Cooperation increases robustness to ecological disturbance in microbial cross-feeding networks

Generating Random Networks

Functions to calculate Entropy and Assortativity

4. Liebig’s Law model

Solving the system of ODE

The function “fNewLiebigK” solves the ODE system and gives the population at steady state of the system. The function “fNewLiebig” receives a network and a disturbance value as arguments.

In[6025]:=

```
fNewLiebigK[Net_, Dh_] := (

  dB₁ =
   B₁[t] (-B₁[t] κ₁ + Min[{nuK M₁[t]/(denK + M₁[t]), nuK M₂[t]/(denK + M₂[t]), nuK M₃[t]/(denK + M₃[t]), nuK M₄[t]/(denK + M₄[t]),
        nuK M₅[t]/(denK + M₅[t])}]) - (c₁,₁ + c₁,₂ + c₁,₃ + c₁,₄ + c₁,₅ + Dh) B₁[t];
  dB₂ = B₂[t] (-B₂[t] κ₂ + Min[{nuK M₁[t]/(denK + M₁[t]), nuK M₂[t]/(denK + M₂[t]), nuK M₃[t]/(denK + M₃[t]),
        nuK M₄[t]/(denK + M₄[t]), nuK M₅[t]/(denK + M₅[t])}]) - (c₂,₁ + c₂,₂ + c₂,₃ + c₂,₄ + c₂,₅ + Dh) B₂[t];
  dB₃ = B₃[t] (-B₃[t] κ₃ + Min[{nuK M₁[t]/(denK + M₁[t]), nuK M₂[t]/(denK + M₂[t]), nuK M₃[t]/(denK + M₃[t]),
        nuK M₄[t]/(denK + M₄[t]), nuK M₅[t]/(denK + M₅[t])}]) - (c₃,₁ + c₃,₂ + c₃,₃ + c₃,₄ + c₃,₅ + Dh) B₃[t];
  dB₄ = B₄[t] (-B₄[t] κ₄ + Min[{nuK M₁[t]/(denK + M₁[t]), nuK M₂[t]/(denK + M₂[t]), nuK M₃[t]/(denK + M₃[t]),
        nuK M₄[t]/(denK + M₄[t]), nuK M₅[t]/(denK + M₅[t])}]) - (c₄,₁ + c₄,₂ + c₄,₃ + c₄,₄ + c₄,₅ + Dh) B₄[t];
```

```
dB5 = B5[t] (-B5[t] κ5 + Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]),
        nuK M4[t]/(denK + M4[t]), nuK M5[t]/(denK + M5[t])}]) - (c5,1 + c5,2 + c5,3 + c5,4 + c5,5 + Dh) B5[t];

dM1 = -M1[t] q1 +
   (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]), nuK M5[t]/(denK + M5[t])}])
    (-B1[t] d1,1 - B2[t] d1,2 - B3[t] d1,3 - B4[t] d1,4 - B5[t] d1,5) +
   B1[t] Ω1,1 + B2[t] Ω1,2 + B3[t] Ω1,3 + B4[t] Ω1,4 + B5[t] Ω1,5;
dM2 = -M2[t] q2 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d2,1 - B2[t] d2,2 - B3[t] d2,3 - B4[t] d2,4 - B5[t] d2,5) +
   B1[t] Ω2,1 + B2[t] Ω2,2 + B3[t] Ω2,3 + B4[t] Ω2,4 + B5[t] Ω2,5;
dM3 = -M3[t] q3 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d3,1 - B2[t] d3,2 - B3[t] d3,3 - B4[t] d3,4 - B5[t] d3,5) +
   B1[t] Ω3,1 + B2[t] Ω3,2 + B3[t] Ω3,3 + B4[t] Ω3,4 + B5[t] Ω3,5;
dM4 = -M4[t] q4 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d4,1 - B2[t] d4,2 - B3[t] d4,3 - B4[t] d4,4 - B5[t] d4,5) +
   B1[t] Ω4,1 + B2[t] Ω4,2 + B3[t] Ω4,3 + B4[t] Ω4,4 + B5[t] Ω4,5;
dM5 = -M5[t] q5 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d5,1 - B2[t] d5,2 - B3[t] d5,3 - B4[t] d5,4 - B5[t] d5,5) +
   B1[t] Ω5,1 + B2[t] Ω5,2 + B3[t] Ω5,3 + B4[t] Ω5,4 + B5[t] Ω5,5;

KK = 0.2;
cc = 0.05;
qq = 0.3;
dd = 0.0015;
OM = 1;
nu = 200;
den = 2;

tmax = 1000;
par = {
   κ1 → KK, κ2 → KK, κ3 → KK, κ4 → KK, κ5 → KK,

   c1,1 → cc Net[[1]][[1]], c1,2 → cc Net[[1]][[2]],
```

```
  c_{1,3} → cc Net[[1]][[3]], c_{1,4} → cc Net[[1]][[4]], c_{1,5} → cc Net[[1]][[5]],
  c_{2,1} → cc Net[[2]][[1]], c_{2,2} → cc Net[[2]][[2]], c_{2,3} → cc Net[[2]][[3]],
  c_{2,4} → cc Net[[2]][[4]], c_{2,5} → cc Net[[2]][[5]],
  c_{3,1} → cc Net[[3]][[1]], c_{3,2} → cc Net[[3]][[2]], c_{3,3} → cc Net[[3]][[3]],
  c_{3,4} → cc Net[[3]][[4]], c_{3,5} → cc Net[[3]][[5]],
  c_{4,1} → cc Net[[4]][[1]], c_{4,2} → cc Net[[4]][[2]], c_{4,3} → cc Net[[4]][[3]],
  c_{4,4} → cc Net[[4]][[4]], c_{4,5} → cc Net[[4]][[5]],
  c_{5,1} → cc Net[[5]][[1]], c_{5,2} → cc Net[[5]][[2]], c_{5,3} → cc Net[[5]][[3]],
  c_{5,4} → cc Net[[5]][[4]], c_{5,5} → cc Net[[5]][[5]],

  q_1 → qq, q_2 → qq, q_3 → qq, q_4 → qq, q_5 → qq,

  d_{1,1} → dd, d_{1,2} → dd, d_{1,3} → dd, d_{1,4} → dd, d_{1,5} → dd,
  d_{2,1} → dd, d_{2,2} → dd, d_{2,3} → dd, d_{2,4} → dd, d_{2,5} → dd,
  d_{3,1} → dd, d_{3,2} → dd, d_{3,3} → dd, d_{3,4} → dd, d_{3,5} → dd,
  d_{4,1} → dd, d_{4,2} → dd, d_{4,3} → dd, d_{4,4} → dd, d_{4,5} → dd,
  d_{5,1} → dd, d_{5,2} → dd, d_{5,3} → dd, d_{5,4} → dd, d_{5,5} → dd,

  Ω_{1,1} → OM Net[[1]][[1]], Ω_{1,2} → OM Net[[1]][[2]],
  Ω_{1,3} → OM Net[[1]][[3]], Ω_{1,4} → OM Net[[1]][[4]], Ω_{1,5} → OM Net[[1]][[5]],
  Ω_{2,1} → OM Net[[2]][[1]], Ω_{2,2} → OM Net[[2]][[2]], Ω_{2,3} → OM Net[[2]][[3]],
  Ω_{2,4} → OM Net[[2]][[4]], Ω_{2,5} → OM Net[[2]][[5]],
  Ω_{3,1} → OM Net[[3]][[1]], Ω_{3,2} → OM Net[[3]][[2]], Ω_{3,3} → OM Net[[3]][[3]],
  Ω_{3,4} → OM Net[[3]][[4]], Ω_{3,5} → OM Net[[3]][[5]],
  Ω_{4,1} → OM Net[[4]][[1]], Ω_{4,2} → OM Net[[4]][[2]], Ω_{4,3} → OM Net[[4]][[3]],
  Ω_{4,4} → OM Net[[4]][[4]], Ω_{4,5} → OM Net[[4]][[5]],
  Ω_{5,1} → OM Net[[5]][[1]], Ω_{5,2} → OM Net[[5]][[2]], Ω_{5,3} → OM Net[[5]][[3]],
  Ω_{5,4} → OM Net[[5]][[4]], Ω_{5,5} → OM Net[[5]][[5]],
  nuK → nu,
  denK → den


 };


B10 = 1500;
B20 = 1500;
B30 = 1500;
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
M30 = 10;
M40 = 10;
```

```
M50 = 10;


sol =
 NDSolve[
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,

    M1'[t] == dM1,
    M2'[t] == dM2,
    M3'[t] == dM3,
    M4'[t] == dM4,
    M5'[t] == dM5,


    B1[0] == B10,
    B2[0] == B20,
    B3[0] == B30,
    B4[0] == B40,
    B5[0] == B50,
    M1[0] == M10,
    M2[0] == M20,
    M3[0] == M30,
    M4[0] == M40,
    M5[0] == M50

   } /. par,
  {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
  {t, 0, tmax}];

{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
   M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par


)
```

In[6026]:=
```
NetK = {
   {0, 1, 0, 1, 0},
   {1, 0, 1, 1, 0},
   {1, 0, 1, 0, 1},
   {0, 1, 0, 1, 0},
   {0, 0, 0, 0, 1}
  };
```

In[6027]:=
```
fNewLiebigK[NetK, 1]
```

Out[6027]=
```
{{660.263, 660.013, 660.013, 660.263,
  660.513, 2203.19, 4404.07, 4404.9, 2203.19, 3.98377}}
```

In[6028]:=

fNewLiebig[Net_, Dh_] := (

$$dB_1 = B_1[t]\left(-B_1[t]\,\varkappa_1 + \mathrm{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4} + c_{1,5} + Dh\right) B_1[t];$$

$$dB_2 = B_2[t]\left(-B_2[t]\,\varkappa_2 + \mathrm{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4} + c_{2,5} + Dh\right) B_2[t];$$

$$dB_3 = B_3[t]\left(-B_3[t]\,\varkappa_3 + \mathrm{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4} + c_{3,5} + Dh\right) B_3[t];$$

$$dB_4 = B_4[t]\left(-B_4[t]\,\varkappa_4 + \mathrm{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4} + c_{4,5} + Dh\right) B_4[t];$$

