Series: PHYSICS

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The Lorentz force is a consequence of the system of Maxwell equations

Annotation

A new variational principle is formulated and it is proved that Maxwell's equations are a consequence of this principle. Symmetric Maxwell's equations, in which, along with electric potentials and charges, there are magnetic potentials and charges are also a consequence of this principle. Thermal losses from conduction currents are also taken into account in this principle. Maxwell's equations, supplemented by the Lorentz force formula, are also a consequence of this principle. Finally, Maxwell's equations, supplemented by the formula for the force arising from the movement of magnetic charges in an electric field, similar to the Lorentz formula, are also a consequence of this principle. This allows the author to conclude that the Lorentz formula and its analogue are also a consequence of the extended symmetric system of Maxwell's equations.

Contents

1. Introduction

Usually, the system of Maxwell equations and the Lorentz force formula are considered as the foundations of electrodynamics, as two independent components of these foundations. Further, it is shown that the Lorentz formula is a consequence of the system of Maxwell's equations.

But for this, Maxwell's symmetric equations are first considered, in which, along with electric potentials and charges, there are magnetic potentials and charges, which is known, but (I would say) not advertised. These equations include the Lorentz formula.

Then a new variational principle is formulated and it is proved that these equations are derived from the proposed variational principle. This allows the author to conclude that the Lorentz formula is a consequence of Maxwell's system of equations.

At the same time, it is proved that along with the Lorentz magnetic force, which relates the magnetic induction and the speed of the electric charge, there also exists the Lorentz electric force, which relates the electric induction and the speed of the magnetic charge. This (as far as the author knows) has not been established experimentally.

2. Symmetric Maxwell equations

It is known that Heaviside was the first to introduce magnetic charges and magnetic currents into Maxwell's electrodynamics [2]. We also note that the pole of a long magnet in mathematical terms can be identified with a magnetic charge. In this case, substances with high magnetic permeability behave approximately like magnetic conductors [3]. Symmetric Maxwell's equations in this consideration are a system of 8 equations with 8 unknown functions - 6 strengths and 2 scalar potentials with known electric and magnetic charges.

In this edition, only the Cartesian coordinate system is considered, where the oz axis is the direction of wave propagation.

Such a system of Maxwell's equations allows solving problems in which electric and/or magnetic charges are given. They can be given by step and impulse functions, as well as by Dirac functions. In this case, the strengths and scalar potentials are determined, i.e. conduction currents - electric and / or magnetic.

3. On the variational principle

It is known [4, 5] that Maxwell's equations are derived from the principle of least action. For this, the concept of the existence of a vector potential is used, as a consequence of Maxwell's equations, then some functional is formulated with respect to such a potential and a scalar electric potential, called action. By varying the action with respect to the vector magnetic potential and the scalar potential, the condition for the minimum of this functional is found. This conclusion is incomplete, since the functional used does not include thermal energy losses arising from conduction currents.

It is important to note for what follows that the solution of Maxwell's equations is not a wave function [6]. It cannot be a solution to the Maxwell equations, since it does not satisfy the law of conservation of energy - this issue is analyzed in detail in [7]. The vector potential is compatible only with the wave equation and, therefore, its use also contradicts the law of conservation of energy - see [8]. However, as mentioned, it was the vector potential that made it possible to obtain the derivation of Maxwell's equations from the proposed functional. Since the existence of a vector potential contradicts the fundamental physical law, the resulting derivation of Maxwell's equations cannot be considered satisfactory.

The matter is further complicated by the fact that in the symmetric form of Maxwell's equations (in the presence of both magnetic and electric charges), the electromagnetic field cannot be described using a vector potential that is continuous throughout space. Therefore, Maxwell's symmetric equations are not derived from the variational principle of least action, where the action is the integral of the difference between the kinetic and potential energies.

Thus, to derive the Maxwell equations from the variational principle, another functional must be found that does not involve the use of a vector potential and allows one to take into account the energy dissipation.

The author proposed <u>the full action extremum principle</u>, which also takes into account heat losses. This principle is described in [1]. There is also a functional for which the complete system of symmetric Maxwell equations is a necessary and sufficient condition for the existence of a unique optimum.

In addition, the proposed functional can be used to solve Maxwell's equations. The fact is that the functional used in this or that principle is an integral. It is possible to construct an algorithm for moving along the surface described by the integrand in the direction of the optimal line. When the optimum is reached, <u>the equations are thus solved, which are the conditions for the existence of this optimum</u>.

Thus, the search for a functional for some area of physics is

1. a method for deriving equations for this area,
2. a method for solving these equations.

