# Cooperation increases robustness to ecological disturbance in microbial cross-feeding networks

## Generating Random Networks

## Functions to calculate Entropy and Assortativity

### Entropy

### Assortativity

## 1. Colimitation model

### Solving the system of ODE

The function “fNewSaitoK” solves the ODE system and gives the population at steady state of the system. The function “fNewSaitoK” receives a network and a disturbance value as arguments.

In[7090]:=

```
fNewSaitoBthK[Net_, Dh_] := (

  dB1 =
   B1[t] (-B1[t] κ1 + nuK * M1[t]/(denK + M1[t]) * M2[t]/(denK + M2[t]) * M3[t]/(denK + M3[t]) * M4[t]/(denK + M4[t]) *
        M5[t]/(denK + M5[t])) - (c1,1 + c1,2 + c1,3 + c1,4 + c1,5 + Dh) B1[t];
  dB2 = B2[t] (-B2[t] κ2 + nuK * M1[t]/(denK + M1[t]) * M2[t]/(denK + M2[t]) * M3[t]/(denK + M3[t]) *
        M4[t]/(denK + M4[t]) * M5[t]/(denK + M5[t])) - (c2,1 + c2,2 + c2,3 + c2,4 + c2,5 + Dh) B2[t];
  dB3 = B3[t] (-B3[t] κ3 + nuK * M1[t]/(denK + M1[t]) * M2[t]/(denK + M2[t]) * M3[t]/(denK + M3[t]) *
```

$$\frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Big) - (c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4} + c_{3,5} + Dh)\, B_3[t];$$

$$dB_4 = B_4[t]\Big(-B_4[t]\,\kappa_4 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Big) - (c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4} + c_{4,5} + Dh)\, B_4[t];$$

$$dB_5 = B_5[t]\Big(-B_5[t]\,\kappa_5 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Big) - (c_{5,1} + c_{5,2} + c_{5,3} + c_{5,4} + c_{5,5} + Dh)\, B_5[t];$$

$$dM_1 = -M_1[t]\,(Dh + q_1) + \Big(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Big) (-B_1[t]\, d_{1,1} - B_2[t]\, d_{1,2} - B_3[t]\, d_{1,3} - B_4[t]\, d_{1,4} - B_5[t]\, d_{1,5}) + B_1[t]\,\Omega_{1,1} + B_2[t]\,\Omega_{1,2} + B_3[t]\,\Omega_{1,3} + B_4[t]\,\Omega_{1,4} + B_5[t]\,\Omega_{1,5};$$

$$dM_2 = -M_2[t]\,(Dh + q_2) + \Big(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Big) (-B_1[t]\, d_{2,1} - B_2[t]\, d_{2,2} - B_3[t]\, d_{2,3} - B_4[t]\, d_{2,4} - B_5[t]\, d_{2,5}) + B_1[t]\,\Omega_{2,1} + B_2[t]\,\Omega_{2,2} + B_3[t]\,\Omega_{2,3} + B_4[t]\,\Omega_{2,4} + B_5[t]\,\Omega_{2,5};$$

$$dM_3 = -M_3[t]\,(Dh + q_3) + \Big(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Big) (-B_1[t]\, d_{3,1} - B_2[t]\, d_{3,2} - B_3[t]\, d_{3,3} - B_4[t]\, d_{3,4} - B_5[t]\, d_{3,5}) + B_1[t]\,\Omega_{3,1} + B_2[t]\,\Omega_{3,2} + B_3[t]\,\Omega_{3,3} + B_4[t]\,\Omega_{3,4} + B_5[t]\,\Omega_{3,5};$$

$$dM_4 = -M_4[t]\,(Dh + q_4) + \Big(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Big) (-B_1[t]\, d_{4,1} - B_2[t]\, d_{4,2} - B_3[t]\, d_{4,3} - B_4[t]\, d_{4,4} - B_5[t]\, d_{4,5}) + B_1[t]\,\Omega_{4,1} + B_2[t]\,\Omega_{4,2} + B_3[t]\,\Omega_{4,3} + B_4[t]\,\Omega_{4,4} + B_5[t]\,\Omega_{4,5};$$

