Method for Adding a new Bosonic Particle to the Equation of State in MESA

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This document describes in detail our method for adding a new bosonic particle to the equation of state module in MESA version 12778. Example working directories are included with our reproduction package. We welcome questions via email.

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I. INTRODUCTION

Our method of updating the MESA equation of state (EOS) to include a new bosonic particle ϕ (not necessarily a scalar) of mass m_ϕ and degeneracy $g_\phi = 2s + 1$ is a three-stage process. First, we create tables of the relevant thermodynamic quantities e.g. pressure, specific entropy. Second, we produce analytic fitting functions for these tables. Third, we recompute the EOS in MESA using the `other_eos` hook in `run_star_extras`.

Our implementation is completely general and there is no need to perform further modifications to `run_star_extras` when the mass and degeneracy are changed. Given a a value of m_ϕ and g_ϕ, our Mathematica script (`boson_in_MESA.nb`, provided with this reproduction package) will generate an inlist of extra controls corresponding the coefficients of the fitting functions that is read at the start of each run and used in the `other_eos` routine, which already has the functional form of the fitting functions implemented. Our method is to first call the default MESA EOS from within the `other_eos` subroutine (called `heavy_DM_eosDT_get` in the code) by calling the subroutine `eos_DTget` and then modifying the EOS by passing the MESA pointer containing the EOS variables (`res(:)`) to the subroutine `add_DM_DT`. The modified form of `res(:)` is then passed to MESA by `heavy_DM_eosDT_get`. After the simulation, the user should check that the temperature evolution ($\dot{T}/T$) does not exceed the production rate of the particle in the star.

We now describe each of the three steps above in turn.

II. THERMODYNAMIC QUANTITIES FOR THE NEW BOSONIC PARTICLE

The goal of this section is to derive the pressure, density, internal energy, and specific entropy of the new bosonic particle. This is achieved by integrating over the Bose-Einstein distribution. We define

$$C_\phi = \frac{1}{\pi^2}\left(\frac{m_\phi c}{\hbar}\right)^3, \quad \text{and} \quad \beta(T) = \frac{m_\phi c^2}{k_B T}. \tag{1}$$

Note that we are assuming that the new particles are not charged under any gauge groups and therefore have zero chemical potential. The thermodynamic quantities are

$$P_\phi(\beta) = m_\phi c^2 C_\phi \left(\frac{g_\phi}{2}\right) H_1(\beta) \tag{2}$$

$$\rho_\phi(\beta) = m_\phi C_\phi \left(\frac{g_\phi}{2}\right) H_2(\beta) \tag{3}$$

$$u_\phi(\beta) = m_\phi c^2 C_\phi \left(\frac{g_\phi}{2}\right) H_3(\beta) \tag{4}$$

$$s_\phi = \frac{k_B C_\phi \beta}{\rho}\left(\frac{g_\phi}{2}\right)\left[H_1(\beta) + H_3(\beta)\right], \tag{5}$$

where

$$H_1(\beta) = \int_{\varepsilon=\beta}^{\infty} G\left(\frac{\varepsilon}{\beta}\right) B(\varepsilon)\frac{\mathrm{d}\varepsilon}{\beta} \tag{6}$$

$$H_2(\beta) = \int_{\varepsilon=\beta}^{\infty} G'\left(\frac{\varepsilon}{\beta}\right) B(\varepsilon)\frac{\mathrm{d}\varepsilon}{\beta} \tag{7}$$

$$H_3(\beta) = \int_{\varepsilon=\beta}^{\infty} \varepsilon G'\left(\frac{\varepsilon}{\beta}\right) B(\varepsilon)\frac{\mathrm{d}\varepsilon}{\beta^2} \tag{8}$$

$$B(\varepsilon) = \frac{1}{e^\varepsilon - 1} \tag{9}$$

$$G(x) = \frac{1}{3}(x^2 - 1)^{\frac{3}{2}}. \tag{10}$$

These are evaluated numerically to high-precision in our Mathematica script discussed in the next section.

