Geometrical optics: Meridional and Skew Rays

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Abstract

In the framework of geometrical optics, let us study meridional and skew rays, those rays that we can find in optical fibres. The study is based on the book by Keigo Iizuka, entitled Engineering Optics.

Keywords: **Optics, Geometrical Optics, Fibers, Rays.**

Subject Areas: **Physics, Optics.**

1. Introduction

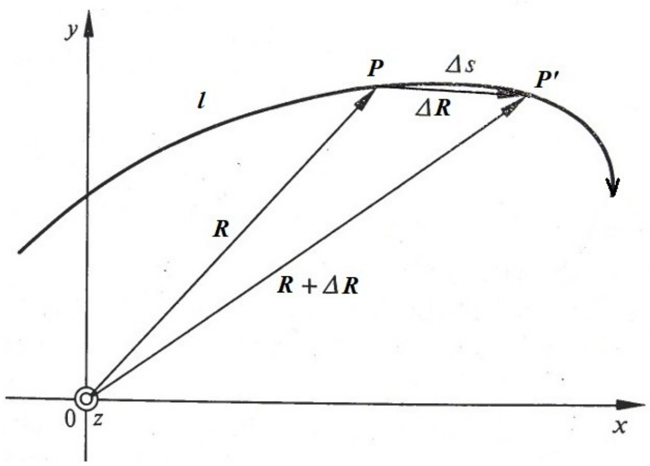
Geometrical optics, or ray optics, puts the emphasis on the light paths. A particular focus is in the calculation of the light path in inhomogeneous media and in the design of optical instruments [1]. Here we will study in particular two rays that we can find in optical fibers: the meridional ray and the skew ray. The study is based on the book by Keigo Iizuja, entitled Engineering Optics [1].

Before the specific study, let us remember some useful calculus.

2. Tangent unit vector and curvature

Some mathematical expressions are necessary for the description of the path of light.

Let us consider the following figure, where the position vector $\boldsymbol{R}$ is given.



The unit tangent vector is given by:

$$\hat{\mathbf{s}}=\frac{d\boldsymbol{R}}{ds}$$

In this formula, the curvilinear co-ordinate s is used. This is the arc length on the curve, given from a fixed origin on it.

Using rectangular coordinates: $\boldsymbol{R}=\hat{\boldsymbol{i}}\,x+\hat{\boldsymbol{j}}\,y+\hat{\boldsymbol{k}}\,z$, the unit tangent vector is:

$$\hat{\mathbf{s}}=\hat{\boldsymbol{i}}\frac{dx}{ds}+\hat{\boldsymbol{j}}\frac{dy}{ds}+\hat{\boldsymbol{k}}\frac{dz}{ds}$$

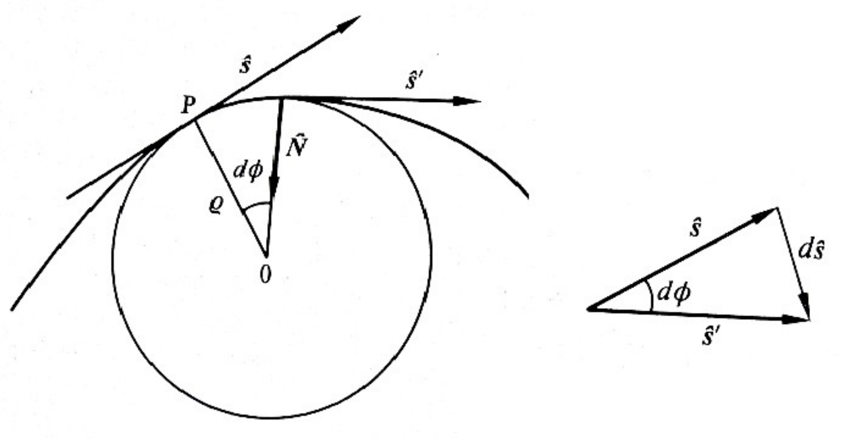
In cylindrical coordinate: $\boldsymbol{R}=\hat{\boldsymbol{r}}\,r+\hat{\boldsymbol{k}}\,z$, $\hat{\boldsymbol{r}}=\hat{\boldsymbol{i}}\cos\phi+\hat{\boldsymbol{j}}\sin\phi$, $\hat{\boldsymbol{\phi}}=-\hat{\boldsymbol{i}}\sin\phi+\hat{\boldsymbol{j}}\cos\phi$

