Note on the Riemann Hypothesis

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Abstract

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2011, Solé and and Planat stated that the Riemann hypothesis is true if and only if the inequality $\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n)$ is satisfied for all primes $q_n > 3$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\zeta(x)$ is the Riemann zeta function. We call this inequality as the Dedekind inequality. We can deduce from that paper, if the Riemann hypothesis is false, then the Dedekind inequality is not satisfied for infinitely many prime numbers q_n. Using this argument, we prove the Riemann hypothesis is true when $\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$ holds for a sufficiently large prime number q_n. We show this is equivalent to show that the Riemann hypothesis is true when $(1 - \frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}} \geq 1 + \frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x}$ is always satisfied for every sufficiently large positive number x. Using the Puiseux series, we check by computer that $(1 - \frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}}$ is $1 + \frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x} + O\left(\left(\frac{1}{x}\right)^2\right)$ in the series expansion at $x = \infty$.

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1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [2]. We know the following properties for the Chebyshev function:

Theorem 1.1. *For all $n \geq 2$, we have [3]:*

$$n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}) \leq \frac{\theta(q_n)}{\log q_{n+1}}.$$

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Theorem 1.2. *For every* $x \geq 19035709163$*, we have [4]:*

$$\theta(x) > (1 - \frac{0.15}{\log^3 x}) \times x.$$

Besides, we define the prime counting function $\pi(x)$ as

$$\pi(x) = \sum_{p \leq x} 1.$$

We also know this property for the prime counting function:

Theorem 1.3. *[5]. For* $x \geq 599$*:*

$$\pi(x) > (1 + \frac{1}{\log x}) \times \frac{x}{\log x}.$$

In mathematics, $\Psi = n \times \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. Say $\mathsf{Dedekind}(q_n)$ holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm and $\zeta(x)$ is the Riemann zeta function. The importance of this inequality is:

Theorem 1.4. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann hypothesis is true [6].*

We define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [7]. We know from the constant H, the following formula:

Theorem 1.5. *[8].*

$$\sum_{q} \left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) = \gamma - B = H.$$

We know this value of the Riemann zeta function:

Theorem 1.6. *[9].*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

We check the following result from the web site https://www.wolframalpha.com/input:

Theorem 1.7. *Using the Puiseux series, we have that* $(1 - \frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}}$ *is* $1 + \frac{\log(1 - \frac{0.15}{\log^3 x}) + \log x}{x} + O\left(\left(\frac{1}{x}\right)^2\right)$ *in the series expansion at* $x = \infty$ *[10].*

Putting all together yields another evidence for the Riemann hypothesis using the Chebyshev function.

2. Results

Theorem 2.1. *If the Riemann hypothesis is false, then there are infinitely many prime numbers q_n for which* $\mathsf{Dedekind}(q_n)$ *do not hold.*

Proof. If the Riemann hypothesis is false, then we consider the function [6]:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(x) \times \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the Riemann hypothesis is false, if there exists some x_0 such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$ [6]. We know the bound [6]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f is introduced in the Nicolas paper [2]:

$$f(x) = e^{\gamma} \times \log \theta(x) \times \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

From the same paper [2], we know when the Riemann hypothesis is false, then there is a $0 < b < 1$ such that $\limsup x^{-b} \times f(x) > 0$ and hence $\limsup \log f(x) \gg \log x$, where the symbol $\gg$ means "much greater than" [6]. In this way, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \log x$ [2], [6]. Since $\frac{2}{x} = o(\log x)$, the result follows because there would be infinitely many x_0 such that $\log g(x_0) > 0$ [6]. □

The following is a key theorem.

Theorem 2.2.

$$\sum_{q} \left(\frac{1}{q} - \log(1 + \frac{1}{q})\right) = \log(\zeta(2)) - H.$$

Proof. If we add H to

$$\sum_{q} \left(\frac{1}{q} - \log(1 + \frac{1}{q})\right)$$

then we obtain that

$$
\begin{aligned}
H + \sum_q \left(\frac{1}{q} - \log(1 + \frac{1}{q}) \right) &= H + \sum_q \left(\frac{1}{q} - \log(\frac{q+1}{q}) \right) \\
&= \sum_q \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right) + \sum_q \left(\frac{1}{q} - \log(\frac{q+1}{q}) \right) \\
&= \sum_q \left(\log(\frac{q}{q-1}) - \log(\frac{q+1}{q}) \right) \\
&= \sum_q \left(\log(\frac{q}{q-1}) + \log(\frac{q}{q+1}) \right) \\
&= \sum_q \left(\log(\frac{q^2}{(q-1) \times (q+1)}) \right) \\
&= \sum_q \left(\log(\frac{q^2}{(q^2-1)}) \right) \\
&= \log(\frac{\pi^2}{6}) \\
&= \log(\zeta(2))
\end{aligned}
$$

according to the theorems 1.5 and 1.6. Therefore, the proof is done. □

This is the main insight.

