hat{\boldsymbol{i}}\sin\phi+\hat{\boldsymbol{j}}\cos\phi$

$$\hat{\mathbf{s}}=\hat{\boldsymbol{r}}\frac{dr}{ds}+r\frac{d\hat{\boldsymbol{r}}}{ds}+\hat{\boldsymbol{k}}\frac{dz}{ds} \quad \text{, where} \quad \frac{d\hat{\boldsymbol{r}}}{ds}=(-\hat{\boldsymbol{i}}\sin\phi+\hat{\boldsymbol{j}}\cos\phi)\frac{d\phi}{ds}=\hat{\boldsymbol{\phi}}\frac{d\phi}{ds} \quad .$$

The tangent expressed in cylindrical coordinates becomes:

$$\hat{\mathbf{s}}=\hat{\boldsymbol{r}}\frac{dr}{ds}+\hat{\boldsymbol{\phi}}\,r\frac{d\phi}{ds}+\hat{\boldsymbol{k}}\frac{dz}{ds}$$

The curvature of a curve is: $\frac{1}{\rho}=|d\,\hat{\mathbf{s}}/ds|$ or $\frac{1}{\rho}=|d^2\boldsymbol{R}/ds^2|$.



The inverse of the radius of curvature is the magnitude of the first derivative of the tangent vector, or the second derivative of the position vector.

4. Mathematics: principal unit vector, binormal and torsion

We have seen that the curvature is: $\frac{1}{\rho}=|d\,\hat{\mathbf{s}}/ds|$ or $\frac{1}{\rho}=|d^2\hat{\boldsymbol{R}}/ds^2|$.

Let us consider the time: $\frac{ds}{dt}=|d\,\boldsymbol{R}/dt|$. We have that $\hat{\mathbf{s}}=\frac{d\,\boldsymbol{R}/dt}{|d\,\boldsymbol{R}/dt|}$, and then the curvature turns out to be: $\frac{1}{\rho}=\frac{|d\,\hat{\mathbf{s}}/dt|}{|d\,\boldsymbol{R}/dt|}$.

$\frac{d\,\hat{\mathbf{s}}}{ds}$ is also called the "curvature vector". The Principal Unit Normal is defined as:

$$\hat{\boldsymbol{N}}=\frac{d\,\hat{\mathbf{s}}/ds}{|d\,\hat{\mathbf{s}}/ds|}=\frac{d\,\hat{\mathbf{s}}/dt}{|d\,\hat{\mathbf{s}}/dt|}$$, which is orthogonal to $\hat{\mathbf{s}}$.

A third vector is the "binormal vector":

$$\hat{\boldsymbol{B}}=\hat{\mathbf{s}}\times\hat{\boldsymbol{N}}$$

It is another unit vector.

Altogether, we obtain a frame, which is known as the ***Frenet frame*** given by vectors $\hat{\mathbf{s}},\hat{\boldsymbol{N}},\hat{\boldsymbol{B}}$. These vectors are of unit length and orthogonal to each other.

The Frenet- Serrat formula is:

$$\frac{d\,\hat{\mathbf{s}}}{ds}=\frac{\hat{\boldsymbol{N}}}{\rho} ,$$

where vector $\hat{\boldsymbol{N}}$ is the unit normal vector to the curve.

About the binormal $\hat{\boldsymbol{B}}$, we have that it is a vector parallel to the unit normal $\hat{\boldsymbol{N}}$.

Being a unit vector, $d\,\hat{\boldsymbol{B}}/ds$ is perpendicular to $\hat{\boldsymbol{B}}$.