$$\hat{\mathbf{s}}=\hat{\boldsymbol{r}}\frac{dr}{ds}+r\frac{d\hat{\boldsymbol{r}}}{ds}+\hat{\boldsymbol{k}}\frac{dz}{ds} \quad \text{, where} \quad \frac{d\hat{\boldsymbol{r}}}{ds}=(-\hat{\boldsymbol{i}}\sin\phi+\hat{\boldsymbol{j}}\cos\phi)\frac{d\phi}{ds}=\hat{\boldsymbol{\phi}}\frac{d\phi}{ds} \quad .$$

The tangent expressed in cylindrical coordinates becomes:

$$\hat{\mathbf{s}}=\hat{\boldsymbol{r}}\frac{dr}{ds}+\hat{\boldsymbol{\phi}}\,r\frac{d\phi}{ds}+\hat{\boldsymbol{k}}\frac{dz}{ds}$$

The curvature of a curve is: $\frac{1}{\rho}=|d\hat{\mathbf{s}}/ds|$ or $\frac{1}{\rho}=|d^2\boldsymbol{R}/ds^2|$.



The inverse of the radius of curvature is the magnitude of the first derivative of the tangent vector, or the second derivative of the position vector.

3. Principal unit vector, binormal and torsion

We have seen that the curvature is: $\frac{1}{\rho}=|d\,\hat{\mathbf{s}}/ds|$ or $\frac{1}{\rho}=|d^2\,\hat{\mathbf{R}}/ds^2|$.

Let us consider the time: $\frac{ds}{dt}=|d\,\mathbf{R}/dt|$. We have that $\hat{\mathbf{s}}=\frac{d\,\mathbf{R}/dt}{|d\,\mathbf{R}/dt|}$, and then the curvature turns out to be: $\frac{1}{\rho}=\frac{|d\,\hat{\mathbf{s}}/dt|}{|d\,\mathbf{R}/dt|}$

$\frac{d\,\hat{\mathbf{s}}}{ds}$ is also called the "curvature vector". The Principal Unit Normal is defined as:

$$\hat{\mathbf{N}}=\frac{d\,\hat{\mathbf{s}}/ds}{|d\,\hat{\mathbf{s}}/ds|}=\frac{d\,\hat{\mathbf{s}}/dt}{|d\,\hat{\mathbf{s}}/dt|}$$, which is orthogonal to $\hat{\mathbf{s}}$.

A third vector is the "binormal vector":

$$\hat{\mathbf{B}}=\hat{\mathbf{s}}\times\hat{\mathbf{N}}$$

It is another unit vector.

Altogether, we obtain a frame, which is known as the Frenet frame $\hat{\boldsymbol{s}}, \hat{\boldsymbol{N}}, \hat{\boldsymbol{B}}$. These vectors are of unit length and orthogonal to each other.

The Frenet- Serrat formula is:

$$\frac{d\hat{\mathbf{s}}}{ds}=\frac{\hat{\mathbf{N}}}{\rho} \quad ,$$

where vector $\hat{\boldsymbol{N}}$ is the unit normal vector to the curve.

About the binormal $\hat{\mathbf{B}}$, we have that it is a vector parallel to the unit normal $\hat{\boldsymbol{N}}$.

Being a unit vector, $d\hat{\mathbf{B}}/ds$ is perpendicular to $\hat{\boldsymbol{B}}$.