III. FITTING FUNCTIONS

As we will see in section IV, we require fitting functions for the following quantities:

$$P_\phi,\, u_\phi,\, s_\phi,\, \frac{\partial P_\phi}{\partial T},\, \frac{\partial E_\phi}{\partial T},\, \frac{\partial s_\phi}{\partial T},\, \frac{\partial s_\phi}{\partial \rho},\, \frac{\partial^2 E_\phi}{\partial T^2},\, \frac{\partial^2 E_\phi}{\partial T \partial \rho},\, \frac{\partial^2 P_\phi}{\partial T^2},\, \frac{\partial^2 s_\phi}{\partial T^2},\, \frac{\partial^2 s_\phi}{\partial \rho^2},\, \frac{\partial^2 s_\phi}{\partial T \partial \rho}, \tag{11}$$

where $E_\phi = u_\phi/\rho$. Since P_ϕ, u_ϕ, and s_ϕ are analytic in ρ, it is convenient to fit for quantities that are only dependent upon temperature. For example, rather than fitting for E_ϕ, which depends on ρ, we fit for $u_\phi(T) = \rho E_\phi$ and then code the ρ-dependence into MESA. The ρ-dependence of the fitting functions is given in the inlists containing the coefficients for the fitting functions.

We fit each of the quantities in equation 11 (in some cases multiplied by powers of ρ), which we denote collectively by $\{X_i\}$, to an eighth-order polynomial of the form

$$\log_{10}(X_i) = \sum_{n=0}^{8} c_{i,n} \log_{10}^n(T_8) \qquad \text{where} \qquad T_8 = T/(10^8\text{K}), \tag{12}$$

over a range spanning $T_{8,\text{initial}} \leq T_8 \leq T_{8,\text{final}}$ where $X_i(T_{8,\text{initial}}) \ll \bar{X}_i$ with $\bar{X}_i$ the MESA default (such that the contribution of ϕ to the EOS is subdominant) and $T_{8,\text{final}}$ an upper temperature of the simulation. In practise, we find we may use an initial temperature $k_B T_{8,\text{initial}} = 10^{-2.5} m_\phi c^2/10^8\text{K}$ and $T_{8,\text{final}} = 10^2$. The coefficients $c_{i,n}$ need to be determined to a number of decimal places depending on the order of the polynomial and the temperature range over which the new controls are switched on. For example, for a temperature range of three orders of magnitude the coefficients need to be determined with a precision of four or more decimals. In the supplied notebook, the coefficients are generated with 14 digit precision. The coefficients are then written to extra controls (x_ctrl()) in a new inlist called inlist_$m_\phi$keV_g_ϕDOF, which can directly be used in the supplied MESA module. For example, the coefficients for the heavy axion with mass $m_a = m_e = 511$ keV studied in our paper would be output to inlist_511kev_1DOF.

IV. INCORPORATING THE NEW BOSON INTO THE MESA EOS

The results of the default MESA EOS are contained in a pointer calles `res(:)`. Information on this is given in `$MESA_DIR/eos/public/eos_def.f90`. There are 16 entries corresponding to the 16 EOS variables used by MESA. These are shown in table I. Three of these, μ, μ_e, and η, are unaffected by the inclusion of new particles so we will not discuss them further. This leaves 13 quantities to modify. In what follows, we use overbars to denote quantities returned by the default MESA EOS.

Since we are calling the default MESA EOS and modifying it rather than recomputing all quantities from first principles, it is not necessarily the case that we can simply add the bosonic contribution to a quantity to its default value. To exemplify the reason for this, we define two concepts of addition. A quantity is additive if we can algebraically (theoretically) add contributions from different species. An example is the following:

$$\left(\frac{\partial P}{\partial T}\right)_s. \tag{13}$$

Since the pressure P is just the sum of the pressure of each individual species, the quantity above is then found by summing

$$\left(\frac{\partial P_i}{\partial T}\right)_s \tag{14}$$

for each species. Now consider adding a new bosonic particle. If we were calculating this contribution concurrently with all other particle species then we could simply add it to the sum but this is not how our procedure works. The default MESA EOS returns the array `res(:)` defined in each cell (each cell has a specified T and ρ). This means that the value that we start with is

$$\sum_{i\neq \text{DM}} \left(\frac{\partial P_i}{\partial T}\right)_{\bar{s}},(15)$$

i.e. the sum over all species except the new bosonic particle computed at constant $\bar{s}$, where $\bar{s}$ is the entropy of all species excluding the boson. This is where the technicality arises. We cannot simply add an expression calculated for the boson in isolation because the numerical returned by the default MESA EOS is calculated holding a different quantity constant.

The discussion above leads to the second concept of addition. A quantity is MESA-additive if, when passed a numerical `res(i)` EOS quantity by MESA, we can add the DM contribution, possibly up to some multiplicative factor. In the example above, the quantity is additive but not MESA-additive. All MESA-additive quantities are additive but not all additive quantities are MESA-additive. Since T and ρ are defined in each cell, or, equivalently, since we are using T and ρ as the input variables for the EOS, the MESA-additive quantities are P, u, s, and derivatives at constant T and ρ. All other quantities are not MESA additive.