$$dB_5 = B_5[t]\left(-B_5[t]\,\varkappa_5 + \mathrm{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{5,1} + c_{5,2} + c_{5,3} + c_{5,4} + c_{5,5} + Dh\right) B_5[t];$$

```
dM1 = -M1[t] q1 +
   (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]), nuK M5[t]/(denK + M5[t])}])
    (-B1[t] d1,1 - B2[t] d1,2 - B3[t] d1,3 - B4[t] d1,4 - B5[t] d1,5) +
   B1[t] Ω1,1 + B2[t] Ω1,2 + B3[t] Ω1,3 + B4[t] Ω1,4 + B5[t] Ω1,5;
dM2 = -M2[t] q2 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d2,1 - B2[t] d2,2 - B3[t] d2,3 - B4[t] d2,4 - B5[t] d2,5) +
   B1[t] Ω2,1 + B2[t] Ω2,2 + B3[t] Ω2,3 + B4[t] Ω2,4 + B5[t] Ω2,5;
dM3 = -M3[t] q3 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d3,1 - B2[t] d3,2 - B3[t] d3,3 - B4[t] d3,4 - B5[t] d3,5) +
   B1[t] Ω3,1 + B2[t] Ω3,2 + B3[t] Ω3,3 + B4[t] Ω3,4 + B5[t] Ω3,5;
dM4 = -M4[t] q4 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d4,1 - B2[t] d4,2 - B3[t] d4,3 - B4[t] d4,4 - B5[t] d4,5) +
   B1[t] Ω4,1 + B2[t] Ω4,2 + B3[t] Ω4,3 + B4[t] Ω4,4 + B5[t] Ω4,5;
dM5 = -M5[t] q5 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d5,1 - B2[t] d5,2 - B3[t] d5,3 - B4[t] d5,4 - B5[t] d5,5) +
   B1[t] Ω5,1 + B2[t] Ω5,2 + B3[t] Ω5,3 + B4[t] Ω5,4 + B5[t] Ω5,5;

KK = 0.2;
cc = 0.05;
qq = 0.3;
dd = 0.0015;
OM = 1;
nu = 200;
den = 2;

tmax = 1000;
par = {
   κ1 → KK, κ2 → KK, κ3 → KK, κ4 → KK, κ5 → KK,

   c1,1 → cc Net[[1]][[1]], c1,2 → cc Net[[1]][[2]],
   c1,3 → cc Net[[1]][[3]], c1,4 → cc Net[[1]][[4]], c1,5 → cc Net[[1]][[5]],
   c2,1 → cc Net[[2]][[1]], c2,2 → cc Net[[2]][[2]], c2,3 → cc Net[[2]][[3]],
   c2,4 → cc Net[[2]][[4]], c2,5 → cc Net[[2]][[5]],
   c3,1 → cc Net[[3]][[1]], c3,2 → cc Net[[3]][[2]], c3,3 → cc Net[[3]][[3]],
   c3,4 → cc Net[[3]][[4]], c3,5 → cc Net[[3]][[5]],
```

```
  c4,1 → cc Net[[4]][[1]], c4,2 → cc Net[[4]][[2]], c4,3 → cc Net[[4]][[3]],
  c4,4 → cc Net[[4]][[4]], c4,5 → cc Net[[4]][[5]],
  c5,1 → cc Net[[5]][[1]], c5,2 → cc Net[[5]][[2]], c5,3 → cc Net[[5]][[3]],
  c5,4 → cc Net[[5]][[4]], c5,5 → cc Net[[5]][[5]],

  q1 → qq, q2 → qq, q3 → qq, q4 → qq, q5 → qq,

  d1,1 → dd, d1,2 → dd, d1,3 → dd, d1,4 → dd, d1,5 → dd,
  d2,1 → dd, d2,2 → dd, d2,3 → dd, d2,4 → dd, d2,5 → dd,
  d3,1 → dd, d3,2 → dd, d3,3 → dd, d3,4 → dd, d3,5 → dd,
  d4,1 → dd, d4,2 → dd, d4,3 → dd, d4,4 → dd, d4,5 → dd,
  d5,1 → dd, d5,2 → dd, d5,3 → dd, d5,4 → dd, d5,5 → dd,

  Ω1,1 → OM Net[[1]][[1]], Ω1,2 → OM Net[[1]][[2]],
  Ω1,3 → OM Net[[1]][[3]], Ω1,4 → OM Net[[1]][[4]], Ω1,5 → OM Net[[1]][[5]],
  Ω2,1 → OM Net[[2]][[1]], Ω2,2 → OM Net[[2]][[2]], Ω2,3 → OM Net[[2]][[3]],
  Ω2,4 → OM Net[[2]][[4]], Ω2,5 → OM Net[[2]][[5]],
  Ω3,1 → OM Net[[3]][[1]], Ω3,2 → OM Net[[3]][[2]], Ω3,3 → OM Net[[3]][[3]],
  Ω3,4 → OM Net[[3]][[4]], Ω3,5 → OM Net[[3]][[5]],
  Ω4,1 → OM Net[[4]][[1]], Ω4,2 → OM Net[[4]][[2]], Ω4,3 → OM Net[[4]][[3]],
  Ω4,4 → OM Net[[4]][[4]], Ω4,5 → OM Net[[4]][[5]],
  Ω5,1 → OM Net[[5]][[1]], Ω5,2 → OM Net[[5]][[2]], Ω5,3 → OM Net[[5]][[3]],
  Ω5,4 → OM Net[[5]][[4]], Ω5,5 → OM Net[[5]][[5]],
  nuK → nu,
  denK → den


 };



B10 = 1500;
B20 = 1500;
B30 = 1500;
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
M30 = 10;
M40 = 10;
M50 = 10;



sol =
 NDSolve[
```

```
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,

    M1'[t] == dM1,
    M2'[t] == dM2,
    M3'[t] == dM3,
    M4'[t] == dM4,
    M5'[t] == dM5,



    B1[0] == B10,
    B2[0] == B20,
    B3[0] == B30,
    B4[0] == B40,
    B5[0] == B50,
    M1[0] == M10,
    M2[0] == M20,
    M3[0] == M30,
    M4[0] == M40,
    M5[0] == M50

   } /. par,
  {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
  {t, 0, tmax}];

 {B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
   M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par;

 Min[{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax]} /. sol /. par]

)
```

```
In[6029]:= robustnessNewLiebig[NetTop_] := (
  n1 = 1;
  n2 = 5000;
  mid = (n1 + n2) / 2;

  While[(n1 ≠ mid && n2 ≠ mid),
   (If[fNewLiebig[NetTop, mid] < 1, n2 = mid, n1 = mid];
    mid = Floor[N[(n1 + n2) / 2]];); {n1, n2, mid}]; mid

 )
```

As an example let's take the following Network

```
In[6030]:= NetK = {
    {0, 1, 0, 1, 0},
    {1, 0, 1, 1, 0},
    {1, 0, 1, 0, 1},
    {0, 1, 0, 1, 0},
    {0, 0, 0, 0, 1}
   };
```

Using the function f we can calculate the smallest value of a bacterial population in the community for a given disturbance vale. For example, let's take Disturbance value 1 and 70:

```
In[6031]:= fNewLiebig[NetK, 1]
Out[6031]= 660.013

In[6032]:= fNewLiebig[NetK, 4]
Out[6032]= 644.992
```

Using the function g3 we can calculate Robustness of the Network:

```
In[6033]:= robustnessNewLiebig[NetK]
Out[6033]= 124
```

We can calculate the (Relative) Entropy and the Assortativity:

```
In[6034]:= RelatEntrop5[NetK]
Out[6034]= 0.960956

In[6035]:= assortativity[NetK]
Out[6035]= -0.113228
```

We can calculate the robustness of the previously generated random networks with different number of auxotrophies:

In[6053]:=
```
AuxoComm6Liebig = Parallelize[robustnessNewLiebig /@ hk6];
AuxoComm7Liebig = Parallelize[robustnessNewLiebig /@ hk7];
AuxoComm8Liebig = Parallelize[robustnessNewLiebig /@ hk8];
AuxoComm9Liebig = Parallelize[robustnessNewLiebig /@ hk9];
AuxoComm10Liebig = Parallelize[robustnessNewLiebig /@ hk10];
AuxoComm11Liebig = Parallelize[robustnessNewLiebig /@ hk11];
AuxoComm12Liebig = Parallelize[robustnessNewLiebig /@ hk12];
AuxoComm13Liebig = Parallelize[robustnessNewLiebig /@ hk13];
AuxoComm14Liebig = Parallelize[robustnessNewLiebig /@ hk14];
AuxoComm15Liebig = Parallelize[robustnessNewLiebig /@ hk15];
AuxoComm16Liebig = Parallelize[robustnessNewLiebig /@ hk16];
AuxoComm17Liebig = Parallelize[robustnessNewLiebig /@ hk17];
```

In[6051]:=
```
Timing[robustnessNewLiebig /@ hk6[[1 ;; 3]];]
```

Out[6051]=
```
{0.575675, Null}
```

In[6052]:=
```
Timing[Parallelize[robustnessNewLiebig /@ hk6[[1 ;; 3]];]]
```

Out[6052]=
```
{0.060792, Null}
```

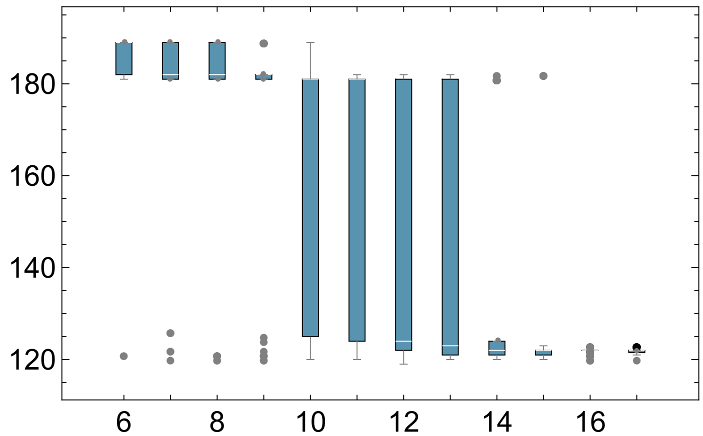
In[6065]:=
```
LikLiebig = {AuxoComm6Liebig, AuxoComm7Liebig, AuxoComm8Liebig, AuxoComm9Liebig,
   AuxoComm10Liebig, AuxoComm11Liebig, AuxoComm12Liebig, AuxoComm13Liebig,
   AuxoComm14Liebig, AuxoComm15Liebig, AuxoComm16Liebig, AuxoComm17Liebig};
```

In[6066]:=
```
coco = RGBColor[0.34509803921568627, 0.5803921568627451, 0.6901960784313725]
```

Out[6066]=

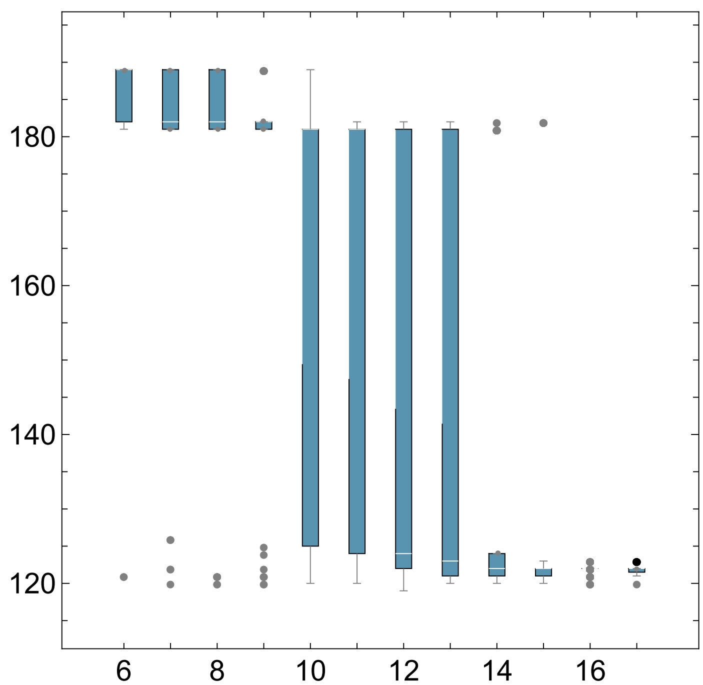
In[6067]:=
```
BoxWhiskerChart[LikLiebig, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{coco}}, Frame → True,
 ChartLabels → {"6", "", "8", "", "10", "", "12", "", "14", "", "16", ""},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15]]
```

Out[6067]=



In[6068]:=
```
BoxWhiskerChart[LikLiebig, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{coco}}, Frame → True,
 ChartLabels → {"6", "", "8", "", "10", "", "12", "", "14", "", "16", ""},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15], AspectRatio → 1]
```

Out[6068]=



In[•]:=
```
AuxoComm8Liebig
```
Out[•]=
```
{181, 181, 182, 189, 181, 181, 121, 189, 189, 182, 181, 181, 182, 181, 189, 182, 181, 181,
 181, 121, 181, 181, 182, 182, 181, 189, 182, 121, 189, 182, 181, 121, 182, 189, 189,
 189, 182, 189, 189, 189, 181, 189, 189, 189, 189, 181, 181, 181, 189, 182, 181, 182,
 182, 121, 189, 181, 120, 189, 189, 182, 182, 182, 189, 189, 182, 181, 181, 189,
 189, 189, 181, 182, 189, 189, 189, 189, 181, 181, 181, 189, 181, 189, 182, 182,
 181, 189, 189, 182, 181, 181, 120, 189, 189, 189, 189, 182, 181, 182, 181, 189}
```

We can study the correlation between Relative entropy and assortativity with Robustness for Networks with 7 auxotrophies.

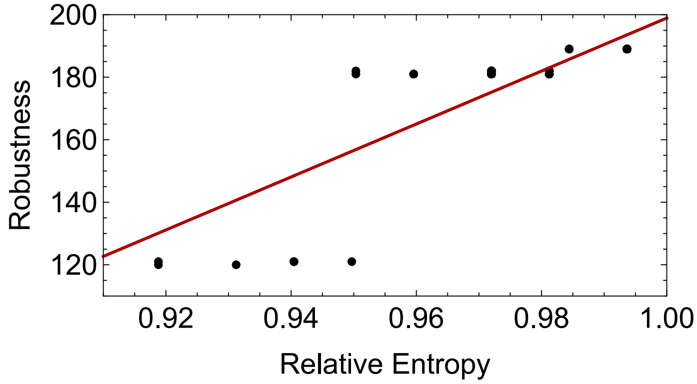
In[•]:=
```
Entropy13 = RelatEntrop5 /@ hk8;
```
In[•]:=
```
Assort13 = assortativity /@ hk8;
```
In[•]:=
```
RobustNewSaito8bLieb = AuxoComm8Liebig;
```

In[•]:=
```
Length[Entropy8]
Length[Assort8]
Length[RobustNewSaito8bLieb]
```
Out[•]=
```
100
```
Out[•]=
```
100
```
Out[•]=
```
100
```

In[•]:=
```
{Min[Entropy8], Max[Entropy8]}
{Min[Assort8], Max[Assort8]}
```
Out[•]=
```
{0.9188, 0.993652}
```
Out[•]=
```
{-0.520325, 0.235702}
```
In[•]:=
```
Position[Entropy8, Min[Entropy8]]
```
Out[•]=
```
{{54}, {91}}
```
In[•]:=
```
RobustNewSaito8bLieb[[#]] & /@ {1, 2, 24}
```
Out[•]=
```
{181, 181, 182}
```
In[•]:=
```
{Min[RobustNewSaito8bLieb], {Max[RobustNewSaito8bLieb]}}
```
Out[•]=
```
{120, {189}}
```