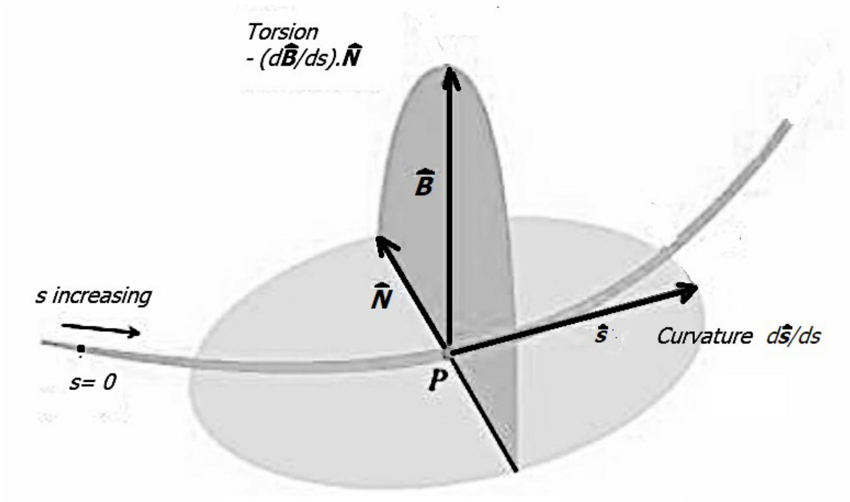
Moreover, $\hat{\boldsymbol{B}}\cdot\hat{\mathbf{s}}=0$, therefore:

$$\mathbf{0}=\frac{d\,\hat{\boldsymbol{B}}}{ds}\cdot\hat{\mathbf{s}}+\hat{\boldsymbol{B}}\cdot\frac{d\,\hat{\mathbf{s}}}{ds}=\frac{d\,\hat{\boldsymbol{B}}}{ds}\cdot\hat{\mathbf{s}}+\hat{\boldsymbol{B}}\cdot\frac{1}{\rho}\hat{\boldsymbol{N}}=\frac{d\,\hat{\boldsymbol{B}}}{ds}\cdot\hat{\mathbf{s}}$$

Since $\frac{d\hat{\boldsymbol{B}}}{ds}\cdot\hat{\boldsymbol{B}}=0$ and $\frac{d\hat{\boldsymbol{B}}}{ds}\cdot\hat{\boldsymbol{s}}=0$, $\frac{d\hat{\boldsymbol{B}}}{ds}$ is a multiple of $\hat{\boldsymbol{N}}$.

The multiple is the "torsion", defined as:

$$\frac{d\hat{\boldsymbol{B}}}{ds}=-\tau\,\hat{\boldsymbol{N}} \quad \text{or} \quad \tau=-\frac{d\hat{\boldsymbol{B}}}{ds}\cdot\hat{\boldsymbol{N}}$$



5. Level surfaces and gradients

Let us consider a level surface $L(x,y,z)=C$. An example is the equipotential function in electrostatics. Let us move the observation point P so that we have an increment:

$$\Delta L=\frac{\partial L}{\partial x}\Delta x+\frac{\partial L}{\partial y}\Delta y+\frac{\partial L}{\partial z}\Delta z+\text{infinitesimals of higher order}$$

The directional derivative $\nabla_s L$ is:

$$\nabla_s L=\lim_{\Delta l\to 0}\frac{\Delta L}{\Delta l}=\frac{\partial L}{\partial x}\frac{\Delta x}{\Delta l}+\frac{\partial L}{\partial y}\frac{\Delta y}{\Delta l}+\frac{\partial L}{\partial z}\frac{\Delta z}{\Delta l}+\text{infinitesimals of higher order}$$

$\Delta x/\Delta l,\ \Delta y/\Delta l,\ \Delta z/\Delta l$ are the direction cosine $\cos\alpha$, $\cos\beta$, $\cos\gamma$ of the movement Δl .

$$\nabla_s L=\frac{dL}{dl}=\frac{\partial L}{\partial x}\cos\alpha+\frac{\partial L}{\partial y}\cos\beta+\frac{\partial L}{\partial z}\cos\gamma$$

We have also that:

$$\nabla L=\hat{\boldsymbol{i}}\frac{\partial L}{\partial x}+\hat{\boldsymbol{j}}\frac{\partial L}{\partial y}+\hat{\boldsymbol{k}}\frac{\partial L}{\partial z} \quad , \quad \hat{\boldsymbol{M}}=\hat{\boldsymbol{i}}\cos\alpha+\hat{\boldsymbol{j}}\cos\beta+\hat{\boldsymbol{k}}\cos\gamma$$