Our procedure for incorporating the bosonic particle into the MESA EOS is then as follow. First, we use the MESA-additive quantities passed by MESA and update them to include the boson. All quantities that are not MESA-additive can then be recalculated in full from these additive quantities, which now correspond to the combined fluid.

We now detail the calculation of each quantity in `res(i)` excluding those that are not modified. To begin, we define $Y = \rho T c_V$ and $X = P\chi_T$ for later convenience. In what follows, one should assume that all barred quantities are known since they are passed by the default MESA EOS, and that any previous quantities have already been calculated.

res(1): Gas pressure. We calculate this as $P_{\mathrm{g}} = \bar{P}_{\mathrm{g}} + P_\phi$. We have a fitting function for P_ϕ.

res(2): Specific internal energy per gram. This is given by

$$E = \sum_{\mathrm{species}\,i} \frac{u_i}{\rho} \tag{16}$$

so we calculate this as:

$$E = \bar{E} + \frac{u_\phi}{\rho}. \tag{17}$$

We have a fitting function for u_ϕ.

res(3): Specific entropy. We calculate this as $s = \bar{s} + s_\phi$. We have a fitting fucntion for s_ϕ

res(11): c_V. This is MESA-additive since it is a derivative at constant ρ and one therefore has

$$c_V = \left(\frac{\partial E}{\partial T}\right)_\rho = \left(\frac{\partial \bar{E}}{\partial T}\right)_\rho + \left(\frac{\partial E_\phi}{\partial T}\right)_\rho. \tag{18}$$

We this means we can simply take the MESA value and add

$$\frac{1}{\rho}\left(\frac{\partial u_\phi}{\partial T}\right)_\rho = -k_B C_\phi \frac{g}{2}\frac{\beta^2}{\rho}\frac{\mathrm{d}H_3(\beta)}{\mathrm{d}\beta}. \tag{19}$$

We have a fitting function for $\partial u_\phi/\partial T$.

res(8): χ_ρ. From the definition in table I we have

$$\chi_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T = \frac{\rho}{P}\left(\frac{\partial P}{\partial \rho}\right)_T. \tag{20}$$

Now $\left(\frac{\partial P}{\partial \rho}\right)$ is MESA-additive but P_ϕ only depends on T there is no correction. Note however that what MESA passes is

$$\bar{\chi}_\rho = \left(\frac{\partial \ln \bar{P}}{\partial \ln \rho}\right)_T = \frac{\rho}{\bar{P}}\left(\frac{\partial \bar{P}}{\partial \rho}\right)_T. \tag{21}$$

so it is necessary to alter this entry because we need the total pressure (including the boson) in the denominator. This change is

$$\chi_\rho = \frac{\bar{P}}{P}\bar{\chi}_\rho. \tag{22}$$

$\bar{P}$ is passed to us by MESA and P has already been calculated above.

res(9): χ_T. One has

$$\chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho = \frac{T}{P}\left[\left(\frac{\partial \bar{P}}{\partial T}\right)_\rho + \left(\frac{\partial P_\phi}{\partial T}\right)_\rho\right] \tag{23}$$

with

$$\left(\frac{\partial P_\phi}{\partial T}\right)_\rho = -k_B C_{\rm d}\frac{g}{2}\beta^2\frac{{\rm d}H_1(\beta)}{{\rm d}\beta}. \tag{24}$$

The complete calculation is

$$\chi_T = \frac{\bar{P}}{P}\bar{\chi}_T + \frac{T}{P}\left(\frac{\partial P_\phi}{\partial T}\right)_\rho, \tag{25}$$

where the first term accounts for the fact that the pressure included in the value of χ_T that MESA passes us does not include the new boson.

res(13): This is

$$\left(\frac{\partial s}{\partial T}\right)_\rho = \left(\frac{\partial \bar{s}}{\partial T}\right)_\rho + \left(\frac{\partial s_\phi}{\partial T}\right)_\rho. \tag{26}$$

One finds

$$\left(\frac{\partial s_\phi}{\partial T}\right)_\rho = -\frac{k_B C_\phi}{\rho}\frac{g}{2}\frac{\beta}{T}\left[H_1(\beta) + H_3(\beta) + \beta\left(\frac{{\rm d}H_1(\beta)}{{\rm d}\beta} + \frac{{\rm d}H_3(\beta)}{{\rm d}\beta}\right)\right].$$

We have a fitting function for this quantity.

res(12): This is

$$\left(\frac{\partial E}{\partial \rho}\right)_T = \left(\frac{\partial \bar{E}}{\partial \rho}\right)_T - \frac{u_\phi}{\rho^2} \tag{27}$$

where we used the fact that u_ϕ is independent of ρ.