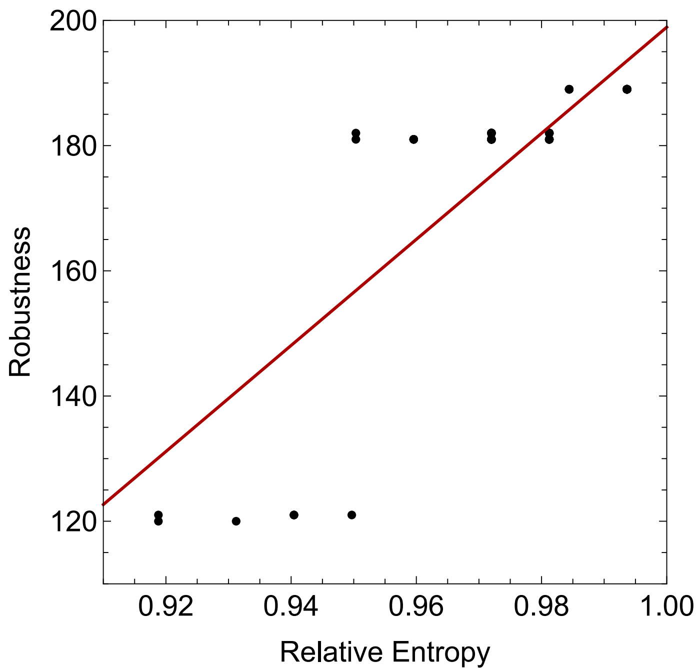
In[◦]:=
```
linerobustnessNewSaito25 =
  Fit[Partition[Riffle[Entropy8, RobustNewSaito8bLieb], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Entropy8, RobustNewSaito8bLieb], {2}],
  Frame → True, FrameLabel → {"Relative Entropy", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{0.91, 1}, {110, 200}},
  AspectRatio → 0.5], Plot[linerobustnessNewSaito25,
  {x, 0.91, 1}, AspectRatio → 0.5, PlotStyle → Darker[Red]]]
```

Out[◦]=



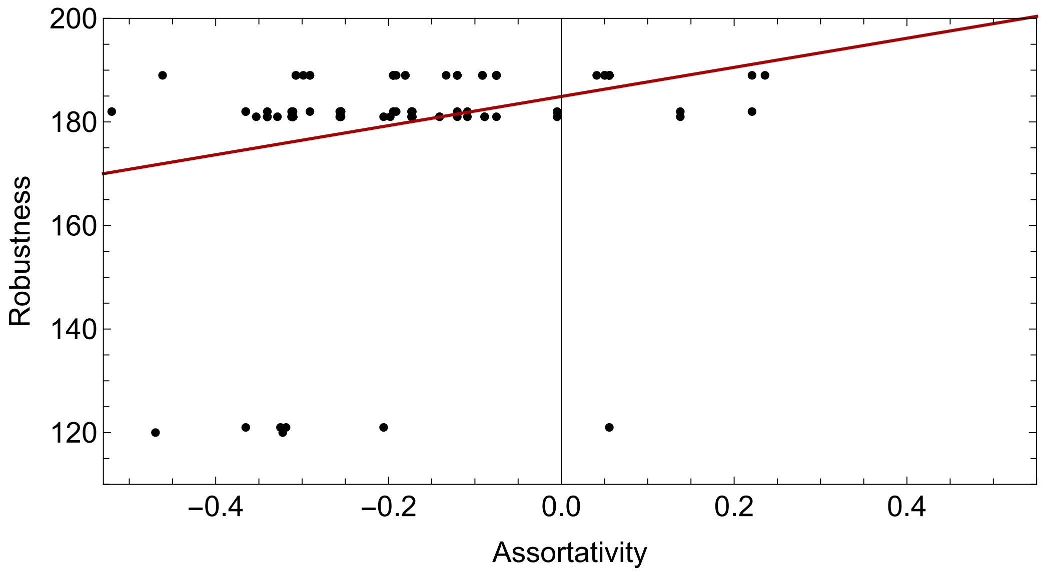
In[∘]:=
```
linerobustnessNewSaito25 =
  Fit[Partition[Riffle[Entropy8, RobustNewSaito8bLieb], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Entropy8, RobustNewSaito8bLieb], {2}],
  Frame → True, FrameLabel → {"Relative Entropy", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{0.91, 1}, {110, 200}},
  AspectRatio → 1], Plot[linerobustnessNewSaito25,
  {x, 0.91, 1}, PlotStyle → Darker[Red], AspectRatio → 1]]
```

Out[∘]=



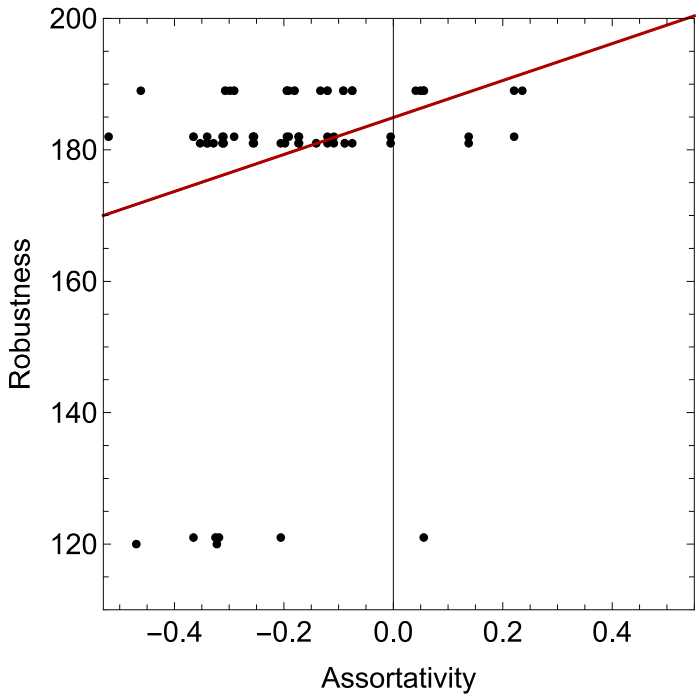
In[◦]:=
```
lineAssoRobrobustnessNewSaito25 =
  Fit[Partition[Riffle[Assort8, RobustNewSaito8bLieb], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Assort8, RobustNewSaito8bLieb], {2}],
  Frame → True, FrameLabel → {"Assortativity", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{-0.53, 0.55}, {110, 200}},
  AspectRatio → 0.5], Plot[lineAssoRobrobustnessNewSaito25,
  {x, -0.53, 0.55}, AspectRatio → 0.5, PlotStyle → Darker[Red]]]
```

Out[◦]=



In[∘]:=
```
lineAssoRobrobustnessNewSaito25 =
  Fit[Partition[Riffle[Assort8, RobustNewSaito8bLieb], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Assort8, RobustNewSaito8bLieb], {2}],
  Frame → True, FrameLabel → {"Assortativity", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{-0.53, 0.55}, {110, 200}},
  AspectRatio → 1], Plot[lineAssoRobrobustnessNewSaito25,
  {x, -0.53, 0.55}, AspectRatio → 1, PlotStyle → Darker[Red]]]
```

Out[∘]=



In[∘]:=
```
SpearmanRankTest[Entropy8, RobustNewSaito8bLieb, "TestDataTable"]
```

Out[∘]=

	Statistic	P-Value
Spearman Rank	0.869096	1.01251×10^{-31}

In[∘]:=
```
SpearmanRankTest[Assort8, RobustNewSaito8bLieb, "TestDataTable"]
```

Out[∘]=

	Statistic	P-Value
Spearman Rank	0.442665	3.99612×10^{-6}

Solving the system of ODE with Overproduction

In[6158]:=

$$\texttt{fNewLiebigOV[Net\_, Dh\_, coop\_] := }\Big($$

$$dB_1 = B_1[t]\left(-B_1[t]\,\varkappa_1 + \text{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4} + c_{1,5} + Dh\right) B_1[t];$$

$$dB_2 = B_2[t]\left(-B_2[t]\,\varkappa_2 + \text{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4} + c_{2,5} + Dh\right) B_2[t];$$

$$dB_3 = B_3[t]\left(-B_3[t]\,\varkappa_3 + \text{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4} + c_{3,5} + Dh\right) B_3[t];$$

$$dB_4 = B_4[t]\left(-B_4[t]\,\varkappa_4 + \text{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4} + c_{4,5} + Dh\right) B_4[t];$$

$$dB_5 = B_5[t]\left(-B_5[t]\,\varkappa_5 + \text{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) - \left(c_{5,1} + c_{5,2} + c_{5,3} + c_{5,4} + c_{5,5} + Dh\right) B_5[t];$$

$$dM_1 = -M_1[t]\,q_1 + \left(\text{Min}\left[\left\{\frac{nuK\,M_1[t]}{denK + M_1[t]}, \frac{nuK\,M_2[t]}{denK + M_2[t]}, \frac{nuK\,M_3[t]}{denK + M_3[t]}, \frac{nuK\,M_4[t]}{denK + M_4[t]}, \frac{nuK\,M_5[t]}{denK + M_5[t]}\right\}\right]\right) \left(-B_1[t]\,d_{1,1} - B_2[t]\,d_{1,2} - B_3[t]\,d_{1,3} - B_4[t]\,d_{1,4} - B_5[t]\,d_{1,5}\right) + B_1[t]\,\Omega_{1,1} + B_2[t]\,\Omega_{1,2} + B_3[t]\,\Omega_{1,3} + B_4[t]\,\Omega_{1,4} + B_5[t]\,\Omega_{1,5};$$

$$dM_2 = -M_2[t]\, q_2 + \left(\text{Min}\left[\left\{\frac{\text{nuK}\, M_1[t]}{\text{denK} + M_1[t]}, \frac{\text{nuK}\, M_2[t]}{\text{denK} + M_2[t]}, \frac{\text{nuK}\, M_3[t]}{\text{denK} + M_3[t]}, \frac{\text{nuK}\, M_4[t]}{\text{denK} + M_4[t]}, \frac{\text{nuK}\, M_5[t]}{\text{denK} + M_5[t]}\right\}\right]\right) \left(-B_1[t]\, d_{2,1} - B_2[t]\, d_{2,2} - B_3[t]\, d_{2,3} - B_4[t]\, d_{2,4} - B_5[t]\, d_{2,5}\right) + B_1[t]\, \Omega_{2,1} + B_2[t]\, \Omega_{2,2} + B_3[t]\, \Omega_{2,3} + B_4[t]\, \Omega_{2,4} + B_5[t]\, \Omega_{2,5};$$

$$dM_3 = -M_3[t]\, q_3 + \left(\text{Min}\left[\left\{\frac{\text{nuK}\, M_1[t]}{\text{denK} + M_1[t]}, \frac{\text{nuK}\, M_2[t]}{\text{denK} + M_2[t]}, \frac{\text{nuK}\, M_3[t]}{\text{denK} + M_3[t]}, \frac{\text{nuK}\, M_4[t]}{\text{denK} + M_4[t]}, \frac{\text{nuK}\, M_5[t]}{\text{denK} + M_5[t]}\right\}\right]\right) \left(-B_1[t]\, d_{3,1} - B_2[t]\, d_{3,2} - B_3[t]\, d_{3,3} - B_4[t]\, d_{3,4} - B_5[t]\, d_{3,5}\right) + B_1[t]\, \Omega_{3,1} + B_2[t]\, \Omega_{3,2} + B_3[t]\, \Omega_{3,3} + B_4[t]\, \Omega_{3,4} + B_5[t]\, \Omega_{3,5};$$

$$dM_4 = -M_4[t]\, q_4 + \left(\text{Min}\left[\left\{\frac{\text{nuK}\, M_1[t]}{\text{denK} + M_1[t]}, \frac{\text{nuK}\, M_2[t]}{\text{denK} + M_2[t]}, \frac{\text{nuK}\, M_3[t]}{\text{denK} + M_3[t]}, \frac{\text{nuK}\, M_4[t]}{\text{denK} + M_4[t]}, \frac{\text{nuK}\, M_5[t]}{\text{denK} + M_5[t]}\right\}\right]\right) \left(-B_1[t]\, d_{4,1} - B_2[t]\, d_{4,2} - B_3[t]\, d_{4,3} - B_4[t]\, d_{4,4} - B_5[t]\, d_{4,5}\right) + B_1[t]\, \Omega_{4,1} + B_2[t]\, \Omega_{4,2} + B_3[t]\, \Omega_{4,3} + B_4[t]\, \Omega_{4,4} + B_5[t]\, \Omega_{4,5};$$