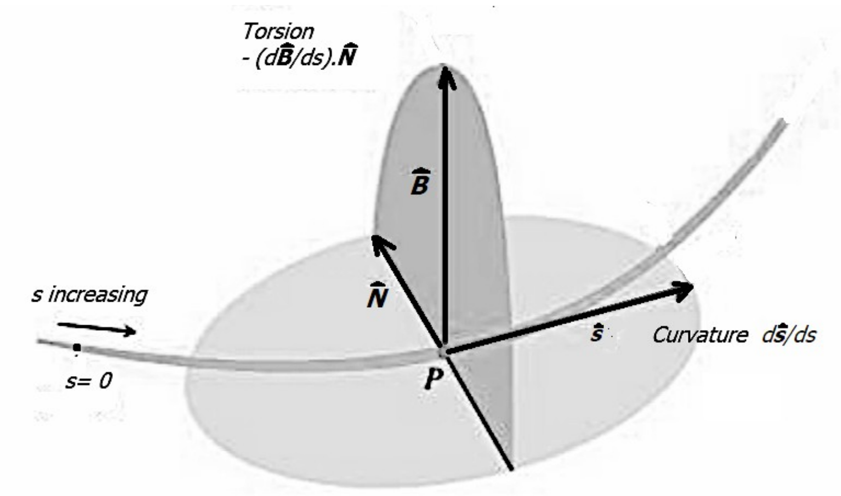
Moreover, $\hat{\boldsymbol{B}}\cdot\hat{\mathbf{s}}=0$, therefore:

$$\mathbf{0}=\frac{d\hat{\mathbf{B}}}{ds}\cdot\hat{\mathbf{s}}+\hat{\mathbf{B}}\cdot\frac{d\hat{\mathbf{s}}}{ds}=\frac{d\hat{\mathbf{B}}}{ds}\cdot\hat{\mathbf{s}}+\hat{\mathbf{B}}\cdot\frac{1}{\rho}\hat{\mathbf{N}}=\frac{d\hat{\mathrm{B}}}{ds}\cdot\hat{\mathbf{s}}$$

Since $\frac{d\hat{\mathbf{B}}}{ds}\cdot\hat{\boldsymbol{B}}=0$ and $\frac{d\hat{\mathbf{B}}}{ds}\cdot\hat{\boldsymbol{s}}=0$, $\frac{d\hat{\mathbf{B}}}{ds}$ is a multiple of $\hat{\mathbf{N}}$.

The multiple is the “torsion”:

$$\frac{d\hat{\mathbf{B}}}{ds}=-\tau\,\hat{\mathbf{N}} \qquad \text{or} \qquad \tau=-\frac{d\hat{\mathbf{B}}}{ds}\cdot\hat{\mathbf{N}}$$



4. Level surfaces and gradients

Let us consider a level surface $L(x,y,z)=C$. An example is the equipotential function in electrostatics. Let us move the observation point P so that we have an increment:

$$\Delta L=\frac{\partial L}{\partial x}\Delta x+\frac{\partial L}{\partial y}\Delta y+\frac{\partial L}{\partial z}\Delta z+\text{infinitesimals of higher order}$$

The directional derivative $\nabla_s L$ is:

$$\nabla_s L=\lim_{\Delta l\to 0}\frac{\Delta L}{\Delta l}=\frac{\partial L}{\partial x}\frac{\Delta x}{\Delta l}+\frac{\partial L}{\partial y}\frac{\Delta y}{\Delta l}+\frac{\partial L}{\partial z}\frac{\Delta z}{\Delta l}+\text{infinitesimals of higher order}$$

$\Delta x/\Delta l,\ \Delta y/\Delta l,\ \Delta z/\Delta l$ are the direction cosine $\cos\alpha,\ \cos\beta,\ \cos\gamma$ of the movement Δl .

$$\nabla_s L=\frac{dL}{dl}=\frac{\partial L}{\partial x}\cos\alpha+\frac{\partial L}{\partial y}\cos\beta+\frac{\partial L}{\partial z}\cos\gamma$$

We have also:

$$\nabla L=\hat{\boldsymbol{i}}\frac{\partial L}{\partial x}+\hat{\boldsymbol{j}}\frac{\partial L}{\partial y}+\hat{\boldsymbol{k}}\frac{\partial L}{\partial z}\quad,\quad \hat{\boldsymbol{M}}=\hat{\boldsymbol{i}}\cos\alpha+\hat{\boldsymbol{j}}\cos\beta+\hat{\boldsymbol{k}}\cos\gamma$$