Once we have all of the additive expressions above, we can use them to calculate the remaining quantities that are not MESA-additive and cannot be found with a simple correction to the numerical results that MESA passes. This has to be done in a specific order because previous results are used to calculate subsequent quantities. Note that since the additive quantities now include the corrections due to the boson, using them to calculate the remaining quantities ensures that the effects of the boson are propagated. At this point, we also have all of the requisite quantities to calculate X and Y. The remaining quantities are then calculated as follows:

ref(16): This is

$$\Gamma_3 = 1 + \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_s = 1 + \frac{X}{Y}. \tag{28}$$

res(15): This is

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_s = \chi_\rho + \chi_T\frac{X}{Y}. \tag{29}$$

res(7): This is

$$\nabla_{\rm ad} = \left(\frac{\partial T}{\partial \ln P}\right)_s = \frac{X}{Y\Gamma_1}. \tag{30}$$

res(10): This is

$$c_P = \frac{c_V}{\chi_\rho}\Gamma_1. \tag{31}$$

A. Derivatives

In addition to the thermodynamic variables discussed above, MESA contains two additional vectors that contain the derivatives of these quantities, which are crucial for determining the time-step. These are `d_dlnRho_const_T(:)` and `d_dlnT_const_Rho(:)`. These correspond to

$$\left(\frac{\partial\ \texttt{res(i)}}{\partial\ln T}\right)_\rho \quad \text{and} \quad \left(\frac{\partial\ \texttt{res(i)}}{\partial\ln\rho}\right)_T \tag{32}$$

respectively. In this section, we describe our implementation of these into MESA. As with the EOS quantities, we first calculate all MESA-additive quantities and then use these to recalculate the remaining derivatives. In this section, the quantities appear in the order in which they appear in `res(:)`.

1. `d_dlnRho_const_T(:)`

1. $(\partial\ln P_{\rm gas}/\partial\ln\rho)_T$. Note that P_ϕ is independent of ρ.

$$\left(\frac{\partial\ln P_{\rm gas}}{\partial\ln\rho}\right)_T = \frac{\rho}{P_{\rm gas}}\left(\frac{\partial P_{\rm gas}}{\partial\rho}\right)_T = \frac{\bar{P}_{\rm gas}}{P_{\rm gas}}\frac{\rho}{\bar{P}_{\rm gas}}\left(\frac{\partial P_{\rm gas}}{\partial\rho}\right)_T = \frac{\bar{P}_{\rm gas}}{P_{\rm gas}}\left(\frac{\partial\ln\bar{P}_{\rm gas}}{\partial\ln\rho}\right)_T. \tag{33}$$

2. $(\partial\ln E_{\rm gas}/\partial\ln\rho)_T$ Note that $E = u/\rho$.

$$\left(\frac{\partial\ln E}{\partial\ln\rho}\right)_T = \frac{\rho}{E}\frac{\partial E}{\partial\rho} = \frac{\bar{E}}{E}\frac{\rho}{\bar{E}}\left(\frac{\partial E}{\partial\rho} + \frac{\partial E_\phi}{\partial\rho}\right) = \frac{\bar{E}}{E}\frac{\partial\ln\bar{E}}{\partial\ln\rho} + \frac{\rho}{E}\frac{\partial E_\phi}{\partial\rho}.$$

   All derivatives are at constant T. We have E from the EOS we calculated above, the barred quantities from the default MESA EOS, and a fitting function for the final derivative.

3. $(\partial\ln s/\partial\ln\rho)_T$. Precisely the same logic as $(\partial\ln E_{\rm gas}/\partial\ln\rho)_T$ but with $E\to s$:

$$\left(\frac{\partial\ln s}{\partial\ln\rho}\right)_T = \frac{\bar{s}}{s}\frac{\partial\ln\bar{s}}{\partial\ln\rho} + \frac{\rho}{s}\frac{\partial s_\phi}{\partial\rho}. \tag{34}$$

   We have a fitting function for $(\partial s_\phi/\partial\rho)$.

4-6. μ, μ_e, η. No change.