$$dM_5 = -M_5[t]\, q_5 + \left(\text{Min}\left[\left\{\frac{\text{nuK}\, M_1[t]}{\text{denK} + M_1[t]}, \frac{\text{nuK}\, M_2[t]}{\text{denK} + M_2[t]}, \frac{\text{nuK}\, M_3[t]}{\text{denK} + M_3[t]}, \frac{\text{nuK}\, M_4[t]}{\text{denK} + M_4[t]}, \frac{\text{nuK}\, M_5[t]}{\text{denK} + M_5[t]}\right\}\right]\right) \left(-B_1[t]\, d_{5,1} - B_2[t]\, d_{5,2} - B_3[t]\, d_{5,3} - B_4[t]\, d_{5,4} - B_5[t]\, d_{5,5}\right) + B_1[t]\, \Omega_{5,1} + B_2[t]\, \Omega_{5,2} + B_3[t]\, \Omega_{5,3} + B_4[t]\, \Omega_{5,4} + B_5[t]\, \Omega_{5,5};$$

```
KK = 0.2;
cc = 0.05;
qq = 0.3;
dd = 0.0015;
OM = 1;
nu = 200;
den = 2;



op = coop; (*Number of links with overExpression*)
posNe = Position[Net, 1];
(*Positions in the matrix where there are links (=1)*)
RaN = RandomSample[posNe, op];
(*Random sample of op links that will be overproduced*)

costincr = 1.3; (*Term multiplying the cost link*)
overprodincr = 1.15;
(*Term multiplying the overproduction link*)



NewNetCost = Net cc;
Table[NewNetCost[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] =
  NewNetCost[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] * costincr, {i, Length[RaN]}];
```

```
NewNetOvProd = Net OM;
Table[NewNetOvProd[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] =
  NewNetOvProd[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] * overprodincr, {i,
  Length[RaN]}];


tmax = 1000;
par = {
  κ1 → KK, κ2 → KK, κ3 → KK, κ4 → KK, κ5 → KK,

  c1,1 → NewNetCost[[1]][[1]],
  c1,2 → NewNetCost[[1]][[2]], c1,3 → NewNetCost[[1]][[3]],
  c1,4 → NewNetCost[[1]][[4]], c1,5 → NewNetCost[[1]][[5]],
  c2,1 → NewNetCost[[2]][[1]], c2,2 → NewNetCost[[2]][[2]],
  c2,3 → NewNetCost[[2]][[3]], c2,4 → NewNetCost[[2]][[4]],
  c2,5 → NewNetCost[[2]][[5]],
  c3,1 → NewNetCost[[3]][[1]], c3,2 → NewNetCost[[3]][[2]],
  c3,3 → NewNetCost[[3]][[3]], c3,4 → NewNetCost[[3]][[4]],
  c3,5 → NewNetCost[[3]][[5]],
  c4,1 → NewNetCost[[4]][[1]], c4,2 → NewNetCost[[4]][[2]],
  c4,3 → NewNetCost[[4]][[3]], c4,4 → NewNetCost[[4]][[4]],
  c4,5 → NewNetCost[[4]][[5]],
  c5,1 → NewNetCost[[5]][[1]], c5,2 → NewNetCost[[5]][[2]],
  c5,3 → NewNetCost[[5]][[3]], c5,4 → NewNetCost[[5]][[4]],
  c5,5 → NewNetCost[[5]][[5]],

  q1 → qq, q2 → qq, q3 → qq, q4 → qq, q5 → qq,

  d1,1 → dd, d1,2 → dd, d1,3 → dd, d1,4 → dd, d1,5 → dd,
  d2,1 → dd, d2,2 → dd, d2,3 → dd, d2,4 → dd, d2,5 → dd,
  d3,1 → dd, d3,2 → dd, d3,3 → dd, d3,4 → dd, d3,5 → dd,
  d4,1 → dd, d4,2 → dd, d4,3 → dd, d4,4 → dd, d4,5 → dd,
  d5,1 → dd, d5,2 → dd, d5,3 → dd, d5,4 → dd, d5,5 → dd,

  Ω1,1 → NewNetOvProd[[1]][[1]],
  Ω1,2 → NewNetOvProd[[1]][[2]], Ω1,3 → NewNetOvProd[[1]][[3]],
  Ω1,4 → NewNetOvProd[[1]][[4]], Ω1,5 → NewNetOvProd[[1]][[5]],
  Ω2,1 → NewNetOvProd[[2]][[1]], Ω2,2 → NewNetOvProd[[2]][[2]],
  Ω2,3 → NewNetOvProd[[2]][[3]], Ω2,4 → NewNetOvProd[[2]][[4]],
  Ω2,5 → NewNetOvProd[[2]][[5]],
  Ω3,1 → NewNetOvProd[[3]][[1]], Ω3,2 → NewNetOvProd[[3]][[2]],
  Ω3,3 → NewNetOvProd[[3]][[3]], Ω3,4 → NewNetOvProd[[3]][[4]],
  Ω3,5 → NewNetOvProd[[3]][[5]],
```

```
    Ω4,1 → NewNetOvProd[[4]][[1]], Ω4,2 → NewNetOvProd[[4]][[2]],
    Ω4,3 → NewNetOvProd[[4]][[3]], Ω4,4 → NewNetOvProd[[4]][[4]],
    Ω4,5 → NewNetOvProd[[4]][[5]],
    Ω5,1 → NewNetOvProd[[5]][[1]], Ω5,2 → NewNetOvProd[[5]][[2]],
    Ω5,3 → NewNetOvProd[[5]][[3]], Ω5,4 → NewNetOvProd[[5]][[4]],
    Ω5,5 → NewNetOvProd[[5]][[5]],
    nuK → nu,
    denK → den


   };



B10 = 1500;
B20 = 1500;
B30 = 1500;
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
M30 = 10;
M40 = 10;
M50 = 10;


sol =
 NDSolve[
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,

    M1'[t] == dM1,
    M2'[t] == dM2,
    M3'[t] == dM3,
    M4'[t] == dM4,
    M5'[t] == dM5,


    B1[0] == B10,
    B2[0] == B20,
    B3[0] == B30,
```

```
      B4[0] == B40,
      B5[0] == B50,
      M1[0] == M10,
      M2[0] == M20,
      M3[0] == M30,
      M4[0] == M40,
      M5[0] == M50

     } /. par,
    {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
    {t, 0, tmax}];

  {B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
     M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par;

  Min[{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax]} /. sol /. par]

 )
```

In[6159]:=

```
robustnessNewLiebigOV[NetTop_, coop_] := (
  n1 = 1;
  n2 = 5000;
  mid = (n1 + n2) / 2;

  While[(n1 ≠ mid && n2 ≠ mid),
   (If[fNewLiebigOV[NetTop, mid, coop] < 1, n2 = mid, n1 = mid];
    mid = Floor[N[(n1 + n2) / 2]];); {n1, n2, mid}]; mid

 )
```

In[6160]:=

```
NetK = {
   {0, 1, 0, 1, 0},
   {1, 0, 1, 1, 0},
   {1, 0, 1, 0, 1},
   {0, 1, 0, 1, 0},
   {0, 0, 0, 0, 1}
  };
```

Compare the Robustness with and without (5 links) overproduction (ratio cost/production = 1.3/1.15)

In[6161]:=
```
fNewLiebig[NetK, 1]
```
Out[6161]=
```
660.013
```

In[6165]:=
```
fNewLiebigOV[NetK, 0, 5]
```
Out[6165]=
```
764.437
```

In[6169]:=
```
AuxoComm8Liebig
```
Out[6169]=
```
{181, 181, 182, 189, 181, 181, 121, 189, 189, 182, 181, 181, 182, 181, 189, 182, 181, 181,
 181, 121, 181, 181, 182, 182, 181, 189, 182, 121, 189, 182, 181, 121, 182, 189, 189,
 189, 182, 189, 189, 189, 181, 189, 189, 189, 189, 181, 181, 181, 189, 182, 181, 182,
 182, 121, 189, 181, 120, 189, 189, 182, 182, 182, 189, 189, 182, 181, 181, 189,
 189, 189, 181, 182, 189, 189, 189, 189, 181, 181, 181, 189, 181, 189, 182, 182,
 181, 189, 189, 182, 181, 181, 120, 189, 189, 189, 189, 182, 181, 182, 181, 189}
```

In[6170]:=
```
coop5to15Li =
  {Table[robustnessNewLiebigOV[#, 5], {20}], Table[robustnessNewLiebigOV[#, 10],
     {20}], Table[robustnessNewLiebigOV[#, 15], {20}]} &;
```

In[6171]:=
```
wf8Li = Parallelize[coop5to15Li /@ hk8];
```

In[6172]:=
```
wf8NormalizedLi = N[wf8Li[[#]] / AuxoComm8Liebig[[#]]] & /@ Range[100]
```

In[6173]:=
```
wf8NormalizedWith5CoopLi = wf8NormalizedLi[[#]][[1]] & /@ Range[100]
```

In[6174]:=
```
wf8NormalizedWith10CoopLi = wf8NormalizedLi[[#]][[2]] & /@ Range[100]
```

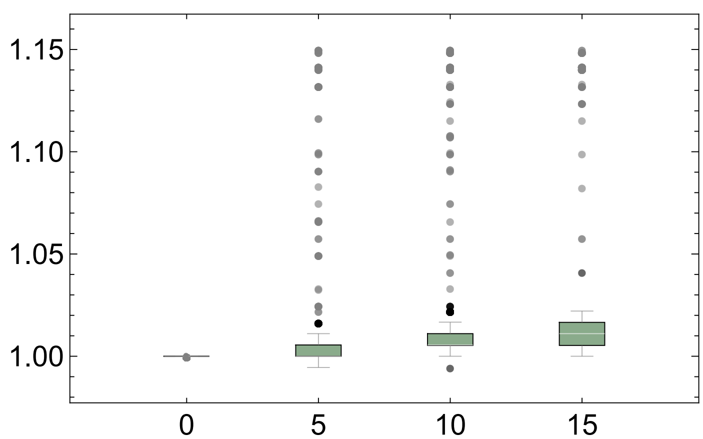
In[6175]:=
```
wf8NormalizedWith15CoopLi = wf8NormalizedLi[[#]][[3]] & /@ Range[100]
```

In[6176]:=
```
allcoopWith8AuxoLi = {Flatten[wf8NormalizedWith5CoopLi],
   Flatten[wf8NormalizedWith10CoopLi], Flatten[wf8NormalizedWith15CoopLi]}
```

In[6177]:=
```
allcoopWith8AuxoPlusAuxoLi = Join[{ConstantArray[1, {2000}]}, allcoopWith8AuxoLi]
```

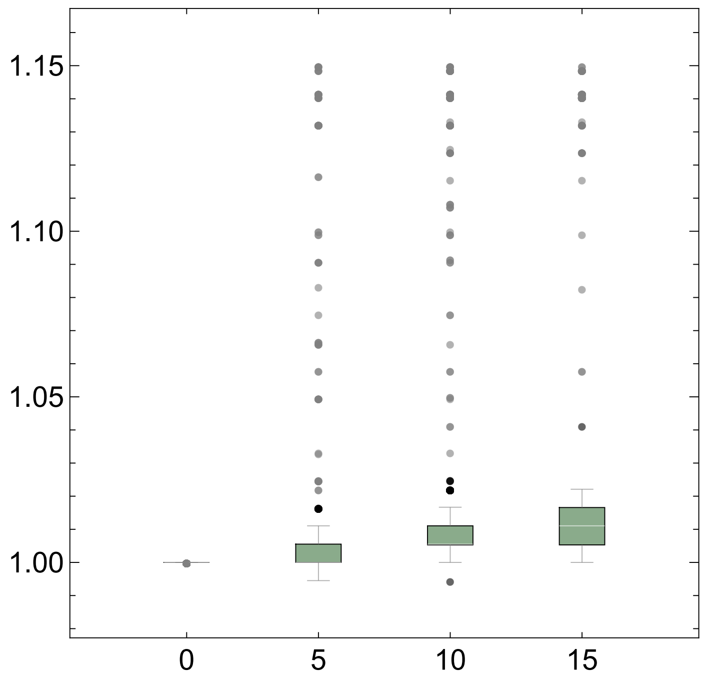
In[6295]:=
```
BoxWhiskerChart[allcoopWith8AuxoPlusAuxoLi, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{gree1}},
 Frame → True, ChartLabels → {"0", "5", "10", "15"},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15]]
```

Out[6295]=



In[6296]:=
```
BoxWhiskerChart[allcoopWith8AuxoPlusAuxoLi, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{gree1}},
 Frame → True, ChartLabels → {"0", "5", "10", "15"}, BarSpacing → 1.9,
 FrameStyle → Directive[Black, FontSize → 15], AspectRatio → 1]
```

Out[6296]=



In[6179]:=
```
allcoopWith8AuxoPlusAuxoLi // Length
```

Out[6179]= 4

In[6180]:=
```
SignedRankTest[allcoopWith8AuxoPlusAuxoLi[[2]], 1]
SignedRankTest[allcoopWith8AuxoPlusAuxoLi[[3]], 1]
SignedRankTest[allcoopWith8AuxoPlusAuxoLi[[4]], 1]
```

Out[6180]= 2.21714×10^{-163}

Out[6181]= 2.25974×10^{-295}

Out[6182]= 0.