Then:

$$\nabla_s L=(\nabla L)\cdot\hat{\boldsymbol{M}} \quad \text{, so we have:} \quad \Delta L=(\nabla L)\cdot\hat{\boldsymbol{M}}\,dl=(\nabla L)\cdot d\boldsymbol{l}$$

$\nabla_s L$ varies with the choice of the direction of the movement $\hat{\boldsymbol{M}}$. The change of L becomes a maximum when the movement is selected in the same direction as that of $\boldsymbol{\nabla} L$. Then, ∇L is the direction that gives the maximum change in L for a given length of the movement.

Let us note that we are evaluating the derivative of a scalar, and that it turns of into the dot product of a gradient and a vector. In this case, the result is a scalar.

The normal $\hat{\boldsymbol{N}}$ to the equi-level surface is:

$$\hat{\boldsymbol{N}}=\frac{\boldsymbol{\nabla} L}{|\boldsymbol{\nabla} L|}$$

In optics, this formula is particularly useful because it determines the optical path from the equi-phase surface.

For more details and examples on directional derivative and gradient, see the following link

libretexts.org/Univ_California_Davis/...Directional_Derivatives_and_Gradient_Vectors

6. Eikonal equation

Let $\boldsymbol{E}(x,y,z)$ representing a light wave that needs to satisfy the wave equation:

$$(\nabla^2+\omega^2\mu\epsilon)\boldsymbol{E}(x,y,z)=0$$

In the case of an isotropic medium, E_x, E_y, E_z equations are identical.

$$(\nabla^2+\omega^2\mu\epsilon)u(x,y,z)=0 \quad (*)$$

where $\omega^2\mu\epsilon=[k\,n(x,y,z)]^2$, k is the free space propagation constant and n the refraction index.

Let us assume the solution: $u(x,y,z)=A(x,y,z)\exp\{j[k\,L(x,y,z)-\omega t]\}$.

Functions $A(x,y,z)$ and $L(x,y,z)$ are unknown. They have to be determined in such a way to satisfy (*). Then:

$$n^2k^2u+\nabla^2 u =$$

$$e^{j(kL-\omega t)}\{k^2[n^2-|\nabla L|^2]A+\nabla^2 A+j\,k\,A\,\nabla^2 L+j\,2k(\nabla A)\cdot(\nabla L)\}=0$$

$|\nabla L|^2$ means the sum of the squares of i,j,k components of ∇L .

If the wavelength of light is much shorter that the dimensions of the associated structure:

$$|\nabla L|^2=n^2 \quad \text{, then:} \quad \left(\frac{\partial L}{\partial x}\right)^2+\left(\frac{\partial L}{\partial y}\right)^2+\left(\frac{\partial L}{\partial z}\right)^2=n^2$$

This is the eikonal equation of the optical path. The wave front L itself is called the eikonal or optical path.

Let us remember that $\Delta L=(\nabla L)\cdot d\boldsymbol{l}$. Then:

$$L=\int(\nabla L)\cdot d\boldsymbol{l} \quad .$$

If the movement is restricted to the normal of the equi-level surface:

$$L = \int_{along\ normal} |\nabla L| ds = \int_{along\ normal} n\, ds \quad .$$

The direction of the normal to the equi-phase surface is $\nabla L / |\nabla L|$. The normal is called the "wave normal".

7. A glass slab

Consider a glass slab whose index of refraction is variable in the *x*-direction but is constant in both the *y*- and *z*- directions, so that $n = n(x)$. $L(x, y, z)$ is assumed to be separable so that $L(x, y, z) = f(x) + g(y) + h(z)$.

$$|\nabla L|^2 = n^2 \quad \text{, then} \quad \{[f'(x)]^2 - [n(x)]^2\} + [g'(y)]^2 + [h'(z)]^2 = 0$$

Let us consider: $\{[f'(x)]^2 - [n(x)]^2\} = a^2$, $[g'(y)]^2 = b^2$, $[h'(z)]^2 = c^2$, so that $a^2 + b^2 + c^2 = 0$.