Then:

$$\nabla_s L=(\nabla L)\cdot\hat{\boldsymbol{M}}$$, so we have: $$\Delta L=(\nabla L)\cdot\hat{\boldsymbol{M}}\,dl=(\nabla L)\cdot d\boldsymbol{l}$$

$\nabla_s L$ varies with the choice of the direction of the movement. The change of L becomes a maximum when the movement is selected in the same direction as that of ∇L . Then, ∇L is the direction that gives the maximum change in L for a given length of the movement.

The normal $\hat{\boldsymbol{N}}$ to the equi-level surface is:

$$\hat{\boldsymbol{N}}=\frac{\nabla L}{|\nabla L|}$$

In optics, this formula is particularly useful because it determines the optical path from the equi-phase surface.

5. Eikonal equation

Let $\boldsymbol{E}(x,y,z)$ representing a light wave that needs to satisfy the wave equation:

$$(\nabla^2+\omega^2\mu\epsilon)\boldsymbol{E}(x,y,z)=0$$

In the case of an isotropic medium, E_x, E_y, E_z equations are identical.

$$(\nabla^2+\omega^2\mu\epsilon)u(x,y,z)=0 \quad (*)$$

where $\omega^2\mu\epsilon=[k\,n(x,y,z)]^2$, k is the free space propagation constant and n the refraction index.

Let us assume the solution: $u(x,y,z)=A(x,y,z)\exp\{j[k\,L(x,y,z)-\omega t]\}$.

Functions $A(x,y,z)$ and $L(x,y,z)$ are unknown. They have to be determined in such a way to satisfy (*). Then:

$$n^2k^2u+\nabla^2u \;=$$

$$e^{j(kL-\omega t)}\,\{k^2[n^2-|\nabla L|^2]A+\nabla^2 A+j\,k\,A\,\nabla^2 L+j\,2k(\nabla A)\cdot(\nabla L)\}=0$$

$|\nabla L|^2$ means the sum of the squares of i,j,k components of ∇L .

If the wavelength of light is much shorter that the dimensions of the associated structure:

$$|\nabla L|^2=n^2 \;\;,\;\text{then:}\;\; \left(\frac{\partial L}{\partial x}\right)^2+\left(\frac{\partial L}{\partial y}\right)^2+\left(\frac{\partial L}{\partial z}\right)^2=n^2$$

This is the eikonal equation of the optical path. The wave front L itself is called the eikonal or optical path.

Let us remember that $\Delta L=(\nabla L)\cdot d\boldsymbol{l}$. Then:

$$L=\int(\nabla L)\cdot d\boldsymbol{l} \quad .$$

If the movement is restricted to the normal of the equi-level surface:

$$L=\int_{along\ normal}|\nabla L|ds=\int_{along\ normal} n\,ds \quad .$$

The direction of the normal to the equi-phase surface is $\nabla L/|\nabla L|$. The normal is called the "wave normal".

6. A glass slab

Consider a glass slab whose index of refraction is variable in the *x*-direction but is constant in both the *y*- and *z*- directions, so that $n=n(x)$. $L(x,y,z)$ is assumed to be separable so that $L(x,y,z)=f(x)+g(y)+h(z)$.

$$|\nabla L|^2=n^2 \quad \text{, then} \quad \{[f'(x)]^2-[n(x)]^2\}+[g'(y)]^2+[h'(z)]^2=0$$

Let us consider: $\{[f'(x)]^2-[n(x)]^2\}=a^2$, $[g'(y)]^2=b^2$, $[h'(z)]^2=c^2$, so that $a^2+b^2+c^2=0$.

Solutions are:

$$f(x)=\pm\int_0^x\sqrt{[n(x)]^2-(b^2+c^2)}\,dx \quad , \quad g(y)=\pm by+m_1 \quad , \quad h(z)=\pm cz+m_2$$

$$L(x,y,z)=\pm\int_m^x\sqrt{[n(x)]^2-(b^2+c^2)}\,dx\pm by\pm cz$$

Constants m_1 and m_2 are considered in the lower limit of the integration. $b,\ c$ are determined by boundary condition, such as launch position and angle.