7. $(\partial c_v/\partial\ln\rho)_T$. $c_v = (\partial E/\partial\rho)T$. We have a fitting function for the final derivative.

$$\left(\frac{\partial c_v}{\partial\ln\rho}\right)_T = \rho\left(\left(\frac{\partial\bar{c}_v}{\partial\rho}\right)_T + \frac{\partial^2 E_\phi}{\partial T\partial\rho}\right) = \left(\frac{\partial\bar{c}_v}{\partial\ln\rho}\right)_T + \rho\frac{\partial^2 E_\phi}{\partial T\partial\rho}. \tag{35}$$

8. $(\partial\chi_\rho/\partial\ln\rho)_T$. $\chi_\rho = (\partial\ln P/\partial\ln\rho)_T$.

$$\left(\frac{\partial\chi_\rho}{\partial\ln\rho}\right)_T = \rho\frac{\partial}{\partial\rho}\left(\frac{\partial\ln P}{\partial\ln\rho}\right) = \rho\frac{\partial}{\partial\rho}\left(\frac{\rho}{P}\frac{\partial P}{\partial\rho}\right) = -\frac{\rho^2}{P^2}\left[\frac{\partial P}{\partial\rho}\right]^2 + \frac{\rho^2}{P}\frac{\partial^2 P}{\partial\rho^2} + \frac{\rho}{P}\frac{\partial P}{\partial\rho} = \chi_\rho - \chi_\rho^2 + \frac{\rho^2}{P}\frac{\partial^2\bar{P}}{\partial\rho^2}. \tag{36}$$

   All derivatives at constant T. Note that in the final equality we used the fact that P_ϕ is independent of ρ. Note further that $P_{\rm rad}$ is independent of ρ. To calculate this, we can use the value of χ_ρ that we calculated in the previous section, but we need $\partial^2\bar{P}/\partial\rho^2$. MESA does not output this so we need an expression for it in terms of quantities MESA does provide. To find this, we use equation (36) and replace all quantities with their barred version and re-arrange to find

$$\frac{\partial^2\bar{P}}{\partial\bar{\rho}^2} = \frac{\bar{P}}{\rho^2}\left(\bar{\chi}_\rho^2 - \bar{\chi}_\rho + \frac{\partial\bar{\chi}_\rho}{\partial\ln\rho}\right). \tag{37}$$

   MESA gives us both expressions on the right.

9. $(\partial\chi_T/\partial\ln\rho)_T$. $\chi_T = (\partial\ln P/\partial\ln T)_\rho$.

$$\left(\frac{\partial\chi_T}{\partial\ln\rho}\right)_T = \rho\left(\frac{\partial}{\partial\rho}\frac{\partial\ln P}{\partial\ln T}\right) = \rho\frac{\partial}{\partial\rho}\left(\frac{T}{P}\frac{\partial P}{\partial T}\right) = \rho T\left(-\frac{1}{P^2}\frac{\partial P}{\partial\rho}\frac{\partial P}{\partial T} + \frac{1}{P}\frac{\partial^2 P}{\partial T\partial\rho}\right)$$
$$= -\frac{\rho}{P}\frac{P}{\rho}\frac{\partial T}{\partial P}\frac{\partial P}{\partial T} + \frac{\rho T}{P}\frac{\partial^2 P}{\partial T\partial\rho} = -\chi_T\chi_\rho + \frac{\rho T}{P}\frac{\partial^2\bar{P}}{\partial T\partial\rho}. \tag{38}$$

Derivatives with respect to ρ are at constant T and vice-versa. The final quantity is not computed in MESA so we use the method introduced above where we replace all quantities with their barred versions to find it in terms of quantities MESA does compute:

$$\frac{\partial^2\bar{P}}{\partial T\partial\rho} = \frac{\bar{P}}{\rho T}\left(\frac{\partial\bar{\chi}_T}{\partial\ln\rho} + \bar{\chi}_T\bar{\chi}_\rho\right). \tag{39}$$

10. $(\partial\Gamma_3/\partial\ln\rho)_T$. $\Gamma_3 = 1 + X/Y$, $X = P\chi_T$, $Y = \rho T c_v$.

$$\left(\frac{\partial\Gamma_3}{\partial\ln\rho}\right)_T = \frac{\partial}{\partial\ln\rho}\left(\frac{P\chi_T}{\rho T c_v}\right) = -\frac{X}{Y} + \frac{\chi_T}{\rho T c_v}\frac{\partial P}{\partial\ln\rho} + \frac{P}{\rho T c_v}\frac{\partial\chi_T}{\partial\ln\rho} - \frac{\chi_T P}{\rho T c_v^2}\frac{\partial c_v}{\partial\ln\rho} \tag{40}$$
$$= \frac{X}{Y}\left[-1 + \frac{1}{\chi_T}\frac{\partial\chi_T}{\partial\ln\rho} - \frac{1}{c_v}\frac{\partial c_v}{\partial\ln\rho} + \frac{1}{P}\frac{\partial P}{\partial\ln\rho}\right].$$