Solutions are:

$$f(x) = \pm \int_0^x \sqrt{[n(x)]^2 - (b^2 + c^2)}\; dx \quad , \quad g(y) = \pm by + m_1 \quad , \quad h(z) = \pm cz + m_2$$

$$L(x, y, z) = \pm \int_m^x \sqrt{[n(x)]^2 - (b^2 + c^2)}\; dx \pm by \pm cz$$

Constants m_1 and m_2 are considered in the lower limit of the integration. b, c are determined by boundary condition, such as launch position and angle.

The direction of the propagation is the direction of the wave normal: $\mathbf{N} = \nabla L / n$. The normal is in the same direction of the unit tangent vector $\hat{\mathbf{s}}$ to the light path.

$$\frac{d\boldsymbol{R}}{ds}=\frac{\nabla L}{n} \quad \text{then:} \quad n\frac{dx}{ds}=\frac{\partial L}{\partial x} \quad , \quad n\frac{dy}{ds}=\frac{\partial L}{\partial y} \quad , \quad n\frac{dz}{ds}=\frac{\partial L}{\partial z}$$

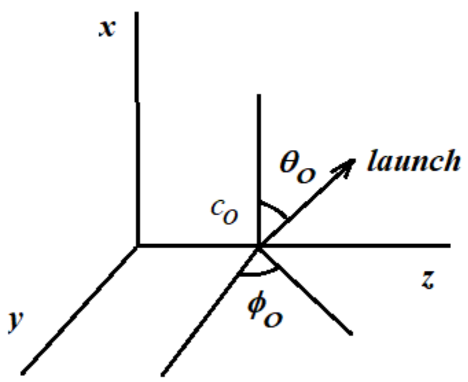
Let us consider the optical path of light launched into a medium having $n=n(x)$.

$$n(x)\frac{dx}{ds}=\sqrt{[n(x)]^2-(b^2+c^2)} \quad , \quad n(x)\frac{dy}{ds}=b \quad , \quad n(x)\frac{dz}{ds}=c$$

$$\frac{dy}{dx}=\frac{b}{c} \quad \text{, then} \quad y=\frac{b}{c}\, z+d \quad (*)$$

There exists a unique plane perpendicular to the *y-z* plane, defined in (*).

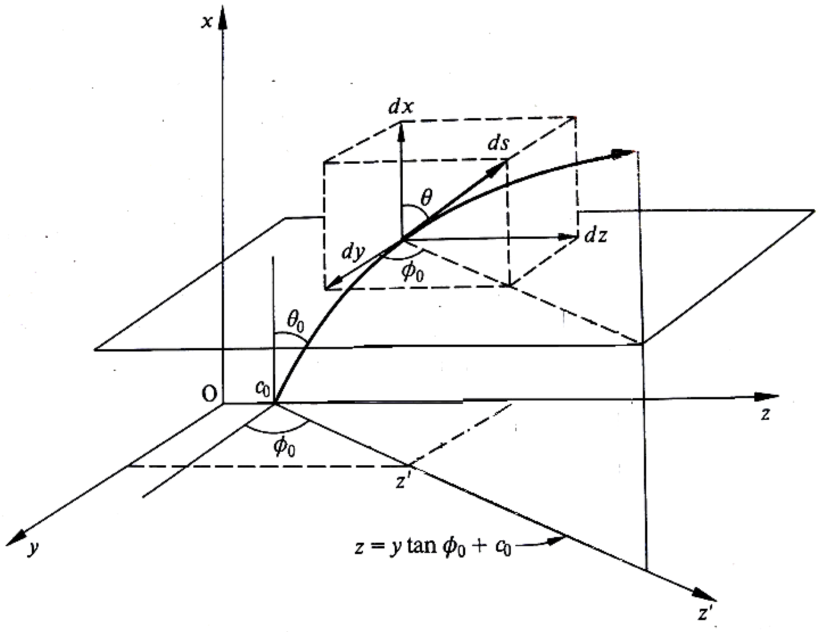
A ray entering a medium characterized by an *x*- dependent refractive index will always remain in this plane, regardless of launching conditions. The projection of the ray to y-z plane is a straight line. The constants are determined by the launching point and angle.