The direction of the propagation is the direction of the wave normal: $\boldsymbol{N}=\nabla L/n$. The normal is in the same direction of the unit tangent vector $\hat{\mathbf{s}}$ to the light path.

$$\frac{d\mathbf{R}}{ds}=\frac{\nabla L}{n} \quad \text{then:} \quad n\frac{dx}{ds}=\frac{\partial L}{\partial x} \quad , \quad n\frac{dy}{ds}=\frac{\partial L}{\partial y} \quad , \quad n\frac{dz}{ds}=\frac{\partial L}{\partial z}$$

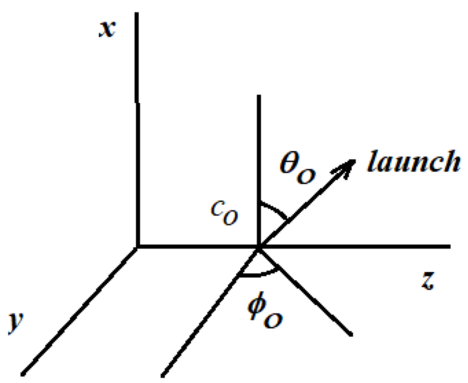
Let us consider the optical path of light launched into a medium having $n=n(x)$.

$$n(x)\frac{dx}{ds}=\sqrt{[n(x)]^2-(b^2+c^2)} \quad , \quad n(x)\frac{dy}{ds}=b \quad , \quad n(x)\frac{dz}{ds}=c$$

$$\frac{dy}{dx}=\frac{b}{c} \quad \text{, then} \quad y=\frac{b}{c}\ z+d \quad (*)$$

There exists a unique plane perpendicular to the *y-z* plane, defined in (*).

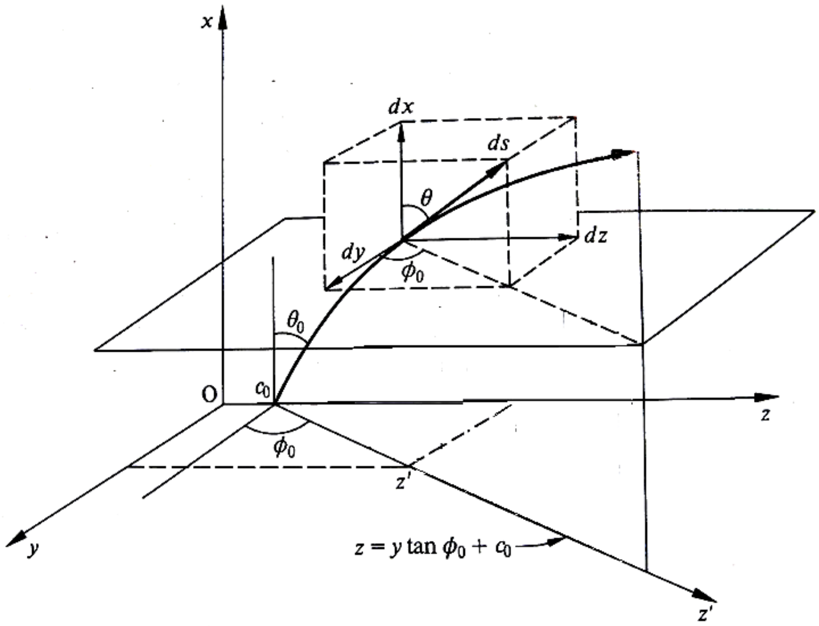
A ray entering a medium characterized by an *x*- dependent refractive index will always remain in this plane, regardless of launching conditions. The projection of the ray to y-z plane is a straight line. The constants are determined by the launching point and angle.



Launching point $(0,0,c_o)$. ϕ_o and θ_o as in the figure.

$z=y\tan\phi_o+c_o$ is the projection line.