4. Functional for rotors of Maxwell's equations

Next, we consider three-dimensional vectors in a vector space with coordinate axes $0x, 0y, 0z$ and unit vectors of these axes i, j, k, respectively. Further, the vector H is denoted as $H = \left(H_x, H_y, H_z\right)$, where its coordinates are indicated in brackets. Consider the functional proposed in [1]:

$$\Phi_o = \oint_z \left\{ \oint_y \left\{ \oint_x \left\{ \begin{array}{l} H_x \dfrac{\partial E_z}{\partial y} - H_x \dfrac{\partial E_y}{\partial z} + \\ + H_y \dfrac{\partial E_x}{\partial z} - H_y \dfrac{\partial E_z}{\partial x} + \\ + H_z \dfrac{\partial E_y}{\partial x} + H_z \dfrac{\partial E_x}{\partial y} + \\ - E_x \dfrac{\partial H_z}{\partial y} + E_x \dfrac{\partial H_y}{\partial z} + \\ - E_y \dfrac{\partial H_x}{\partial z} + E_y \dfrac{\partial H_z}{\partial x} + \\ - E_z \dfrac{\partial H_y}{\partial x} + E_z \dfrac{\partial H_x}{\partial y} + \end{array} \right\} dx \right\} dy \right\} dz, \tag{1}$$

from the functions $H_x, H_y, H_z, E_x, E_y, E_z$ of three variables x, y, z and show that the extremals of this functional are equations of the form

$$\text{rot} H = 0, \tag{2}$$

$$\text{rot} E = 0. \tag{3}$$

Necessary conditions for the extremum of a functional of functions of several independent variables - the Ostrogradsky equations [9] for each function have the form

$$\frac{\partial f}{\partial v} - \sum_{a=x,y,z,t} \left[\frac{\partial}{\partial a}\left(\frac{\partial f}{\partial (dv/da)}\right)\right] = 0, \tag{4}$$

where f is an integrand, $v(x,y,z,t)$ is a variable function, a is an independent variable. For this functionality, they take the following form:

- for the variable H_x (see terms 1, 2, 9, 12):

  $2\frac{\partial E_z}{\partial y} - 2\frac{\partial E_y}{\partial z} = 0,$
- for the variable H_y (see terms 3, 4, 8, 11):

  $2\frac{\partial E_x}{\partial z} - 2\frac{\partial E_z}{\partial x} = 0,$
- for the variable H_z (see terms 5, 6, 7, 10):

$$2\frac{\partial E_y}{\partial x} - 2\frac{\partial E_x}{\partial y} = 0,$$

- for the variable E_x (see terms 3, 6, 7, 8):

$$2\frac{\partial H_z}{\partial y} - 2\frac{\partial H_y}{\partial z} = 0,$$

- for the variable E_y (see terms 2, 5, 9, 10):

$$2\frac{\partial H_z}{\partial y} - 2\frac{\partial H_y}{\partial z} = 0,$$

- for the variable E_z (see terms 1, 4, 11, 12):

$$2\frac{\partial H_y}{\partial x} - 2\frac{\partial H_x}{\partial y} = 0.$$

Hence it follows that the necessary conditions for the extremum of the functional (1) are the equations

- for the variable E:

$$2 \cdot \mathrm{rot}H = 0, \tag{5}$$

- for the variable H:

$$2 \cdot \mathrm{rot}E = 0. \tag{6}$$

For the convenience of further presentation, the integrand in (1) will be denoted as $\Im(H, E)$. In this case, functional (1) takes the form

$$\Phi_0 = \oint_z \left\{ \oint_y \{ \oint_x \{ \Im(H, \quad E) \} \, dx \} dy \right\} dz, \tag{7}$$

It can be seen that

$$\Im(H, E) = H \cdot \mathrm{rot}(E) - E \cdot \mathrm{rot}(H). \tag{8}$$

Here, each factor is considered as a three-component vector in the sense of matrix algebra. Thus, fair

Lemma 1. The necessary conditions for the extremum of the functional (7, 8) are equations (2, 3).

5. Construction of a functional for Maxwell's equations

Consider a functional that differs from the one proposed in [1] in that the last 2 lines are added to it:

$$\Phi = \int_{t=0}^{T} \left\{ \int_z \left\{ \int_y \left\{ \int_x (\Phi_1 dx) \right\} dy \right\} dz \right\} dt \tag{1}$$