Solving the system of ODE Random parametrization

In[6620]:=
```
Knum = 0.2;
ccrnum = 0.05;
qqrnum = 0.3;
ddrnum = 0.0015;
OMrnum = 1;
nurum = 200;
den2rum = 2;

corrpar0 = 10^3;
corrpar1 = 10^4;
corrpar2 = 10^6;

KKr := RandomVariate[
    GammaDistribution[corrpar0 Sqrt[Knum], (1/corrpar0) Sqrt[Knum]], 1][[1]];
ccr := RandomVariate[GammaDistribution[corrpar1 Sqrt[ccrnum],
      (1/corrpar1) Sqrt[ccrnum]], 1][[1]];
qqr := RandomVariate[GammaDistribution[corrpar0 Sqrt[qqrnum],
      (1/corrpar0) Sqrt[qqrnum]], 1][[1]];
ddr := RandomVariate[GammaDistribution[corrpar1 Sqrt[ddrnum],
      (1/corrpar1) Sqrt[ddrnum]], 1][[1]];
OMr := RandomVariate[GammaDistribution[corrpar0 Sqrt[OMrnum],
      (1/corrpar0) Sqrt[OMrnum]], 1][[1]];
nur := RandomVariate[GammaDistribution[corrpar2 Sqrt[nurum],
      (1/corrpar2) Sqrt[nurum]], 1][[1]] ;
denr2 := RandomVariate[GammaDistribution[corrpar2 Sqrt[den2rum],
      (1/corrpar2) Sqrt[den2rum]], 1][[1]] ;


parR = Join[Table[KKr, {5}], Table[ccr, {25}],
  Table[qqr, {5}], Table[ddr, {25}], Table[OMr, {25}], {nur}, {denr2}]
```

Out[6637]= {0.174065, 0.192716, 0.201013, 0.199342, 0.205467, 0.051267, 0.0509792, 0.0492509, 0.0505965, 0.0480712, 0.0487571, 0.0516892, 0.0511037, 0.0493342, 0.0499709, 0.0520139, 0.0512811, 0.0500806, 0.048457, 0.0504627, 0.0510088, 0.048274, 0.049034, 0.0489444, 0.0509747, 0.0502627, 0.0509413, 0.0500432, 0.0498128, 0.0514343, 0.274148, 0.295, 0.29056, 0.3092, 0.304498, 0.00158584, 0.00155599, 0.00155081, 0.0016041, 0.00146101, 0.00140389, 0.00148479, 0.00149624, 0.00147291, 0.00158423, 0.0016865, 0.00150169, 0.00155989, 0.00146453, 0.00148963, 0.0016086, 0.00145168, 0.00146253, 0.00143854, 0.00151936, 0.00158187, 0.00159033, 0.00161823, 0.0016764, 0.00143317, 0.959485, 0.991248, 0.989999, 0.962826, 0.994657, 1.01296, 1.111, 1.01255, 0.98884, 0.945443, 1.00391, 1.00769, 1.03404, 0.962685, 1.02578, 1.01396, 0.973462, 0.973176, 0.952024, 0.972961, 1.00321, 1.03881, 0.927777, 1.0447, 1.04131, 200.038, 2.00091}

```
parR = {0.17406491963765225`, 0.19271552318731072`, 0.20101265981745065`,
   0.1993416825951762`, 0.20546747418997402`, 0.05126697346911639`,
   0.050979209585780075`, 0.049250907922295285`, 0.050596494945335864`,
   0.04807123965645414`, 0.04875707371061734`, 0.05168919593748627`,
   0.05110374775894862`, 0.04933415213279622`, 0.049970862334809504`,
   0.052013918924004365`, 0.051281069200719044`, 0.05008059124653844`,
   0.048456986155650895`, 0.05046268834028382`, 0.05100881647129115`,
   0.04827398693541804`, 0.049033976435347196`, 0.04894443311184237`,
   0.05097473744162672`, 0.05026272693204548`, 0.0509412937337713`,
   0.050043183079927396`, 0.04981275493116924`, 0.05143431351532053`,
   0.2741484220752475`, 0.29500048735377876`, 0.29055979190752795`,
   0.30919969983773093`, 0.30449818646425864`, 0.0015858418050202845`,
   0.0015559938026471695`, 0.0015508063247152385`, 0.0016041035713506373`,
   0.0014610080434572011`, 0.0014038903176297444`, 0.0014847851291227255`,
   0.0014962435725992124`, 0.0014729115074682852`, 0.0015842336090410236`,
   0.0016865044671100866`, 0.001501687307936476`, 0.0015598859023090983`,
   0.00146452620048039 2`, 0.0014896327356137065`, 0.0016086000253588925`,
   0.0014516848408536902`, 0.0014625348014891348`, 0.0014385370012988826`,
   0.0015193617217833212`, 0.0015818702687085818`, 0.0015903285844332212`,
   0.0016182333378202214`, 0.0016763973379304588`, 0.0014331690573307755`,
   0.959484902075753`, 0.9912481178882356`, 0.9899993730630222`,
   0.9628256622169967`, 0.9946569905494801`, 1.0129632879517105`,
   1.1109986104234093`, 1.0125450635832367`, 0.988840015814362`,
   0.94544275526312`, 1.003913112496818`, 1.0076856430130368`,
   1.0340447722669786`, 0.9626849557677253`, 1.0257791881087754`,
   1.013963295639842`, 0.9734619708145545`, 0.973176386033725`,
   0.9520239235604386`, 0.9729607432853318`, 1.0032062732410925`,
   1.038812253989961 1`, 0.9277774829512535`, 1.0446966704519374`,
   1.041314808105069 6`, 200.03791321285763`, 2.0009063754198024`};
```

In[6638]:=

```
fNewLiebigR[Net_, Dh_] := (

  dB1 =
   B1[t] (-B1[t] κ1 + Min[{(nuK M1[t])/(denK + M1[t]), (nuK M2[t])/(denK + M2[t]), (nuK M3[t])/(denK + M3[t]), (nuK M4[t])/(denK + M4[t]),
        (nuK M5[t])/(denK + M5[t])}]) - (c1,1 + c1,2 + c1,3 + c1,4 + c1,5 + Dh) B1[t];
  dB2 = B2[t] (-B2[t] κ2 + Min[{(nuK M1[t])/(denK + M1[t]), (nuK M2[t])/(denK + M2[t]), (nuK M3[t])/(denK + M3[t]),
        (nuK M4[t])/(denK + M4[t]), (nuK M5[t])/(denK + M5[t])}]) - (c2,1 + c2,2 + c2,3 + c2,4 + c2,5 + Dh) B2[t];
  dB3 = B3[t] (-B3[t] κ3 + Min[{(nuK M1[t])/(denK + M1[t]), (nuK M2[t])/(denK + M2[t]), (nuK M3[t])/(denK + M3[t]),
        (nuK M4[t])/(denK + M4[t]), (nuK M5[t])/(denK + M5[t])}]) - (c3,1 + c3,2 + c3,3 + c3,4 + c3,5 + Dh) B3[t];
  dB4 = B4[t] (-B4[t] κ4 + Min[{(nuK M1[t])/(denK + M1[t]), (nuK M2[t])/(denK + M2[t]), (nuK M3[t])/(denK + M3[t]),
        (nuK M4[t])/(denK + M4[t]), (nuK M5[t])/(denK + M5[t])}]) - (c4,1 + c4,2 + c4,3 + c4,4 + c4,5 + Dh) B4[t];
  dB5 = B5[t] (-B5[t] κ5 + Min[{(nuK M1[t])/(denK + M1[t]), (nuK M2[t])/(denK + M2[t]), (nuK M3[t])/(denK + M3[t]),
        (nuK M4[t])/(denK + M4[t]), (nuK M5[t])/(denK + M5[t])}]) - (c5,1 + c5,2 + c5,3 + c5,4 + c5,5 + Dh) B5[t];

  dM1 = -M1[t] q1 +
    (Min[{(nuK M1[t])/(denK + M1[t]), (nuK M2[t])/(denK + M2[t]), (nuK M3[t])/(denK + M3[t]), (nuK M4[t])/(denK + M4[t]), (nuK M5[t])/(denK + M5[t])}])
     (-B1[t] d1,1 - B2[t] d1,2 - B3[t] d1,3 - B4[t] d1,4 - B5[t] d1,5) +
    B1[t] Ω1,1 + B2[t] Ω1,2 + B3[t] Ω1,3 + B4[t] Ω1,4 + B5[t] Ω1,5;
  dM2 = -M2[t] q2 + (Min[{(nuK M1[t])/(denK + M1[t]), (nuK M2[t])/(denK + M2[t]), (nuK M3[t])/(denK + M3[t]), (nuK M4[t])/(denK + M4[t]),
        (nuK M5[t])/(denK + M5[t])}]) (-B1[t] d2,1 - B2[t] d2,2 - B3[t] d2,3 - B4[t] d2,4 - B5[t] d2,5) +
    B1[t] Ω2,1 + B2[t] Ω2,2 + B3[t] Ω2,3 + B4[t] Ω2,4 + B5[t] Ω2,5;
  dM3 = -M3[t] q3 + (Min[{(nuK M1[t])/(denK + M1[t]), (nuK M2[t])/(denK + M2[t]), (nuK M3[t])/(denK + M3[t]), (nuK M4[t])/(denK + M4[t]),
        (nuK M5[t])/(denK + M5[t])}]) (-B1[t] d3,1 - B2[t] d3,2 - B3[t] d3,3 - B4[t] d3,4 - B5[t] d3,5) +
    B1[t] Ω3,1 + B2[t] Ω3,2 + B3[t] Ω3,3 + B4[t] Ω3,4 + B5[t] Ω3,5;
```

```
dM4 = -M4[t] q4 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d4,1 - B2[t] d4,2 - B3[t] d4,3 - B4[t] d4,4 - B5[t] d4,5) +
   B1[t] Ω4,1 + B2[t] Ω4,2 + B3[t] Ω4,3 + B4[t] Ω4,4 + B5[t] Ω4,5;
dM5 = -M5[t] q5 + (Min[{nuK M1[t]/(denK + M1[t]), nuK M2[t]/(denK + M2[t]), nuK M3[t]/(denK + M3[t]), nuK M4[t]/(denK + M4[t]),
        nuK M5[t]/(denK + M5[t])}]) (-B1[t] d5,1 - B2[t] d5,2 - B3[t] d5,3 - B4[t] d5,4 - B5[t] d5,5) +
   B1[t] Ω5,1 + B2[t] Ω5,2 + B3[t] Ω5,3 + B4[t] Ω5,4 + B5[t] Ω5,5;



tmax = 1000;
par = {
   κ1 → parR[[1]], κ2 → parR[[2]], κ3 → parR[[3]], κ4 → parR[[4]], κ5 → parR[[5]],

   c1,1 → parR[[6]] × Net[[1]][[1]],
   c1,2 → parR[[7]] × Net[[1]][[2]], c1,3 → parR[[8]] × Net[[1]][[3]],
   c1,4 → parR[[9]] × Net[[1]][[4]], c1,5 → parR[[10]] × Net[[1]][[5]],
   c2,1 → parR[[11]] × Net[[2]][[1]], c2,2 → parR[[12]] × Net[[2]][[2]],
   c2,3 → parR[[13]] × Net[[2]][[3]], c2,4 → parR[[14]] × Net[[2]][[4]],
   c2,5 → parR[[15]] × Net[[2]][[5]],
   c3,1 → parR[[16]] × Net[[3]][[1]], c3,2 → parR[[17]] × Net[[3]][[2]],
   c3,3 → parR[[18]] × Net[[3]][[3]], c3,4 → parR[[19]] × Net[[3]][[4]],
   c3,5 → parR[[20]] × Net[[3]][[5]],
   c4,1 → parR[[21]] × Net[[4]][[1]], c4,2 → parR[[22]] × Net[[4]][[2]],
   c4,3 → parR[[23]] × Net[[4]][[3]], c4,4 → parR[[24]] × Net[[4]][[4]],
   c4,5 → parR[[25]] × Net[[4]][[5]],
   c5,1 → parR[[26]] × Net[[5]][[1]], c5,2 → parR[[27]] × Net[[5]][[2]],
   c5,3 → parR[[28]] × Net[[5]][[3]], c5,4 → parR[[29]] × Net[[5]][[4]],
   c5,5 → parR[[30]] × Net[[5]][[5]],

   q1 → parR[[31]], q2 → parR[[32]],
   q3 → parR[[33]], q4 → parR[[34]], q5 → parR[[35]],

   d1,1 → parR[[36]], d1,2 → parR[[37]],
   d1,3 → parR[[38]], d1,4 → parR[[39]], d1,5 → parR[[40]],
   d2,1 → parR[[41]], d2,2 → parR[[42]], d2,3 → parR[[43]],
   d2,4 → parR[[44]], d2,5 → parR[[45]],
   d3,1 → parR[[46]], d3,2 → parR[[47]], d3,3 → parR[[48]],
   d3,4 → parR[[49]], d3,5 → parR[[50]],
   d4,1 → parR[[51]], d4,2 → parR[[52]], d4,3 → parR[[53]],
   d4,4 → parR[[54]], d4,5 → parR[[55]],
   d5,1 → parR[[56]], d5,2 → parR[[57]], d5,3 → parR[[58]],
```

```
  d5,4 → parR[[59]], d5,5 → parR[[60]],

  Ω1,1 → parR[[61]] × Net[[1]][[1]],
  Ω1,2 → parR[[62]] × Net[[1]][[2]], Ω1,3 → parR[[63]] × Net[[1]][[3]],
  Ω1,4 → parR[[64]] × Net[[1]][[4]], Ω1,5 → parR[[65]] × Net[[1]][[5]],
  Ω2,1 → parR[[66]] × Net[[2]][[1]], Ω2,2 → parR[[67]] × Net[[2]][[2]],
  Ω2,3 → parR[[68]] × Net[[2]][[3]], Ω2,4 → parR[[69]] × Net[[2]][[4]],
  Ω2,5 → parR[[70]] × Net[[2]][[5]],
  Ω3,1 → parR[[71]] × Net[[3]][[1]], Ω3,2 → parR[[72]] × Net[[3]][[2]],
  Ω3,3 → parR[[73]] × Net[[3]][[3]], Ω3,4 → parR[[74]] × Net[[3]][[4]],
  Ω3,5 → parR[[75]] × Net[[3]][[5]],
  Ω4,1 → parR[[76]] × Net[[4]][[1]], Ω4,2 → parR[[77]] × Net[[4]][[2]],
  Ω4,3 → parR[[78]] × Net[[4]][[3]], Ω4,4 → parR[[79]] × Net[[4]][[4]],
  Ω4,5 → parR[[80]] × Net[[4]][[5]],
  Ω5,1 → parR[[81]] × Net[[5]][[1]], Ω5,2 → parR[[82]] × Net[[5]][[2]],
  Ω5,3 → parR[[83]] × Net[[5]][[3]], Ω5,4 → parR[[84]] × Net[[5]][[4]],
  Ω5,5 → parR[[85]] × Net[[5]][[5]],
  nuK → parR[[86]],
  denK → parR[[87]]


 };


B10 = 1500;
B20 = 1500;
B30 = 1500;
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
M30 = 10;
M40 = 10;
M50 = 10;


sol =
 NDSolve[
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,
```

```
      M1'[t] == dM1,
      M2'[t] == dM2,
      M3'[t] == dM3,
      M4'[t] == dM4,
      M5'[t] == dM5,


      B1[0] == B10,
      B2[0] == B20,
      B3[0] == B30,
      B4[0] == B40,
      B5[0] == B50,
      M1[0] == M10,
      M2[0] == M20,
      M3[0] == M30,
      M4[0] == M40,
      M5[0] == M50

     } /. par,
    {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
    {t, 0, tmax}];

  {B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
     M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par;

  Min[{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax]} /. sol /. par]

 )
```