Launching point $(0,0,c_o)$. ϕ_o and θ_o as in the figure.

$z=y\tan\phi_o+c_o$ is the projection line.

Moreover: $y=\int \frac{b\,dx}{\sqrt{[n(x)]^2-(b^2+c^2)}}$, $z=\int \frac{c\,dx}{\sqrt{[n(x)]^2-(b^2+c^2)}}$.



From the figure, we have that:

$$ds\ \sin\theta\cos\phi_o=dy \quad , \quad ds\ \sin\theta\sin\phi_o=dz$$

$$n(x)\ \sin\theta\cos\phi_o=b \quad , \quad n(x)\ \sin\theta\sin\phi_o=c$$

$$n(x)\sin\theta=\sqrt{b^2+c^2} \quad \text{then} \quad n(x)\sin\theta=constant$$

This holds true throughout the trajectory. $n(x)\sin\theta=constant$ is the Snell Law for a one-dimensional stratified medium.

At $x=0$, $n_o\sin\theta_o=\sqrt{b^2+c^2}$, then:

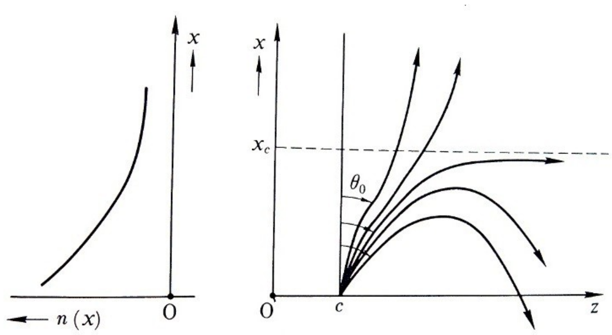
$$y=\int_0^x \frac{n_o\sin\theta_o\cos\phi_o}{\sqrt{[n(x)]^2-n_o^2\sin^2\theta_o}}dx \quad , \quad z-c_o=\int_0^x \frac{n_o\sin\theta_o\sin\phi_o}{\sqrt{[n(x)]^2-n_o^2\sin^2\theta_o}}dx$$

Let us introduce:

$$z' = \int_0^x \frac{n_o \sin\theta_o \sin\phi_o}{\sqrt{[n(x)]^2 - n_o^2 \sin^2\theta_o}} dx$$

When the quantity inside the square root becomes negative, we have that z' becomes negative and the light does not propagate. The light will reflect at the point where $[n(x)]^2 - n_o^2 \sin^2\theta_o$ becomes negative, i.e. $n(x) - n_o \sin\theta_o$.

When the refractive index is decreasing monotonically with *x*, the light will not propagate beyond $x = x_o$, where $n(x_o) = n_o \sin\theta_o$. The location of the total reflection is a function of the launching angle θ_o .



8. Selfoc fiber

As told in [6], in November 1968, "the first optical fiber for communication use, named Selfoc, was press-released. This fiber was initially made by ion-exchange method; a compound glass rod 1 mm in diameter and 1 m in length was immersed in a dilute nitric salt bath for several hundreds hours and then drawn to a fiber. This rod itself was just a graded refractive index (GRIN) lens. This was the first optical fiber for communication use. In 1970, Corning Glass Works disclosed a low loss silica glass fiber with step index type. This type of Silica fiber was made by using vapor phase or MOCVD". About the Selfoc fiber, see [7].