Theorem 2.3. Dedekind(q_n) *holds for all prime numbers* $q_n > 3$ *if and only if the inequality*

$$\sum_q \frac{1}{q} > B + \sum_{q > q_n} \log(1 + \frac{1}{q}) + \log \log \theta(q_n)$$

is satisfied for all prime numbers $q_n > 3$.

Proof. We start from the inequality:

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q} \right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n).$$

If we apply the logarithm to the both sides of the inequality, then

$$\log(\zeta(2)) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > \gamma + \log \log \theta(q_n).$$

This is the same as

$$\log(\zeta(2)) - H + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log \log \theta(q_n)$$

which is

$$\sum_q \left(\frac{1}{q} - \log(1 + \frac{1}{q}) \right) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log \log \theta(q_n)$$

according to the theorem 2.2. Let's distribute the elements of the inequality to obtain that

$$\sum_{q} \frac{1}{q} > B + \sum_{q > q_n} \log(1 + \frac{1}{q}) + \log\log\theta(q_n)$$

when $\mathsf{Dedekind}(q_n)$ holds. The same happens in the reverse implication. □

This is a new criterion based on the Dedekind inequality.

Theorem 2.4. *The Riemann hypothesis is true if the inequality*

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n.

Proof. The inequality

$$\sum_{q} \frac{1}{q} > B + \sum_{q > q_n} \log(1 + \frac{1}{q}) + \log\log\theta(q_n)$$

is satisfied when

$$\sum_{q} \frac{1}{q} > B + \sum_{q \geq q_n} \log(1 + \frac{1}{q}) + \log\log\theta(q_n)$$

is also satisfied. Since in the inequality

$$\sum_{q} \frac{1}{q} > B + \sum_{q \geq q_n} \log(1 + \frac{1}{q}) + \log\log\theta(q_n)$$

only changes the value of

$$\sum_{q \geq q_n} \log(1 + \frac{1}{q}) + \log\log\theta(q_n).$$

Hence, it is enough to show that

$$\log(1 + \frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

for all sufficiently large prime numbers q_n according to the theorems 2.1 and 2.3. Certainly, if the inequality

$$\log(1 + \frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n, then it cannot exist infinitely many prime numbers q_n for which $\mathsf{Dedekind}(q_n)$ do not hold. By contraposition, we know that the Riemann hypothesis should be true. This is the same as

$$\log\left((1 + \frac{1}{q_n}) \times \log\theta(q_n)\right) \geq \log\log\theta(q_{n+1}).$$

That is equivalent to

$$\log\log\theta(q_n)^{1+\frac{1}{q_n}} \geq \log\log\theta(q_{n+1}).$$

Therefore, the Riemann hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. □

Theorem 2.5. *The Riemann hypothesis is true when* $(1 - \frac{0.15}{\log^3 x})^{\frac{1}{x}} \times x^{\frac{1}{x}} \geq 1 + \frac{\log(1-\frac{0.15}{\log^3 x})+\log x}{x}$ *is satisfied for all sufficiently large positive numbers x.*

Proof. Because of the theorem 2.4, we know that the Riemann hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. This is the same as

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_n) + \log(q_{n+1})$$

which is

$$\theta(q_n)^{\frac{1}{q_n}} \geq 1 + \frac{\log(q_{n+1})}{\theta(q_n)}.$$

We use the theorem 1.2 to show that

$$\theta(q_n)^{\frac{1}{q_n}} > (1 - \frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}} \times q_n^{\frac{1}{q_n}}$$

for a sufficiently large prime number q_n. Under our assumption in this theorem, we have that

$$(1 - \frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}} \times q_n^{\frac{1}{q_n}} \geq 1 + \frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n}.$$

Using the theorems 1.1 and 1.3, we only need to show that

$$\begin{aligned} \frac{\theta(q_n)}{\log q_{n+1}} &\geq n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}) \\ &> n \times (1 - \frac{1}{\log n}) \\ &> (1 + \frac{1}{\log q_n}) \times \frac{q_n}{\log q_n} \times (1 - \frac{1}{\log n}) \\ &> \frac{q_n}{\log q_n + \log(1 - \frac{0.15}{\log^3 q_n})} \end{aligned}$$

for a sufficiently large prime number q_n. However, this implies that

$$\frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} > \frac{\log(q_{n+1})}{\theta(q_n)}$$

which is equal to

$$1 + \frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} > 1 + \frac{\log(q_{n+1})}{\theta(q_n)}$$

and finally, the proof is complete. □

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