We have all of the quantities from previous computations, except for the final term in the brackets. We find this by noting that $P = P_{\rm gas} + P_{\rm rad} + P_\phi$ and that only $P_{\rm gas}$ depends on density so that

$$\frac{1}{P}\frac{\partial P}{\partial\ln\rho} = \frac{\bar{P}_{\rm gas}}{P}\frac{1}{P_{\rm gas}}\frac{\partial\bar{P}_{\rm gas}}{\partial\ln\rho} = \frac{\bar{P}_{\rm gas}}{P}\frac{\partial\ln\bar{P}_{\rm gas}}{\partial\ln\rho}. \tag{41}$$

We have all of these quantities computed already.

11. $(\partial\Gamma_1/\partial\ln\rho)_T$. $\Gamma_1 = \chi_\rho + \chi_T X/Y$.

$$\frac{\partial\Gamma_1}{\partial\ln\rho} = \frac{\partial\chi_\rho}{\partial\ln\rho} + \frac{X}{Y}\frac{\partial\chi_T}{\partial\ln\rho} + \chi_T\frac{\partial\Gamma_3}{\partial\ln\rho}. \tag{42}$$

At this stage in the computation, we have all of these.

12. $(\partial\nabla_{\rm ad}/\partial\ln\rho)_T$. $\nabla_{\rm ad} = X/(Y\Gamma_1)$.

$$\frac{\partial\nabla_{\rm ad}}{\partial\ln\rho} = -\frac{1}{\Gamma_1^2}\frac{X}{Y}\frac{\partial\Gamma_1}{\partial\ln\rho} + \frac{1}{\Gamma_1}\frac{\partial}{\partial\ln\rho}\left(\frac{X}{Y}\right) = -\frac{\nabla_{\rm ad}}{\Gamma_1}\frac{\partial\Gamma_1}{\partial\ln\rho} + \frac{1}{\Gamma_1}\frac{\partial\Gamma_3}{\partial\ln\rho}. \tag{43}$$

13. $(\partial c_P/\partial\ln\rho)_T$. $c_P = \Gamma_1 c_v/\chi_\rho$.

$$\frac{\partial c_P}{\partial\ln\rho} = \frac{c_v}{\chi_\rho}\frac{\partial\Gamma_1}{\partial\ln\rho} + \frac{\Gamma_1}{\chi_\rho}\frac{c_v}{\ln\rho} - \frac{c_P}{\chi_\rho}\frac{\partial\chi_\rho}{\partial\ln\rho}. \tag{44}$$

We have all of these quantities.

14. $(\partial/\partial\ln\rho)_T(\partial E/\partial\rho)$. This is equal to

$$\rho\frac{\partial^2 E}{\partial\rho^2} = \rho\frac{\partial^2\bar{E}}{\partial\rho^2} + \rho\frac{\partial^2 E_\phi}{\partial\rho^2}. \tag{45}$$

The first term is computed by MESA, and we have a fitting function for the second.

15. $(\partial/\partial\ln\rho)_T(\partial s/\partial\rho)_T$. Same as above but $E\to s$. This is equal to

$$\rho\frac{\partial^2 s}{\partial\rho^2} = \rho\frac{\partial^2\bar{s}}{\partial\rho^2} + \rho\frac{\partial^2 s_\phi}{\partial\rho^2}. \tag{46}$$

Once again, we have all of the quantities and fitting formulas that we need.

16. $(\partial/\partial\ln\rho)_T(\partial s/\partial T)_\rho$. Identical logic to above but one of the ρ derivatives is replaced by T. The result is

$$\frac{\partial}{\partial\ln\rho}\left(\frac{\partial\bar{s}}{\partial T}\right) + \rho\frac{\partial^2 s}{\partial T\partial\rho}. \tag{47}$$

The first term is computed by MESA and we have a fitting function for the second.

2. `d_dlnT_const_Rho(:)`

Many of these expressions are identical to those above with ρ and T interchanged but a couple are more complicated. This is because both $P_{\rm rad}$ and P_ϕ depend on T but not ρ so extra terms are needed. In cases where no extra terms are needed, we simply quote the results since the formula is the same as in the previous subsection with T and ρ switched and the derivation is identical.