In [1], it is told that the distribution employed by the Selfoc fiber is:

$$n^2 = n_c^2(1-\alpha^2 x^2)$$

$$z' = \int \frac{a\,dx}{\sqrt{n_c^2 - a^2 - \alpha^2 n_c^2 x^2}} \quad , \quad a = n_c \sin^2\theta_o$$

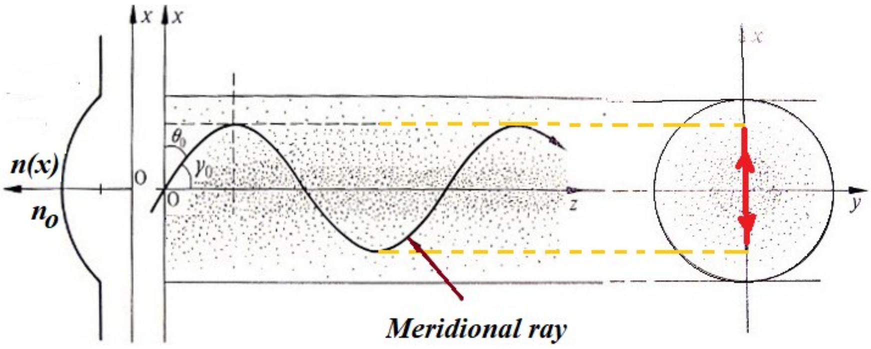
For a launching point at the origin:

$$z = \frac{\sin\theta_o}{\alpha}\int \frac{dx}{\sqrt{(\cos\theta_o/\alpha)^2 - x^2}} = \frac{\sin\theta_o}{\alpha}\sin^{-1}\frac{\alpha x}{\cos\theta_o}$$

$$x = \frac{\sin\gamma_o}{\alpha}\sin\frac{\alpha x}{\cos\gamma_o} \quad , \quad \gamma_o = 90° - \theta_o \quad .$$

The optical path in such a medium is sinusoidal with an oscillating amplitude equal to $\sin\gamma_o/\alpha$ and a one-quarter period $\pi\cos\gamma_o/2\alpha$.

The distribution of the Selfoc fiber is $n^2 = n_c^2(1-\alpha^2 r^2)$. With this distribution, light is confined inside the fiber and propagates for a long distance with very little loss, so that Selfoc fiber plays an important role in the fiber-optical communication.



9. Cylindrical symmetric medium

Let us consider a distribution having a cylindrical symmetry:

$$\nabla L = \hat{\mathbf{r}}\frac{\partial L}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial L}{\partial \phi} + \hat{\mathbf{k}}\frac{\partial L}{\partial z}$$

The eikonal equation is:

$$\left(\frac{\partial L}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial L}{\partial \phi}\right)^2 + \left(\frac{\partial L}{\partial z}\right)^2 = n^2$$

Let us separate variables:

$$L(r,\phi,z) = R(r) + \Phi(\phi) + Z(z)$$

$$(R')^2 + \left(\frac{1}{r}\Phi'\right)^2 + (Z')^2 = n^2$$

If n is cylindrical symmetric:

$$\{(R')^2 - [n(r)]^2\} + \left(\frac{1}{r}\Phi'\right)^2 + (Z')^2 = 0 \quad \text{assuming:} \quad -a^2 + a^2 = 0 \quad :$$

$$(Z')^2 = a^2 \quad , \quad \{(R')^2 - [n(r)]^2\} + \left(\frac{1}{r}\Phi'\right)^2 = -a^2$$

$$r^2\{(R')^2 - [n(r)]^2 + a^2\} + (\Phi')^2 = -a^2$$

Assuming $(\Phi')^2 = c^2$, we have: $(R')^2 = [n(r)]^2 - a^2 - \frac{c^2}{r^2}$.

The eikonal in a cylindrical symmetric medium is:

$$L(r,\phi,z) = \int_m^r \sqrt{[n(r)]^2 - c^2/r^2 - a^2}\; dr + c\phi + az$$

All integration constants are included in the lower limit of the integral.