Moreover: $$y=\int\frac{b\,dx}{\sqrt{[n(x)]^2-(b^2+c^2)}} \quad , \quad z=\int\frac{c\,dx}{\sqrt{[n(x)]^2-(b^2+c^2)}} \quad .$$



From the figure, we have that:

$$ds \ \sin\theta \cos\phi_o = dy \quad , \quad ds \ \sin\theta \sin\phi_o = dz$$

$$n(x) \ \sin\theta \cos\phi_o = b \quad , \quad n(x) \ \sin\theta \sin\phi_o = c$$

$$n(x)\sin\theta = \sqrt{b^2+c^2} \quad \text{then} \quad n(x)\sin\theta = constant$$

This holds true throughout the trajectory. $n(x)\sin\theta = constant$ is the Snell Law for a one-dimensional stratified medium.

At $x=0$, $n_o \sin\theta_o = \sqrt{b^2+c^2}$, then:

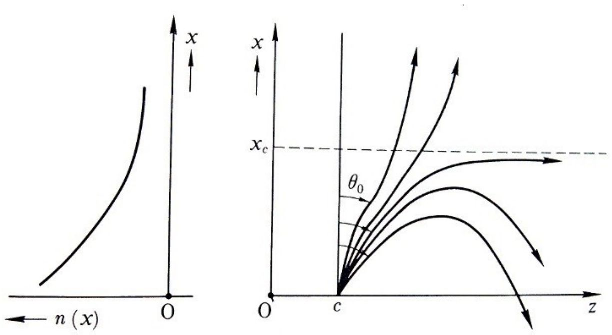
$$y = \int_0^x \frac{n_o \sin\theta_o \cos\phi_o}{\sqrt{[n(x)]^2 - n_o^2 \sin^2\theta_o}} dx \quad , \quad z - c_o = \int_0^x \frac{n_o \sin\theta_o \sin\phi_o}{\sqrt{[n(x)]^2 - n_o^2 \sin^2\theta_o}} dx$$

Let us introduce:

$$z' = \int_0^x \frac{n_o \sin\theta_o \sin\phi_o}{\sqrt{[n(x)]^2 - n_o^2 \sin^2\theta_o}} dx$$

When the quantity inside the square root becomes negative, we have that z' becomes negative and the light does not propagate. The light will reflect at the point where $[n(x)]^2 - n_o^2 \sin^2\theta_o$ becomes negative, i.e. $n(x) - n_o \sin\theta_o$.

When the refractive index is decreasing monotonically with x, the light will not propagate beyond $x = x_o$, where $n(x_o) = n_o \sin\theta_o$. The location of the total reflection is a function of the launching angle θ_o .



7. Selfoc fiber

As told in [2], in November 1968, "the first optical fiber for communication use, named Selfoc, was press-released. This fiber was initially made by ion-exchange method; a compound glass rod 1 mm in diameter and 1 m in length was immersed in a dilute nitric salt bath for several hundreds hours and then drawn to a fiber. This rod itself was just a graded refractive index (GRIN) lens. This was the first optical fiber for communication use. In 1970, Corning Glass Works disclosed a low loss silica glass fiber with step index type. This type of Silica fiber was made by using vapor phase or MOCVD". About the Selfoc fiber, see [3].

In [1], it is told that the distribution employed by the Selfoc fiber is:

$$n^2 = n_c^2 (1 - \alpha^2 x^2)$$

$$z' = \int \frac{a\, dx}{\sqrt{n_c^2 - a^2 - \alpha^2 n_c^2 x^2}} \quad , \quad a = n_c \sin^2\theta_o$$