1. $(\partial \ln P_{\rm gas}/\partial \ln T)_\rho$.

$$\frac{\partial P_{\rm gas}}{\partial \ln T} = \frac{T}{P_{\rm gas}}\frac{\partial P_{\rm gas}}{\partial T} = \frac{\bar{P}_{\rm gas}}{P_{\rm gas}}\frac{\partial \ln P_{\rm gas}}{\partial \ln T} + \frac{T}{P_{\rm gas}}\frac{\partial P_\phi}{\partial T}. \tag{48}$$

The first term is composed of quantities computed by MESA and we have a fitting function for the second.

2. $(\partial \ln E/\partial \ln T)_\rho$. Same as above with $E \to T$:

$$\frac{\partial \ln E}{\partial \ln T} = \frac{\bar{E}}{E}\frac{\partial \ln \bar{E}}{\partial \ln T} + \frac{T}{E}\frac{\partial E_\phi}{\partial \ln T}. \tag{49}$$

3. $(\partial \ln s/\partial \ln T)_\rho$. Same as above with $E \to s$:

$$\frac{\partial \ln s}{\partial \ln T} = \frac{\bar{s}}{s}\frac{\partial \ln \bar{s}}{\partial \ln T} + \frac{T}{s}\frac{\partial s_\phi}{\partial \ln T}. \tag{50}$$

4-6. μ, μ_e, η. No change.

7. $(\partial c_v/\partial \ln T)_\rho$.

$$\left(\frac{\partial c_v}{\partial \ln T}\right)_\rho = \frac{\partial \bar{c}_v}{\partial \ln T} + T\frac{\partial^2 E_\phi}{\partial T^2}. \tag{51}$$

We have a fitting formula for the last quantity.

8.

$$\frac{\partial \chi_\rho}{\partial \ln T} = -\chi_T\chi_\rho + \frac{\rho T}{P}\frac{\partial^2 \bar{P}}{\partial T \partial \rho}. \tag{52}$$

The final quantity can be found using equation (39).

9. $(\partial \chi_T/\partial \ln T)_\rho$. This equation has extra terms compared with its analog in the previous section.

$$\frac{\partial \chi_T}{\partial \ln T} = T\frac{\partial}{\partial T}\left[\frac{T}{P}\frac{\partial P}{\partial T}\right] = \chi_T - \chi_T^2 + \frac{T^2}{P}\frac{\partial^2 P}{\partial T^2} = \chi_T - \chi_T^2 + \frac{T^2}{P}\left(\frac{\partial^2 \bar{P}}{\partial T^2} + \frac{\partial^2 P_\phi}{\partial T^2}\right). \tag{53}$$

We have a fitting function for $\partial^2 P_\phi/\partial^2 T$ so we just need $\partial^2 \bar{P}/\partial T^2$. To find this, we use the equation above and set all quantities equal to their barred (MESA default) versions. This means we have

$$\frac{\partial^2 \bar{P}}{\partial T^2} = \frac{\bar{P}}{T^2}\left(\bar{\chi}_T^2 - \bar{\chi}_T + \frac{\partial \bar{\chi}_T}{\partial \ln T}\right). \tag{54}$$

Thus, we have all the quantities we need.

10. $(\partial \Gamma_3/\partial \ln T)_\rho$. $\Gamma_3 = 1 + X/Y$, $X = \chi_T P$, $Y = \rho T c_v$. Following the same derivation as $(\partial \Gamma_3/\partial \ln \rho)_T$, we find the expression

$$\frac{\partial \Gamma_3}{\partial \ln T} = \frac{X}{Y}\left[-1 + \frac{1}{\chi_T}\frac{\partial \chi_T}{\partial \ln T} - \frac{1}{c_v}\frac{\partial c_v}{\partial \ln T} + \frac{1}{P}\frac{\partial P}{\partial \ln T}\right]. \tag{55}$$

We need to find an expression for the last term in the square brackets. Previously, we used the fact that only $P_{\rm gas}$ depends only on ρ but we can't do that this time since all three contributions depend on T. Using the fact that $P_{\rm rad} = aT^4/3$ we have

$$\frac{\partial P}{\partial \ln T} = \frac{\partial \bar{P}_{\rm gas}}{\partial \ln T} + T\frac{\partial P_{\rm rad}}{\partial T} + T\frac{\partial P_\phi}{\partial T} = \bar{P}_{\rm gas}\frac{\partial \ln \bar{P}_{\rm gas}}{\partial \ln T} + T\frac{\partial P_\phi}{\partial T} + 4P_{\rm rad}. \tag{56}$$

These are quantities that we have from MESA or that we have fitting functions for.