In[6639]:=

```
robustnessNewLiebigR[NetTop_] := (
  n1 = 1;
  n2 = 5000;
  mid = (n1 + n2) / 2;

  While[(n1 ≠ mid && n2 ≠ mid),
   (If[fNewLiebigR[NetTop, mid] < 1, n2 = mid, n1 = mid];
    mid = Floor[N[(n1 + n2) / 2]];); {n1, n2, mid}]; mid

 )
```

As an example let’s take the following Network

In[6454]:=
```
NetK = {
   {0, 1, 0, 1, 0},
   {1, 0, 1, 1, 0},
   {1, 0, 1, 0, 1},
   {0, 1, 0, 1, 0},
   {0, 0, 0, 0, 1}
  };
```

Using the function fNewSaito we can calculate the smallest value of a bacterial population in the community for a given disturbance vale. For example, let’s take Disturbance value 1 and 500:

In[6550]:=
```
fNewLiebig[NetK, 0]
```

Out[6550]= 665.02

In[6431]:=
```
fNewLiebig[NetK, 500]
```

Out[6431]= -1.939×10^{-53}

In[6640]:=
```
fNewLiebigR[NetK, 0]
```

Out[6640]= 604.006

In[6641]:=
```
fNewLiebigR[NetK, 500]
```

Out[6641]= -9.36978×10^{-49}

Using the function fNewSaito we can calculate Robustness of the Network:

In[6506]:=
```
robustnessNewLiebig[NetK]
```

Out[6506]= 124

In[6642]:=
```
robustnessNewLiebigR[NetK]
```

Out[6642]= 116

We can calculate the (Relative) Entropy and the Assortativity:

In[6281]:=
```
RelatEntrop5[NetK]
```
Out[6281]=
```
0.960956
```

In[6282]:=
```
assortativity[NetK]
```
Out[6282]=
```
-0.113228
```

We can calculate the robustness of the previously generated random networks with different number of auxotrophies:

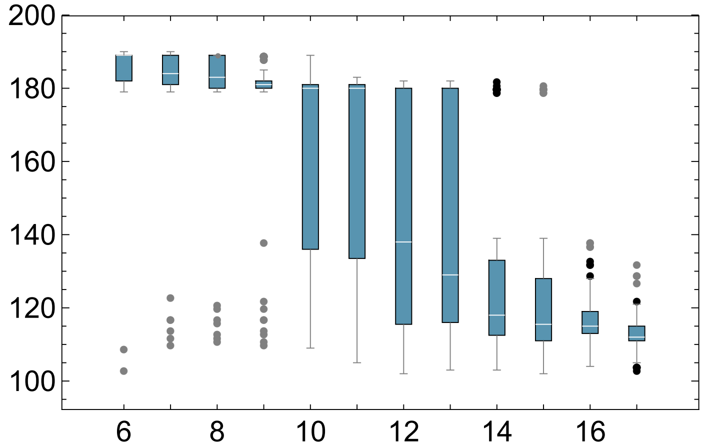
In[6643]:=
```
AuxoComm6RLi = Parallelize[robustnessNewLiebigR /@ hk6];
AuxoComm7RLi = Parallelize[robustnessNewLiebigR /@ hk7];
AuxoComm8RLi = Parallelize[robustnessNewLiebigR /@ hk8];
AuxoComm9RLi = Parallelize[robustnessNewLiebigR /@ hk9];
AuxoComm10RLi = Parallelize[robustnessNewLiebigR /@ hk10];
AuxoComm11RLi = Parallelize[robustnessNewLiebigR /@ hk11];
AuxoComm12RLi = Parallelize[robustnessNewLiebigR /@ hk12];
AuxoComm13RLi = Parallelize[robustnessNewLiebigR /@ hk13];
AuxoComm14RLi = Parallelize[robustnessNewLiebigR /@ hk14];
AuxoComm15RLi = Parallelize[robustnessNewLiebigR /@ hk15];
AuxoComm16RLi = Parallelize[robustnessNewLiebigR /@ hk16];
AuxoComm17RLi = Parallelize[robustnessNewLiebigR /@ hk17];
```

In[6655]:=
```
LikRLi = {AuxoComm6RLi, AuxoComm7RLi, AuxoComm8RLi, AuxoComm9RLi,
   AuxoComm10RLi, AuxoComm11RLi, AuxoComm12RLi, AuxoComm13RLi,
   AuxoComm14RLi, AuxoComm15RLi, AuxoComm16RLi, AuxoComm17RLi};
```

In[6470]:=
```
coco = RGBColor[0.34509803921568627, 0.5803921568627451, 0.6901960784313725]
```
Out[6470]=

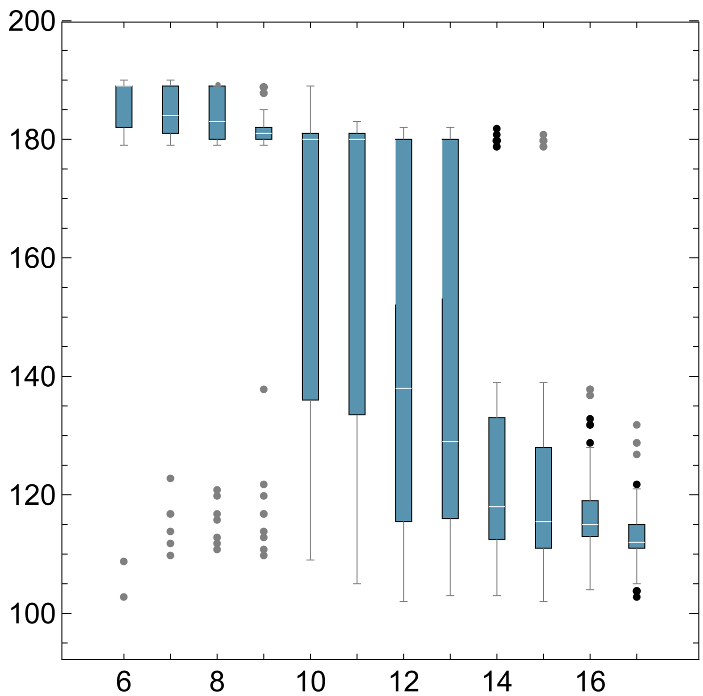
In[6656]:=
```
BoxWhiskerChart[LikRLi, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{coco}}, Frame → True,
 ChartLabels → {"6", "", "8", "", "10", "", "12", "", "14", "", "16", ""},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15]]
```

Out[6656]=



In[6657]:=
```
BoxWhiskerChart[LikRLi, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{coco}}, Frame → True,
 ChartLabels → {"6", "", "8", "", "10", "", "12", "", "14", "", "16", ""},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15], AspectRatio → 1]
```

Out[6657]=



In[6595]:=
```
AuxoComm8RLi
```

Out[6595]=
```
{181, 181, 182, 189, 181, 181, 121, 189, 189, 182, 181, 181, 182, 181, 189, 182, 181, 181,
 181, 121, 181, 181, 182, 182, 181, 189, 182, 121, 189, 182, 181, 121, 182, 189, 189,
 189, 182, 189, 189, 189, 181, 189, 189, 189, 189, 181, 181, 181, 189, 182, 181, 182,
 182, 121, 189, 181, 120, 189, 189, 182, 182, 182, 189, 189, 182, 181, 181, 189,
 189, 189, 181, 182, 189, 189, 189, 189, 181, 181, 181, 189, 181, 189, 182, 182,
 181, 189, 189, 182, 181, 181, 120, 189, 189, 189, 189, 182, 181, 182, 181, 189}
```

We can study the correlation between Relative entropy and assortativity with Robustness for Networks with 7 auxotrophies.

In[6302]:=
```
Entropy8 = RelatEntrop5 /@ hk8;
```

In[6303]:=
```
Assort8 = assortativity /@ hk8;
```

In[6658]:=
```
RobustLiebig8R = AuxoComm8RLi;
```

In[6659]:=
```
Length[Entropy8]
Length[Assort8]
Length[RobustLiebig8R]
```

Out[6659]= 100

Out[6660]= 100

Out[6661]= 100

In[6474]:=
```
{Min[Entropy8], Max[Entropy8]}
{Min[Assort8], Max[Assort8]}
```

Out[6474]= {0.9188, 0.993652}

Out[6475]= {-0.520325, 0.235702}

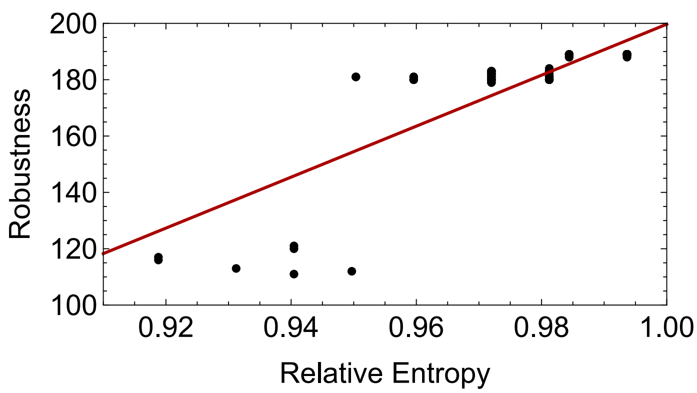
In[6395]:=
```
Position[Entropy8, Min[Entropy8]]
```

Out[6395]= {{54}, {91}}

In[6662]:=
```
RobustLiebig8R[[#]] & /@ {1, 2, 24}
```
Out[6662]=
```
{183, 181, 180}
```

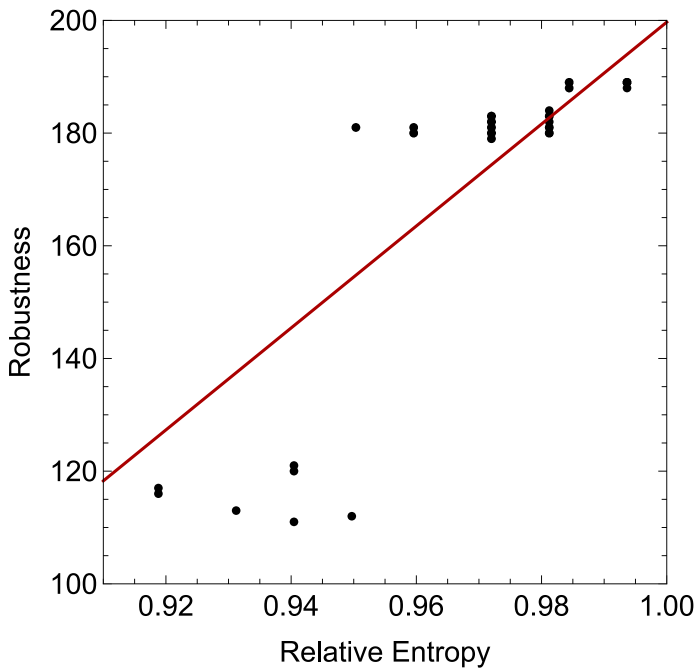
In[6663]:=
```
{Min[RobustLiebig8R], {Max[RobustLiebig8R]}}
```
Out[6663]=
```
{111, {189}}
```

In[6664]:=
```
linerobustnessLiebig8R =
  Fit[Partition[Riffle[Entropy8, RobustLiebig8R], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Entropy8, RobustLiebig8R], {2}],
  Frame → True, FrameLabel → {"Relative Entropy", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{0.91, 1}, {100, 200}},
  AspectRatio → 0.5], Plot[linerobustnessLiebig8R,
  {x, 0.91, 1}, AspectRatio → 0.5, PlotStyle → Darker[Red]]]
```
Out[6665]=