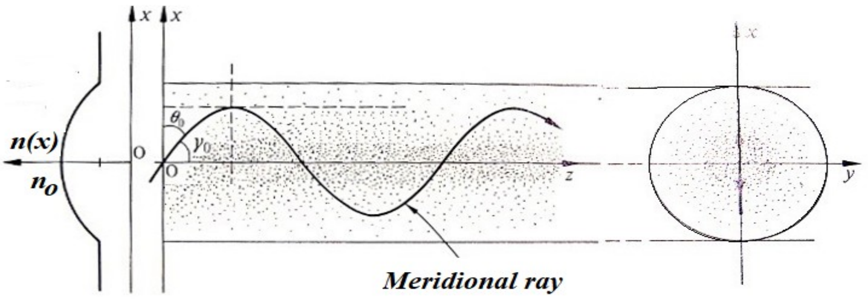
For a launching point at the origin:

$$z=\frac{\sin\theta_o}{\alpha}\int\frac{dx}{\sqrt{(\cos\theta_o/\alpha)^2-x^2}}=\frac{\sin\theta_o}{\alpha}\sin^{-1}\frac{\alpha x}{\cos\theta_o}$$

$$x=\frac{\sin\gamma_o}{\alpha}\sin\frac{\alpha x}{\cos\gamma_o} \quad , \quad \gamma_o=90°-\theta_o \quad .$$

The optical path in such a medium is sinusoidal with an oscillating amplitude equal to $\sin\gamma_o/\alpha$ and a one-quarter period $\pi\cos\gamma_o/2\alpha$.

The distribution of the Selfoc fiber is $n^2=n_c^2(1-\alpha^2 r^2)$. With this distribution, light is confined inside the fiber and propagates for a long distance with very little loss, so that Selfoc fiber plays an important role in the fiber-optical communication.



7. Cylindrical symmetric medium

Let us consider a distribution having a cylindrical symmetry:

$$\nabla L=\hat{\boldsymbol{r}}\frac{\partial L}{\partial r}+\hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial L}{\partial\phi}+\hat{\boldsymbol{k}}\frac{\partial L}{\partial z}$$

The eikonal equation is:

$$\left(\frac{\partial L}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial L}{\partial\phi}\right)^2+\left(\frac{\partial L}{\partial z}\right)^2=n^2$$

Let us separate variables:

$$L(r,\phi,z)=R(r)+\Phi(\phi)+Z(z)$$

$$(R')^2+\left(\frac{1}{r}\Phi'\right)^2+(Z')^2=n^2$$

If n is cylindrical symmetric:

$$\{(R')^2-[n(r)]^2\}+\left(\frac{1}{r}\Phi'\right)^2+(Z')^2=0 \quad \text{assuming:} \quad -a^2+a^2=0 \ :$$

$$(Z')^2=a^2 \quad , \quad \{(R')^2-[n(r)]^2\}+\left(\frac{1}{r}\Phi'\right)^2=-a^2$$

$$r^2\{(R')^2-[n(r)]^2+a^2\}+(\Phi')^2=-a^2$$

Assuming $(\Phi')^2=c^2$, we have: $(R')^2=[n(r)]^2-a^2-\frac{c^2}{r^2}$.

The eikonal in a cylindrical symmetric medium is:

$$L(r,\phi,z)=\int_m^r \sqrt{[n(r)]^2-c^2/r^2-a^2}\ dr+c\phi+az$$

All integration constants are included in the lower limit of the integral.

The following differential equations are used to derive the path from the eikonal.

$$n\frac{dr}{ds}=\frac{\partial L}{\partial r} \qquad \hat{\boldsymbol{r}} \text{ component}$$

$$nr\frac{d\phi}{ds}=\frac{1}{r}\frac{\partial L}{\partial \phi} \qquad \hat{\boldsymbol{\phi}} \text{ component}$$

$$n\frac{dz}{ds}=\frac{\partial L}{\partial z} \qquad \hat{\boldsymbol{k}} \text{ component}$$

$$n\frac{dr}{ds}=\sqrt{[n(r)]^2-c^2/r^2-a^2} \quad , \quad n\frac{d\phi}{ds}=\frac{c}{r} \quad , \quad n\frac{dz}{ds}=a \quad .$$

Constants a, c are to be determined from initial constants. Then:

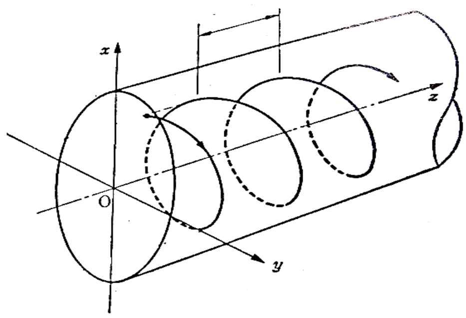
$$\phi=\int\frac{c\,dr}{r^2\sqrt{[n(r)]^2-c^2/r^2-a^2}} \quad , \quad z=\int\frac{a\,dr}{\sqrt{[n(r)]^2-c^2/r^2-a^2}}$$

Launching conditions are: $r=r_o$, $\phi=\phi_o$, $z=0$, $n(r_o)=n_o$, $a=n_o\cos\gamma_o$,

$c = n_o r_o \cos \delta_o$.

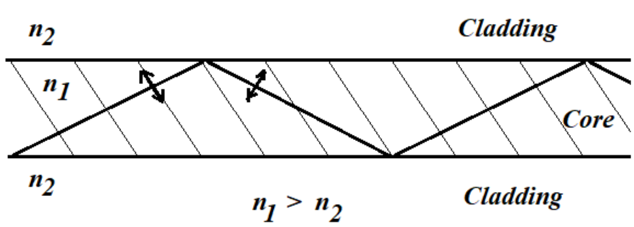
We can simply assume $c = 0,\ \delta_o = 90°$, that is when the launch path is in the plane containing the *z*-axis. ϕ becomes 0 and the path stays in a plane containing the *z*-axis. Such a ray is called a "meridional ray", We are back to the one-dimensional case. The ray that does not stay in this plane, due to a non zero *c*, rotates around the *z*-axis and it is called the "skew ray".

As long as the medium is perfectly symmetric, only the initial launch conditions determines whether the path is meridional or skew. The specific formulas for z and ϕ in the case of the Selfoc fiber are given in [1].



8. Quantization

In [1] it is shown that only those rays that are incident at certain discrete angles can propagate into the fiber. To simplify the discussion, [1] is using a slab (step-index guide).



If the incident angle is smaller enough, we have the total reflection of the ray. The ray is taking a zig-zag path, having a length l_{zz} . However, we have also a direct front-wave, moving of

l_{dir} . The path difference is therefore:

$$n_1 k(l_{zz} - l_{dir}) = n_1 k 4 d \sin\theta$$

d is the thickness of the slab. The angle θ so that:

$$n_1 k 4 d \sin\theta = 2N\pi$$

where N is an integer. This condition can be turned into:

$$\phi_x = N\pi$$

The phase difference ψ_x between the upper and the lower boundaries of the core glass can take only discrete values of an integer multiple of π . The same result is true for a meridional ray in an optical fiber.

In the case of a skew ray, the phase factor ψ must satisfy some conditions, so that:

$$\beta = \frac{\partial \phi}{\partial z} = n_c k \sqrt{1 - 2\alpha\left(\frac{2\mu + \nu}{n_c k}\right)} \quad (*)$$

where μ and ν are integers.

When the end of an optical fiber is excited by a light source, those rays which propagate are incident at discrete angles. For these rays, the propagation constant in the *z*-direction satisfies (*). The skew ray is designated by the pair of integers μ and ν . The ray is in the (μ, ν) mode of propagation.

The condition:

$$1 - 2\alpha\left(\frac{2\mu + \nu}{n_c k}\right) = 0$$

is the cut-off condition. For $(0,0)$ mode, the cut-off is 0. The cut-off of $(0,1)$ mode is 2α . In the case that $k n_c < 2\alpha$, only the $(0,0)$ mode is excited. The optical fiber where only a mode propagates, is a single-mode fiber.

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[2] https://dbnst.nii.ac.jp/english/detail/922

[3] Teiji Uchida, Motoaki Furukawa, Ichiro Kitano, Ken Koizumi, and Hiroyoshi Matsumura 、 Optical Characterisitics of a Light-Focusing Fiber Guide and Its Applications、1970、IEEE Journal of Quantum Electronics, Vol.QE-6, No.10, October 1970