11. $(\partial\Gamma_1/\partial\ln T)_\rho$.

$$\frac{\partial\Gamma_1}{\partial\ln T} = \frac{\partial\chi_\rho}{\partial\ln T} + \frac{X}{Y}\frac{\partial\chi_T}{\partial\ln T} + \chi_T\frac{\partial\Gamma_3}{\partial\ln T}. \tag{57}$$

12. $(\partial\nabla_{\rm ad}/\partial\ln T)_\rho$

$$\frac{\partial\nabla_{\rm ad}}{\partial\ln T} = -\frac{\nabla_{\rm ad}}{\Gamma_1}\frac{\partial\Gamma_1}{\partial\ln T} + \frac{1}{\Gamma_1}\frac{\partial\Gamma_3}{\partial\ln T}. \tag{58}$$

13. $(\partial c_P/\ln T)_\rho$.

$$\frac{\partial c_P}{\partial\ln T} = \frac{c_v}{\chi_\rho}\frac{\partial\Gamma_1}{\partial\ln T} + \frac{\Gamma_1}{\chi_\rho}\frac{c_v}{\ln T} - \frac{c_P}{\chi_\rho}\frac{\partial\chi_\rho}{\partial\ln T}. \tag{59}$$

14. $(\partial/\ln T)_\rho\,(\partial s/\partial T)_\rho$

$$\frac{\partial}{\partial\ln T}\left(\frac{\partial s}{\partial T}\right) = \frac{\partial}{\partial\ln T}\left(\frac{\partial\bar{s}}{\partial T}\right) + T\frac{\partial^2 s_\phi}{\partial T^2} \tag{60}$$

The first term is computed by MESA and we have a fitting formula for the second.

15. $(\partial/\partial\ln T)_\rho\,(\partial s/\partial\rho)_T$

$$\frac{\partial}{\partial\ln T}\left(\frac{\partial s}{\partial\rho}\right) = \frac{\partial}{\partial\ln T}\left(\frac{\partial\bar{s}}{\partial\rho}\right) + T\frac{\partial^2 s_\phi}{\partial T\partial\rho}. \tag{61}$$

The first term is computed by MESA and we have a fitting formula for the second.

16. $(\partial/pa\ln T)_\rho\,(\partial E/\partial\rho)_T$.

$$\frac{\partial}{\partial\ln T}\left(\frac{\partial E}{\partial\rho}\right) = \frac{\partial}{\partial\ln T}\left(\frac{\partial\bar{E}}{\partial\rho}\right) + T\frac{\partial^2 E_\phi}{\partial T\partial\rho}. \tag{62}$$

The first term is computed by MESA we have a fitting formula for the second.

res(i)	Quantity	Definition	Units in MESA
res(1)	$\ln(P_{\rm gas})$	gas pressure	$\rm ergs/cm^3$
res(2)	$\ln(E)$	specific internal energy	ergs/g
res(3)	s	specific entropy	ergs/g/K
res(4)	μ	mean molecular weight per gas particle	none
res(5)	$1/\mu_e$	mean number of free electrons per nucleon	none
res(6)	η	ratio of electron chemical potential to $k_B T$	none
res(11)	c_V	$\left(\frac{\partial E}{\partial T}\right)_\rho$	ergs/g/K
res(8)	χ_ρ	$\left.\frac{\partial \ln P}{\partial \ln \rho}\right\rvert_T$	none
res(9)	χ_T	$\left.\frac{\partial \ln P}{\partial \ln T}\right\rvert_\rho$	none
res(14)	$\left(\frac{\partial s}{\partial T}\right)_\rho$	—	$\rm ergs/g/K^2$
res(13)	$\left(\frac{\partial s}{\partial \rho}\right)_T$	—	$\rm ergs\ cm^3/g^2/K$
res(12)	$\left(\frac{\partial E}{\partial \rho}\right)_T$	—	$\rm ergs\ cm^3/g^2$
res(16)	Γ_3	$1 + \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_s$	none
res(15)	Γ_1	$\left(\frac{\partial \ln P}{\partial \ln \rho}\right)_s$	none
res(7)	$\nabla_{\rm ad}$	$\left(\frac{\partial \ln T}{\partial \ln P}\right)_s$	none
res(10)	c_P	$\left(\frac{\partial h}{\partial T}\right)_P$	ergs/g/K

TABLE I. The 16 MESA EOS variables. The ordering of the rows corresponds to the order in which we compute the variables in our MESA code. The specific enthalpy is $h = E + P/\rho$. The gas pressure is defined as the total of all sources of pressure except the radiation pressure $P_{\rm rad} = aT^4/3$ with a the radiation constant. The total pressure is then $P = P_{\rm gas} + aT^4/3$.