The following differential equations are used to derive the path from the eikonal.

$$n\frac{dr}{ds}=\frac{\partial L}{\partial r} \qquad \hat{\boldsymbol{r}} \text{ component}$$

$$nr\frac{d\phi}{ds}=\frac{1}{r}\frac{\partial L}{\partial \phi} \qquad \hat{\boldsymbol{\phi}} \text{ component}$$

$$n\frac{dz}{ds}=\frac{\partial L}{\partial z} \qquad \hat{\boldsymbol{k}} \text{ component}$$

$$n\frac{dr}{ds}=\sqrt{[n(r)]^2-c^2/r^2-a^2} \quad , \quad n\frac{d\phi}{ds}=\frac{c}{r} \quad , \quad n\frac{dz}{ds}=a \quad .$$

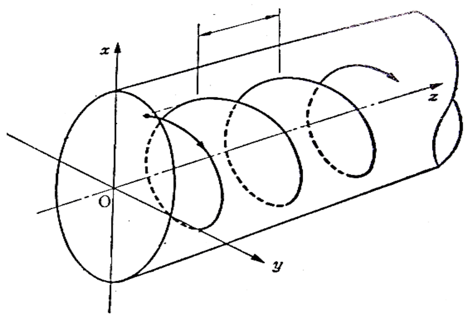
Constants a, c are to be determined from initial constants. Then:

$$\phi=\int\frac{c\,dr}{r^2\sqrt{[n(r)]^2-c^2/r^2-a^2}} \quad , \quad z=\int\frac{a\,dr}{\sqrt{[n(r)]^2-c^2/r^2-a^2}}$$

Launching conditions are: $r=r_o$, $\phi=\phi_o$, $z=0$, $n(r_o)=n_o$, $a=n_o\cos\gamma_o$, $c=n_o r_o\cos\delta_o$.

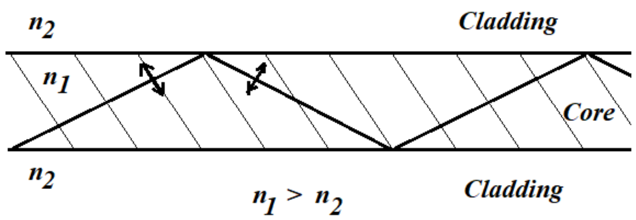
We can simply assume $c=0,\ \delta_o=90°$, that is when the launch path is in the plane containing the *z*-axis. ϕ becomes 0 and the path stays in a plane containing the *z*-axis. Such a ray is called a "meridional ray", We are back to the one-dimensional case. The ray that does not stay in this plane, due to a non zero *c*, rotates around the *z*-axis and it is called the "skew ray".

As long as the medium is perfectly symmetric, only the initial launch conditions determines whether the path is meridional or skew. The specific formulas for z and ϕ in the case of the Selfoc fiber are given in [1].



10. Quantization

In [1] it is shown that only those rays that are incident at certain discrete angles can propagate into the fiber. To simplify the discussion, [1] is using a slab (step-index guide).



If the incident angle is smaller enough, we have the total reflection of the ray. The ray is taking a zig-zag path, having a length l_{zz} . However, we have also a direct front-wave, moving of l_{dir} . The path difference is therefore:

$$n_1 k(l_{zz} - l_{dir}) = n_1 k\, 4\, d \sin\theta$$

d is the thickness of the slab. The angle θ so that:

$$n_1 k\, 4\, d \sin\theta = 2 N \pi$$

where N is an integer. This condition can be turned into:

$$\phi_x = N \pi$$

The phase difference ψ_x between the upper and the lower boundaries of the core glass can take only discrete values of an integer multiple of π. The same result is true for a meridional ray in an optical fiber.

In the case of a skew ray, the phase factor ψ must satisfy some conditions, so that:

$$\beta=\frac{\partial \phi}{\partial z}=n_c k\sqrt{1-2\alpha\left(\frac{2\mu+\nu}{n_c k}\right)} \quad (*)$$

where μ and ν are integers.

When the end of an optical fiber is excited by a light source, those rays which propagate are incident at discrete angles. For these rays, the propagation constant in the *z*-direction satisfies (*). The skew ray is designated by the pair of integers μ and ν. The ray is in the (μ,ν) mode of propagation.