where

$$\Phi_1 = \begin{Bmatrix} +\frac{1}{2}\{\Im(H',E') - \Im(H'',E'')\} \\ +\frac{\mu}{2}\left\{H'\frac{dH''}{dt} - H''\frac{dH'}{dt}\right\} \\ +\frac{\varepsilon}{2}\left\{-E'\frac{dE''}{dt} + E''\frac{dE'}{dt}\right\} \\ +\left\{-K'\left(\mathrm{div}E' - \frac{\rho}{2\varepsilon}\right) + K''\left(\mathrm{div}E'' - \frac{\rho}{2\varepsilon}\right)\right\} \\ +\left\{L'\left(\mathrm{div}H' - \frac{\sigma}{2\mu}\right) - L''\left(\mathrm{div}H'' - \frac{\sigma}{2\mu}\right)\right\} \\ +\frac{\mu}{2}\left\{H'\cdot\frac{\partial}{\partial X}\left[v_\rho \times H''\right] - H''\cdot\left[v_\rho \times H'\right]\right\} \\ -\frac{\varepsilon}{2}\left\{\left\{E'\cdot\frac{\partial}{\partial X}[v_m \times E''] + E''\cdot[v_m \times E']\right\}\right\} \end{Bmatrix};$$

$X = \{x, y, z\}$; v_ρ, v_m are the speed of movement of electric and magnetic charges, respectively. In this functional, all variable functions are represented as sums: $H = H' + H''$, etc. Applying now the above Ostrogradsky equations, we find by differentiating:

- for the variable E':

$$\mathrm{rot}H' - \varepsilon\frac{dE''}{dt} - \mathrm{grad}(K') - \varepsilon\frac{\partial}{\partial X}[v_m \times E''] = 0, \qquad (2)$$

- for the variable E'':

$$-\mathrm{rot}H'' + \varepsilon\frac{dE'}{dt} + \mathrm{grad}(K'') + \varepsilon\frac{\partial}{\partial X}[v_m \times E'] = 0, \qquad (3)$$

- for the variable H':

$$\mathrm{rot}E' + \mu\frac{dH''}{dt} + \mathrm{grad}(L') + \mu\frac{\partial}{\partial X}\left[v_\rho \times H''\right] = 0, \qquad (4)$$

- for the variable H'':

$$-\mathrm{rot}E'' - \mu\frac{dH'}{dt} - \mathrm{grad}(L'') - \mu\frac{\partial}{\partial X}\left[v_\rho \times H'\right] = 0, \qquad (5)$$

- for the variable K', L', K'', L'' respectively:

$$-\left(\mathrm{div}E' - \frac{\rho}{2\varepsilon}\right) = 0,\ \left(\mathrm{div}H' - \frac{\sigma}{2\mu}\right) = 0, \qquad (6)$$

$$\left(\mathrm{div}E'' - \frac{\rho}{2\varepsilon}\right) = 0,\ -\left(\mathrm{div}H'' - \frac{\sigma}{2\mu}\right) = 0. \qquad (7)$$

Due to the symmetry of equations (2-7) we have:

$$E' = E'',\ H' = H'',\ K' = K'',\ L' = L''. \qquad (8)$$

Denote:

$$E = E' + E'',\ H = H' + H'',\ K = K' + K'',\ L = L' + L''. \qquad (9)$$

Subtracting equation (3) from (2), we obtain

$$\mathrm{rot}H - \varepsilon\frac{dE}{dt} - \mathrm{grad}(K) - \varepsilon\frac{\partial}{\partial X}[v_m \times E] = 0. \tag{10}$$

Similarly, subtracting from (5) from (4), we obtain

$$\mathrm{rot}E + \mu\frac{dH}{dt} + \mathrm{grad}(L') + \mu\frac{\partial}{\partial X}[v_\rho \times H] = 0. \tag{11}$$

Similarly, from (6, 7) we obtain

$$(\mathrm{div}E - \rho/\varepsilon) = 0, \tag{12}$$

$$(\mathrm{div}H - \sigma/\mu) = 0. \tag{13}$$

Equations (2) and (3) are necessary conditions for the existence of an extremum of the functional (1) with respect to the function E' and with respect to the function E''. These extrema are of opposite nature (minimum-maximum or maximum-minimum), since equations (2) and (3) differ in the signs of the terms. Consequently, these equations are necessary conditions for the existence of a saddle line for the functions $E'$and E'' in the functional (1).

Similarly, equations (4) and (5) are necessary conditions for the existence of a saddle line in the functions H' and H'' for functional (1).

Similarly, equations (6, 7) are necessary conditions for the existence of a saddle line with respect to the functions K', K'' and a saddle point with respect to the functions L', L'' in functional (1).

Lemma 2. The necessary conditions for the extremum of the functional (1) are equations (9-13).

It can be seen that equations (9-13) are symmetric Maxwell equations, where

E is electric field strength,
H is magnetic field strength,
μ is magnetic permeability,
ε is permittivity,
ρ is electric charge density,
σ is the density of the hypothetical magnetic charge,
$\mathrm{grad}(K)$ is- electric current density,
$\mathrm{grad}(L)$ is the hypothetical magnetic current density.