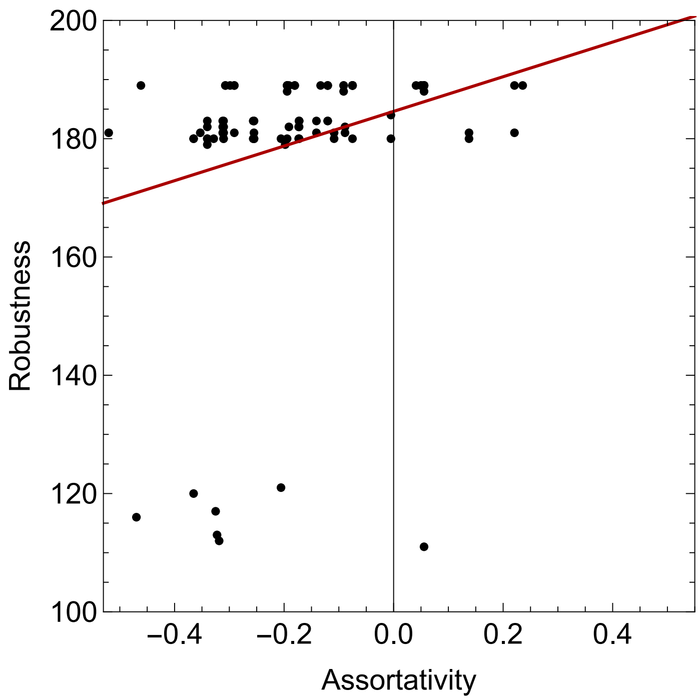
In[6672]:=
```
linerobustnessLiebig8R =
  Fit[Partition[Riffle[Entropy8, RobustLiebig8R], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Entropy8, RobustLiebig8R], {2}],
  Frame → True, FrameLabel → {"Relative Entropy", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]},
  PlotRange → {{0.91, 1}, {100, 200}}, AspectRatio → 1],
 Plot[linerobustnessLiebig8R, {x, 0.91, 1}, AspectRatio → 1, PlotStyle → Darker[Red]]]
```

Out[6673]=



In[7530]:=
```
lineAssoRobrobustnessLiebig8R =
  Fit[Partition[Riffle[Assort8, RobustLiebig8R], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Assort8, RobustLiebig8R], {2}],
  Frame → True, FrameLabel → {"Assortativity", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{-0.53, 0.55}, {100, 200}},
  AspectRatio → 1], Plot[lineAssoRobrobustnessLiebig8R,
  {x, -0.53, 0.55}, AspectRatio → 1, PlotStyle → Darker[Red]]]
```

Out[7531]=



In[6668]:=
```
SpearmanRankTest[Entropy8, RobustLiebig8R, "TestDataTable"]
```

Out[6668]=

	Statistic	P-Value
Spearman Rank	0.854777	1.14252×10^{-29}

In[6669]:=
```
SpearmanRankTest[Assort8, RobustLiebig8R, "TestDataTable"]
```

Out[6669]=

	Statistic	P-Value
Spearman Rank	0.45037	2.58156×10^{-6}

Solving the system of ODE with Overproduction Random parametrization

In[6921]:=

```
parR = {0.17406491963765225`, 0.19271552318731072`, 0.20101265981745065`,
   0.1993416825951762`, 0.20546747418997402`, 0.05126697346911639`,
   0.050979209585780075`, 0.049250907922295285`, 0.050596494945335864`,
   0.0480712396564541 4`, 0.04875707371061734`, 0.0516891959374862 7`,
   0.05110374775894862`, 0.04933415213279622`, 0.049970862334809504`,
   0.052013918924004365`, 0.051281069200719044`, 0.05008059124653844`,
   0.048456986155650895`, 0.05046268834028382`, 0.0510088164712911 5`,
   0.04827398693541804`, 0.049033976435347196`, 0.04894443311184237`,
   0.05097473744162672`, 0.05026272693204548`, 0.0509412937337713`,
   0.050043183079927396`, 0.04981275493116924`, 0.05143431351532053`,
   0.2741484220752475`, 0.29500048735377876`, 0.29055979190752795`,
   0.30919969983773093`, 0.30449818646425864`, 0.0015858418050202845`,
   0.0015559938026471695`, 0.0015508063247152385`, 0.0016041035713506373`,
   0.0014610080434572011`, 0.0014038903176297444`, 0.0014847851291227255`,
   0.0014962435725992124`, 0.0014729115074682852`, 0.0015842336090410236`,
   0.0016865044671100866`, 0.00150168730793647 6`, 0.0015598859023090983`,
   0.00146452620048039 2`, 0.0014896327356137065`, 0.0016086000253588925`,
   0.0014516848408536902`, 0.0014625348014891348`, 0.0014385370012988826`,
   0.0015193617217833212`, 0.0015818702687085818`, 0.0015903285844332212`,
   0.0016182333378202214`, 0.0016763973379304588`, 0.0014331690573307755`,
   0.959484902075753`, 0.9912481178882356`, 0.9899993730630222`,
   0.9628256622169967`, 0.9946569905494801`, 1.0129632879517105`,
   1.1109986104234093`, 1.0125450635832367`, 0.988840015814362`,
   0.94544275526312`, 1.003913112496818`, 1.0076856430130368`,
   1.0340447722669786`, 0.9626849557677253`, 1.0257791881087754`,
   1.013963295639842`, 0.9734619708145545`, 0.973176386033725`,
   0.952023923560438 6`, 0.9729607432853318`, 1.0032062732410925`,
   1.038812253989961 1`, 0.9277774829512535`, 1.0446966704519374`,
   1.0413148081050696`, 200.03791321285763`, 2.0009063754198024`};
```

In[6922]:=

```
fNewLiebigOVR[Net_, Dh_, coop_] := (

  dB1 =
   B1[t] (-B1[t] κ1 + Min[{ (nuK M1[t])/(denK + M1[t]) , (nuK M2[t])/(denK + M2[t]) , (nuK M3[t])/(denK + M3[t]) , (nuK M4[t])/(denK + M4[t]) ,
```

$$\left.\left.\frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) - \left(c_{1,1}+c_{1,2}+c_{1,3}+c_{1,4}+c_{1,5}+\text{Dh}\right) B_1[t];$$

$$\text{dB}_2 = B_2[t]\left(-B_2[t]\,\kappa_2+\text{Min}[\{\frac{\text{nuK}\,M_1[t]}{\text{denK}+M_1[t]}, \frac{\text{nuK}\,M_2[t]}{\text{denK}+M_2[t]}, \frac{\text{nuK}\,M_3[t]}{\text{denK}+M_3[t]}, \frac{\text{nuK}\,M_4[t]}{\text{denK}+M_4[t]}, \frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) - \left(c_{2,1}+c_{2,2}+c_{2,3}+c_{2,4}+c_{2,5}+\text{Dh}\right) B_2[t];$$

$$\text{dB}_3 = B_3[t]\left(-B_3[t]\,\kappa_3+\text{Min}[\{\frac{\text{nuK}\,M_1[t]}{\text{denK}+M_1[t]}, \frac{\text{nuK}\,M_2[t]}{\text{denK}+M_2[t]}, \frac{\text{nuK}\,M_3[t]}{\text{denK}+M_3[t]}, \frac{\text{nuK}\,M_4[t]}{\text{denK}+M_4[t]}, \frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) - \left(c_{3,1}+c_{3,2}+c_{3,3}+c_{3,4}+c_{3,5}+\text{Dh}\right) B_3[t];$$

$$\text{dB}_4 = B_4[t]\left(-B_4[t]\,\kappa_4+\text{Min}[\{\frac{\text{nuK}\,M_1[t]}{\text{denK}+M_1[t]}, \frac{\text{nuK}\,M_2[t]}{\text{denK}+M_2[t]}, \frac{\text{nuK}\,M_3[t]}{\text{denK}+M_3[t]}, \frac{\text{nuK}\,M_4[t]}{\text{denK}+M_4[t]}, \frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) - \left(c_{4,1}+c_{4,2}+c_{4,3}+c_{4,4}+c_{4,5}+\text{Dh}\right) B_4[t];$$

$$\text{dB}_5 = B_5[t]\left(-B_5[t]\,\kappa_5+\text{Min}[\{\frac{\text{nuK}\,M_1[t]}{\text{denK}+M_1[t]}, \frac{\text{nuK}\,M_2[t]}{\text{denK}+M_2[t]}, \frac{\text{nuK}\,M_3[t]}{\text{denK}+M_3[t]}, \frac{\text{nuK}\,M_4[t]}{\text{denK}+M_4[t]}, \frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) - \left(c_{5,1}+c_{5,2}+c_{5,3}+c_{5,4}+c_{5,5}+\text{Dh}\right) B_5[t];$$

$$\text{dM}_1 = -M_1[t]\,q_1 + \left(\text{Min}[\{\frac{\text{nuK}\,M_1[t]}{\text{denK}+M_1[t]}, \frac{\text{nuK}\,M_2[t]}{\text{denK}+M_2[t]}, \frac{\text{nuK}\,M_3[t]}{\text{denK}+M_3[t]}, \frac{\text{nuK}\,M_4[t]}{\text{denK}+M_4[t]}, \frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) \left(-B_1[t]\,d_{1,1}-B_2[t]\,d_{1,2}-B_3[t]\,d_{1,3}-B_4[t]\,d_{1,4}-B_5[t]\,d_{1,5}\right) + B_1[t]\,\Omega_{1,1}+B_2[t]\,\Omega_{1,2}+B_3[t]\,\Omega_{1,3}+B_4[t]\,\Omega_{1,4}+B_5[t]\,\Omega_{1,5};$$

$$\text{dM}_2 = -M_2[t]\,q_2 + \left(\text{Min}[\{\frac{\text{nuK}\,M_1[t]}{\text{denK}+M_1[t]}, \frac{\text{nuK}\,M_2[t]}{\text{denK}+M_2[t]}, \frac{\text{nuK}\,M_3[t]}{\text{denK}+M_3[t]}, \frac{\text{nuK}\,M_4[t]}{\text{denK}+M_4[t]}, \frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) \left(-B_1[t]\,d_{2,1}-B_2[t]\,d_{2,2}-B_3[t]\,d_{2,3}-B_4[t]\,d_{2,4}-B_5[t]\,d_{2,5}\right) + B_1[t]\,\Omega_{2,1}+B_2[t]\,\Omega_{2,2}+B_3[t]\,\Omega_{2,3}+B_4[t]\,\Omega_{2,4}+B_5[t]\,\Omega_{2,5};$$

$$\text{dM}_3 = -M_3[t]\,q_3 + \left(\text{Min}[\{\frac{\text{nuK}\,M_1[t]}{\text{denK}+M_1[t]}, \frac{\text{nuK}\,M_2[t]}{\text{denK}+M_2[t]}, \frac{\text{nuK}\,M_3[t]}{\text{denK}+M_3[t]}, \frac{\text{nuK}\,M_4[t]}{\text{denK}+M_4[t]}, \frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) \left(-B_1[t]\,d_{3,1}-B_2[t]\,d_{3,2}-B_3[t]\,d_{3,3}-B_4[t]\,d_{3,4}-B_5[t]\,d_{3,5}\right) + B_1[t]\,\Omega_{3,1}+B_2[t]\,\Omega_{3,2}+B_3[t]\,\Omega_{3,3}+B_4[t]\,\Omega_{3,4}+B_5[t]\,\Omega_{3,5};$$

$$\text{dM}_4 = -M_4[t]\,q_4 + \left(\text{Min}[\{\frac{\text{nuK}\,M_1[t]}{\text{denK}+M_1[t]}, \frac{\text{nuK}\,M_2[t]}{\text{denK}+M_2[t]}, \frac{\text{nuK}\,M_3[t]}{\text{denK}+M_3[t]}, \frac{\text{nuK}\,M_4[t]}{\text{denK}+M_4[t]}, \frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) \left(-B_1[t]\,d_{4,1}-B_2[t]\,d_{4,2}-B_3[t]\,d_{4,3}-B_4[t]\,d_{4,4}-B_5[t]\,d_{4,5}\right) + B_1[t]\,\Omega_{4,1}+B_2[t]\,\Omega_{4,2}+B_3[t]\,\Omega_{4,3}+B_4[t]\,\Omega_{4,4}+B_5[t]\,\Omega_{4,5};$$