The condition:

$$1-2\alpha\left(\frac{2\mu+\nu}{n_c k}\right)=0$$

is the cut-off condition. For $(0,0)$ mode, the cut-off is 0. The cut-off of $(0,1)$ mode is 2α. In the case that $k n_c<2\alpha$, only the $(0,0)$ mode is excited. The optical fiber where only a mode propagates, is a single-mode fiber.

11. Appendix - Rays

We have defined meridional and skew rays. However other specific rays exists. Then, let us given a list of them ([8]-[11] and other references at this LINK).

In optical systems:

1) meridional ray or tangential ray - a ray that is confined to the plane containing the system's optical axis and the object point from which the ray originated;

2) skew ray – a ray that does not propagate in a plane that contains both the object point and

the optical axis. Such rays do not cross the optical axis anywhere, and are not parallel to it;

3) marginal ray - in an optical system, it is the meridional ray that starts at the point where the object crosses the optical axis, and touches the edge of the aperture stop of the system. The ray crosses the optical axis again at the locations where an image will be formed;

4) principal ray or chief ray - it is the meridional ray that starts at the edge of the object, and passes through the center of the aperture stop. The distance between the principal ray and the optical axis, at an image location, is defining the size of the image. *The marginal and principal rays together define the Lagrange invariant, which is a measure of the light propagating through an optical system. The Lagrange invariant is a constant throughout all space;*

5) sagittal ray or transverse ray from an off-axis object point – it is a ray that propagates in the plane that is perpendicular to the meridional plane and contains the principal ray;

6) paraxial ray - a ray that makes a small angle to the optical axis of the system, and lies close to the axis throughout the system. Such rays can be modeled reasonably well by using the paraxial approximation;

7) finite ray or real ray - a ray that is traced without making the paraxial approximation;

8) parabasal ray - a ray that propagates close to some defined "base ray" rather than the optical axis. This is more appropriate than the paraxial model in systems that lack symmetry about the optical axis.

In fiber optics:

9) the meridional ray is a ray that passes through the axis of an optical fiber;

10) a skew ray is a ray that travels in a non-planar zig-zag path and never crosses the axis of an optical fiber;

11) a guided ray, bound ray, or trapped ray is a ray in a multi-mode optical fiber, which is confined by the core. In the case of a step index fiber, the light entering the fiber will be guided if it makes an angle with the fiber axis that is less than the fiber's acceptance angle;

12) a leaky ray or tunneling ray is a ray in an optical fiber that geometric optics predicts would totally reflect at the boundary between the core and the cladding, but which suffers loss due to

the curved core boundary.

11. Appendix – The directional derivative of a vector

We have mentioned the directional derivative of a scalar. Let us add some words about this derivative for a vector field $\boldsymbol{f}$. The gradient of a vector field is written as $\nabla \boldsymbol{f}=\frac{\partial f_j}{\partial x_i}\boldsymbol{e}_i\boldsymbol{e}_j$, where $\boldsymbol{e}_i$ are the unit vectors of the frame of reference, that is $\hat{\boldsymbol{i}},\hat{\boldsymbol{j}},\hat{\boldsymbol{k}}$ in the case of a Cartesian orthogonal frame.

Then, the directional derivative in the direction given by the unit vector $\boldsymbol{u}$ is:

$$D_{\boldsymbol{u}}\boldsymbol{f}(x,y,z)=\nabla \boldsymbol{f}(x,y,z)\cdot\boldsymbol{u}$$

$$D_{\boldsymbol{u}}=\nabla \boldsymbol{f}\cdot\boldsymbol{u}=\frac{\partial f_j}{\partial x_i}\boldsymbol{e}_i\boldsymbol{e}_j\cdot u_k\boldsymbol{e}_k=u_j\frac{\partial f_j}{\partial x_i}\boldsymbol{e}_i$$

Repeated indices are implicitly summed over.

See the example given at

https://math.stackexchange.com/questions/2995762/directional-derivatives-for-vector-valued-functions

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