Denote:

$$J = \mathrm{grad}(K), \tag{14}$$

$$M = \mathrm{grad}(L). \tag{15}$$

Let us consider the physical meaning of the quantity K. Denote:

ϕ is electric scalar potential,

ϑ is electrical conductivity,

j_x is the projection of the electric current density vector J onto the OX axis.

Then we get $j_x = -\vartheta \frac{d\phi}{dx}$. But from (14) it follows that $j_x = \frac{dK}{dx}$. Consequently,

$$\frac{dK}{dx} = -\vartheta \frac{d\phi}{dx}, \tag{16}$$

i.e.

$$K = -\vartheta\phi. \tag{17}$$

Likewise,

$$\frac{dL}{dx} = -\varsigma \frac{d\varphi}{dx}, \tag{18}$$

$$L = -\varsigma\varphi, \tag{19}$$

where

φ is magnetic scalar potential,

ς is magnetic conductivity,

m_x is the projection of the magnetic current density vector M onto the OX axis.

So, combining equations (10, 11, 14, 15), we get the final form of the extended Maxwell equations:

$$\mathrm{rot}H - \varepsilon\frac{dE}{dt} - \mathrm{J} - \varepsilon\frac{\partial}{\partial X}[v_m \times E] = 0, \tag{20}$$

$$\mathrm{rot}E + \mu\frac{dH}{dt} + \mathrm{M} + \mu\frac{\partial}{\partial X}[v_\rho \times H] = 0, \tag{21}$$

$$(\mathrm{div}E - \rho/\varepsilon) = 0, \tag{22}$$

$$(\mathrm{div}H - \sigma/\mu) = 0. \tag{23}$$

6. Lorentz force

From (5.21) with $\frac{dH}{dt} = 0$, $\mathrm{M} = 0$ we find:

$$\mathrm{rot}E + \mu\frac{\partial}{\partial X}[v_\rho \times H] = 0 \tag{1}$$

or in component form

$$\left\{\begin{array}{l} (\mathrm{rot}E)_x + \mu\frac{\partial}{\partial x}[v_\rho \times H]_x = 0 \\ (\mathrm{rot}E)_y + \mu\frac{\partial}{\partial y}[v_\rho \times H]_y = 0 \\ (\mathrm{rot}E)_{xz} + \mu\frac{\partial}{\partial z}[v_\rho \times H]_z = 0 \end{array}\right\} \tag{2}$$

With $\mathrm{rot}E = 0$, i.e. at $E = \mathrm{const}$, the second terms - derivatives are also equal to zero and then the differentiable expressions become constants, i.e.

$$\begin{Bmatrix} \mu[v_\rho \times H]_x \\ \mu[v_\rho \times H]_y \\ \mu[v_\rho \times H]_z \end{Bmatrix} = \mu[v_\rho \times H] = [v_\rho \times B] = E \tag{3}$$

It is easy to see that we have obtained an expression for the Lorentz magnetic force vector. Similarly, from (5.20) we can obtain an expression of the form

$$\varepsilon[v_m \times E] = [v_m \times D] = F_m, \tag{4}$$

which it is natural to call the expression for the Lorentz electric force vector. It seems that the existence of such a force should not be in doubt. Apparently, it has a small value and therefore has not yet been discovered.

It should be noted that from the same equation (5.21) we can obtain formulas for the Faraday and Lorentz laws:

at $M=0$, $v_\rho = 0$ from (5.21) we find $\mathrm{rot}E = -\frac{dB}{dt}$,

at $M = 0$, $\frac{dH}{dt} = 0$ from (3) we find $E = [v_\rho \times B]$.

At the same time, the contradiction that Feynman spoke about when he discussed the phenomena in which the emf arises is removed: "we do not know of any other such example when a simple and exact law would require for its real understanding of analysis in terms of *two different phenomena*" [12].

7. On sufficient conditions for an extremum

In [1], sufficient conditions for an extremum are considered. We will not repeat this proof here. This proof is essentially the proof of the following theorem.

Theorem 1. The functional Φ defined in (5.1) depending on the functions $Z' = [E', H', K', L']$ and $Z'' = [E'', H'', K'', L'']$, has a global saddle extremal, where a strong minimum is reached on the function Z' and a strong maximum on the function Z''. The functions on this extremal are such that $Z' = Z''$, and their sum $Z = Z' + Z'' = [E, H, K, L]$ satisfies Maxwell's equations.

It is shown in [1] that this functional can be obtained by transforming the well-known equation [10] of the electromagnetic field power balance.

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