$$\text{dM}_5 = -M_5[t]\,q_5 + \left(\text{Min}[\{\frac{\text{nuK}\,M_1[t]}{\text{denK}+M_1[t]}, \frac{\text{nuK}\,M_2[t]}{\text{denK}+M_2[t]}, \frac{\text{nuK}\,M_3[t]}{\text{denK}+M_3[t]}, \frac{\text{nuK}\,M_4[t]}{\text{denK}+M_4[t]}, \frac{\text{nuK}\,M_5[t]}{\text{denK}+M_5[t]}\}]\right) \left(-B_1[t]\,d_{5,1}-B_2[t]\,d_{5,2}-B_3[t]\,d_{5,3}-B_4[t]\,d_{5,4}-B_5[t]\,d_{5,5}\right) + B_1[t]\,\Omega_{5,1}+B_2[t]\,\Omega_{5,2}+B_3[t]\,\Omega_{5,3}+B_4[t]\,\Omega_{5,4}+B_5[t]\,\Omega_{5,5};$$

```
op = coop; (*Number of links with overExpression*)
posNe = Position[Net, 1];
(*Positions in the matrix where there are links (=1)*)
RaN = RandomSample[posNe, op];
(*Random sample of op links that will be overproduced*)

costincr = 1.3; (*Term multiplying the cost link*)
overprodincr = 1.15;
(*Term multiplying the overproduction link*)


NewNetCost = Partition[Flatten[Net] × parR[[6 ;; 30]], {5}];
Table[NewNetCost[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] =
  NewNetCost[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] * costincr, {i, Length[RaN]}];

NewNetOvProd = Partition[Flatten[Net] × parR[[61 ;; 85]], {5}];
Table[NewNetOvProd[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] =
  NewNetOvProd[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] * overprodincr, {i,
  Length[RaN]}];


tmax = 1000;
par = {
  κ1 → parR[[1]], κ2 → parR[[2]], κ3 → parR[[3]], κ4 → parR[[4]], κ5 → parR[[5]],

  c1,1 → NewNetCost[[1]][[1]],
  c1,2 → NewNetCost[[1]][[2]], c1,3 → NewNetCost[[1]][[3]],
  c1,4 → NewNetCost[[1]][[4]], c1,5 → NewNetCost[[1]][[5]],
  c2,1 → NewNetCost[[2]][[1]], c2,2 → NewNetCost[[2]][[2]],
  c2,3 → NewNetCost[[2]][[3]], c2,4 → NewNetCost[[2]][[4]],
  c2,5 → NewNetCost[[2]][[5]],
  c3,1 → NewNetCost[[3]][[1]], c3,2 → NewNetCost[[3]][[2]],
  c3,3 → NewNetCost[[3]][[3]], c3,4 → NewNetCost[[3]][[4]],
  c3,5 → NewNetCost[[3]][[5]],
  c4,1 → NewNetCost[[4]][[1]], c4,2 → NewNetCost[[4]][[2]],
  c4,3 → NewNetCost[[4]][[3]], c4,4 → NewNetCost[[4]][[4]],
  c4,5 → NewNetCost[[4]][[5]],
  c5,1 → NewNetCost[[5]][[1]], c5,2 → NewNetCost[[5]][[2]],
  c5,3 → NewNetCost[[5]][[3]], c5,4 → NewNetCost[[5]][[4]],
  c5,5 → NewNetCost[[5]][[5]],
```

```
r1,1 → parR[[31]], r1,2 → parR[[32]],
r1,3 → parR[[33]], r1,4 → parR[[34]], r1,5 → parR[[35]],
r2,1 → parR[[36]], r2,2 → parR[[37]], r2,3 → parR[[38]],
r2,4 → parR[[39]], r2,5 → parR[[40]],
r3,1 → parR[[41]], r3,2 → parR[[42]], r3,3 → parR[[43]],
r3,4 → parR[[44]], r3,5 → parR[[45]],
r4,1 → parR[[46]], r4,2 → parR[[47]], r4,3 → parR[[48]],
r4,4 → parR[[49]], r4,5 → parR[[50]],
r5,1 → parR[[51]], r5,2 → parR[[52]], r5,3 → parR[[53]],
r5,4 → parR[[54]], r5,5 → parR[[55]],

q1 → parR[[31]], q2 → parR[[32]],
q3 → parR[[33]], q4 → parR[[34]], q5 → parR[[35]],

d1,1 → parR[[36]], d1,2 → parR[[37]],
d1,3 → parR[[38]], d1,4 → parR[[39]], d1,5 → parR[[40]],
d2,1 → parR[[41]], d2,2 → parR[[42]], d2,3 → parR[[43]],
d2,4 → parR[[44]], d2,5 → parR[[45]],
d3,1 → parR[[46]], d3,2 → parR[[47]], d3,3 → parR[[48]],
d3,4 → parR[[49]], d3,5 → parR[[50]],
d4,1 → parR[[51]], d4,2 → parR[[52]], d4,3 → parR[[53]],
d4,4 → parR[[54]], d4,5 → parR[[55]],
d5,1 → parR[[56]], d5,2 → parR[[57]], d5,3 → parR[[58]],
d5,4 → parR[[59]], d5,5 → parR[[60]],

Ω1,1 → NewNetOvProd[[1]][[1]],
Ω1,2 → NewNetOvProd[[1]][[2]], Ω1,3 → NewNetOvProd[[1]][[3]],
Ω1,4 → NewNetOvProd[[1]][[4]], Ω1,5 → NewNetOvProd[[1]][[5]],
Ω2,1 → NewNetOvProd[[2]][[1]], Ω2,2 → NewNetOvProd[[2]][[2]],
Ω2,3 → NewNetOvProd[[2]][[3]], Ω2,4 → NewNetOvProd[[2]][[4]],
Ω2,5 → NewNetOvProd[[2]][[5]],
Ω3,1 → NewNetOvProd[[3]][[1]], Ω3,2 → NewNetOvProd[[3]][[2]],
Ω3,3 → NewNetOvProd[[3]][[3]], Ω3,4 → NewNetOvProd[[3]][[4]],
Ω3,5 → NewNetOvProd[[3]][[5]],
Ω4,1 → NewNetOvProd[[4]][[1]], Ω4,2 → NewNetOvProd[[4]][[2]],
Ω4,3 → NewNetOvProd[[4]][[3]], Ω4,4 → NewNetOvProd[[4]][[4]],
Ω4,5 → NewNetOvProd[[4]][[5]],
Ω5,1 → NewNetOvProd[[5]][[1]], Ω5,2 → NewNetOvProd[[5]][[2]],
Ω5,3 → NewNetOvProd[[5]][[3]], Ω5,4 → NewNetOvProd[[5]][[4]],
Ω5,5 → NewNetOvProd[[5]][[5]],
nuK → parR[[86]],
denK → parR[[87]]
```

```
 };


B10 = 1500;
B20 = 1500;
B30 = 1500;
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
M30 = 10;
M40 = 10;
M50 = 10;


sol =
 NDSolve[
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,

    M1'[t] == dM1,
    M2'[t] == dM2,
    M3'[t] == dM3,
    M4'[t] == dM4,
    M5'[t] == dM5,


    B1[0] == B10,
    B2[0] == B20,
    B3[0] == B30,
    B4[0] == B40,
    B5[0] == B50,
    M1[0] == M10,
    M2[0] == M20,
    M3[0] == M30,
    M4[0] == M40,
    M5[0] == M50

   } /. par,
  {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
```

```
    {t, 0, tmax}];

  {B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
     M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par;

  Min[{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax]} /. sol /. par]

 )
```

In[6923]:=
```
robustnessNewLiebigOVR[NetTop_, coop_] := (
  n1 = 1;
  n2 = 5000;
  mid = (n1 + n2) / 2;

  While[(n1 ≠ mid && n2 ≠ mid),
   (If[fNewLiebigOVR[NetTop, mid, coop] < 1, n2 = mid, n1 = mid];
    mid = Floor[N[(n1 + n2) / 2]];); {n1, n2, mid}]; mid

 )
```

In[6924]:=
```
NetK = {
   {0, 1, 0, 1, 0},
   {1, 0, 1, 1, 0},
   {1, 0, 1, 0, 1},
   {0, 1, 0, 1, 0},
   {0, 0, 0, 0, 1}
  };
```

Compare the Robustness with and without (5 links) overproduction (ratio cost/production = 1.3/1.15)

In[6925]:=
```
fNewLiebigR[NetK, 0]
```

Out[6925]= 604.006

```
In[6926]:= fNewLiebigOVR[NetK, 0, 5]
Out[6926]= 694.357
```

```
In[6927]:= robustnessNewLiebigR[NetK]
Out[6927]= 116
In[6928]:= robustnessNewLiebigOVR[NetK, 5]
Out[6928]= 120
In[6929]:= robustnessNewLiebigOVR[NetK, 10]
Out[6929]= 133
```

```
AuxoComm8RLi
Out[•]= {90, 90, 91, 91, 90, 90, 88, 91, 91, 90, 91, 91, 90, 91, 91, 90, 90, 90, 91, 88,
 90, 90, 91, 90, 90, 91, 91, 88, 91, 91, 91, 88, 91, 91, 91, 91, 90, 91, 91, 91,
 90, 91, 91, 91, 91, 91, 90, 90, 91, 90, 90, 91, 90, 88, 91, 91, 88, 91, 91, 91,
 90, 90, 91, 91, 91, 90, 90, 91, 91, 91, 91, 90, 91, 91, 91, 91, 91, 90, 90, 91,
 90, 91, 91, 90, 90, 91, 91, 91, 90, 90, 87, 91, 91, 91, 91, 91, 90, 90, 91, 91}
```

```
In[6930]:= coop5to15LiROV =
  {Table[robustnessNewLiebigOVR[#, 5], {20}], Table[robustnessNewLiebigOVR[#, 10],
     {20}], Table[robustnessNewLiebigOVR[#, 15], {20}]} &;
In[6931]:= wf8LiOVR = Parallelize[coop5to15LiROV /@ hk8];
In[6932]:= wf8NormalizedLiOVR = N[wf8LiOVR[[#]] / AuxoComm8RLi[[#]]] & /@ Range[100]
```

In[6933]:=
```
wf8NormalizedWith5CoopLiOVR = wf8NormalizedLiOVR[[#]][[1]] & /@ Range[100]
```

In[6934]:=
```
wf8NormalizedWith10CoopLiOVR = wf8NormalizedLiOVR[[#]][[2]] & /@ Range[100]
```

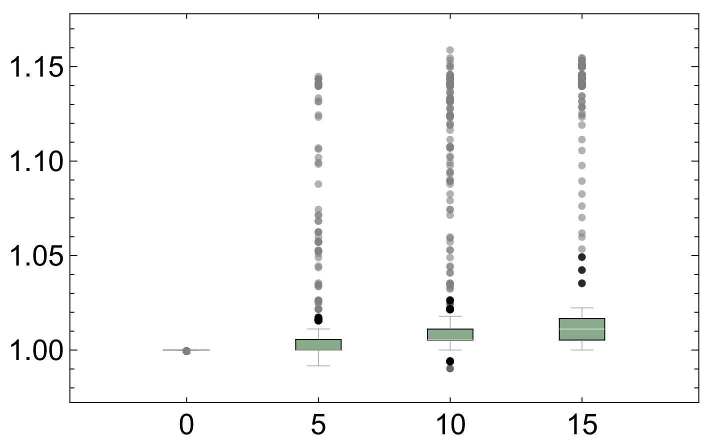
In[6935]:=
```
wf8NormalizedWith15CoopLiOVR = wf8NormalizedLiOVR[[#]][[3]] & /@ Range[100]
```

In[6936]:=
```
allcoopWith8AuxoLiOVR = {Flatten[wf8NormalizedWith5CoopLiOVR],
  Flatten[wf8NormalizedWith10CoopLiOVR], Flatten[wf8NormalizedWith15CoopLiOVR]}
```

In[6937]:=
```
allcoopWith8AuxoPlusAuxoLiOVR =
 Join[{ConstantArray[1, {2000}]}, allcoopWith8AuxoLiOVR]
```

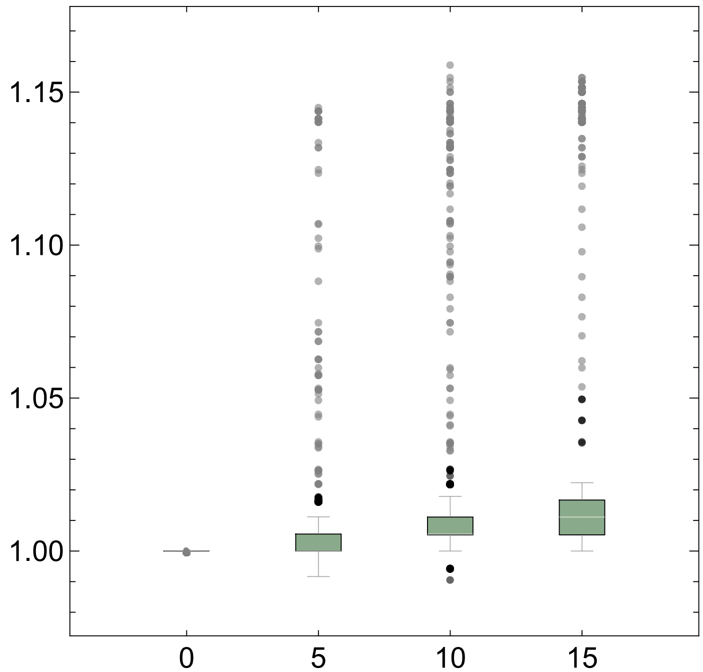
In[6938]:=
```
BoxWhiskerChart[allcoopWith8AuxoPlusAuxoLiOVR, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{gree1}},
 Frame → True, ChartLabels → {"0", "5", "10", "15"},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15]]
```

Out[6938]=



In[6939]:=
```
BoxWhiskerChart[allcoopWith8AuxoPlusAuxoLiOVR, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{gree1}},
 Frame → True, ChartLabels → {"0", "5", "10", "15"}, BarSpacing → 1.9,
 FrameStyle → Directive[Black, FontSize → 15], AspectRatio → 1]
```

Out[6939]=



In[6940]:=
```
allcoopWith8AuxoPlusAuxoLiOVR // Length
```

Out[6940]= 4

In[6941]:=
```
SignedRankTest[allcoopWith8AuxoPlusAuxoLiOVR[[2]], 1]
SignedRankTest[allcoopWith8AuxoPlusAuxoLiOVR[[3]], 1]
SignedRankTest[allcoopWith8AuxoPlusAuxoLiOVR[[4]], 1]
```

Out[6941]= 1.93628×10^{-137}

Out[6942]= 2.80686×10^{-261}

Out[6943]= 0.