$$dM_5 = -M_5[t]\,(Dh + q_5) + \Big(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Big) (-B_1[t]\, d_{5,1} - B_2[t]\, d_{5,2} - B_3[t]\, d_{5,3} - B_4[t]\, d_{5,4} - B_5[t]\, d_{5,5}) + B_1[t]\,\Omega_{5,1} + B_2[t]\,\Omega_{5,2} + B_3[t]\,\Omega_{5,3} + B_4[t]\,\Omega_{5,4} + B_5[t]\,\Omega_{5,5};$$

```
KK = 0.2;
cc = 0.05;
qq = 0.3;
dd = 0.00015;
OM = 1;
nu = 1500;
den = 2;
```

```
tmax = 1000;
par = {
  κ1 → KK, κ2 → KK, κ3 → KK, κ4 → KK, κ5 → KK,

  c1,1 → cc Net[[1]][[1]], c1,2 → cc Net[[1]][[2]],
  c1,3 → cc Net[[1]][[3]], c1,4 → cc Net[[1]][[4]], c1,5 → cc Net[[1]][[5]],
  c2,1 → cc Net[[2]][[1]], c2,2 → cc Net[[2]][[2]], c2,3 → cc Net[[2]][[3]],
  c2,4 → cc Net[[2]][[4]], c2,5 → cc Net[[2]][[5]],
  c3,1 → cc Net[[3]][[1]], c3,2 → cc Net[[3]][[2]], c3,3 → cc Net[[3]][[3]],
  c3,4 → cc Net[[3]][[4]], c3,5 → cc Net[[3]][[5]],
  c4,1 → cc Net[[4]][[1]], c4,2 → cc Net[[4]][[2]], c4,3 → cc Net[[4]][[3]],
  c4,4 → cc Net[[4]][[4]], c4,5 → cc Net[[4]][[5]],
  c5,1 → cc Net[[5]][[1]], c5,2 → cc Net[[5]][[2]], c5,3 → cc Net[[5]][[3]],
  c5,4 → cc Net[[5]][[4]], c5,5 → cc Net[[5]][[5]],

  q1 → qq, q2 → qq, q3 → qq, q4 → qq, q5 → qq,

  d1,1 → dd, d1,2 → dd, d1,3 → dd, d1,4 → dd, d1,5 → dd,
  d2,1 → dd, d2,2 → dd, d2,3 → dd, d2,4 → dd, d2,5 → dd,
  d3,1 → dd, d3,2 → dd, d3,3 → dd, d3,4 → dd, d3,5 → dd,
  d4,1 → dd, d4,2 → dd, d4,3 → dd, d4,4 → dd, d4,5 → dd,
  d5,1 → dd, d5,2 → dd, d5,3 → dd, d5,4 → dd, d5,5 → dd,

  Ω1,1 → OM Net[[1]][[1]], Ω1,2 → OM Net[[1]][[2]],
  Ω1,3 → OM Net[[1]][[3]], Ω1,4 → OM Net[[1]][[4]], Ω1,5 → OM Net[[1]][[5]],
  Ω2,1 → OM Net[[2]][[1]], Ω2,2 → OM Net[[2]][[2]], Ω2,3 → OM Net[[2]][[3]],
  Ω2,4 → OM Net[[2]][[4]], Ω2,5 → OM Net[[2]][[5]],
  Ω3,1 → OM Net[[3]][[1]], Ω3,2 → OM Net[[3]][[2]], Ω3,3 → OM Net[[3]][[3]],
  Ω3,4 → OM Net[[3]][[4]], Ω3,5 → OM Net[[3]][[5]],
  Ω4,1 → OM Net[[4]][[1]], Ω4,2 → OM Net[[4]][[2]], Ω4,3 → OM Net[[4]][[3]],
  Ω4,4 → OM Net[[4]][[4]], Ω4,5 → OM Net[[4]][[5]],
  Ω5,1 → OM Net[[5]][[1]], Ω5,2 → OM Net[[5]][[2]], Ω5,3 → OM Net[[5]][[3]],
  Ω5,4 → OM Net[[5]][[4]], Ω5,5 → OM Net[[5]][[5]],
  nuK → nu,
  denK → den


 };


B10 = 1500;
B20 = 1500;
B30 = 1500;
```

```
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
M30 = 10;
M40 = 10;
M50 = 10;


sol =
 NDSolve[
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,

    M1'[t] == dM1,
    M2'[t] == dM2,
    M3'[t] == dM3,
    M4'[t] == dM4,
    M5'[t] == dM5,



    B1[0] == B10,
    B2[0] == B20,
    B3[0] == B30,
    B4[0] == B40,
    B5[0] == B50,
    M1[0] == M10,
    M2[0] == M20,
    M3[0] == M30,
    M4[0] == M40,
    M5[0] == M50

   } /. par,
  {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
  {t, 0, tmax}];

{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
   M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par

(*Min[{B1[tmax],B2[tmax],B3[tmax],B4[tmax],B5[tmax]}/.sol/.par]*)
```

```
)
```

As an example let’s take the following Network

In[7091]:=
```
NetK = {
    {0, 1, 0, 1, 0},
    {1, 0, 1, 1, 0},
    {1, 0, 1, 0, 1},
    {0, 1, 0, 1, 0},
    {0, 0, 0, 0, 1}
  };
```

In[7092]:=
```
fNewSaitoBthK[NetK, 0]
fNewSaitoBthK[NetK, 1]
```

Out[7092]=
```
{{6661.68, 6661.43, 6661.43, 6661.68,
  6661.93, 22 219.9, 44 425.5, 44 426.3, 22 219.9, 15.9422}}
```

Out[7093]=
```
{{6640.93, 6640.68, 6640.68, 6640.93,
  6641.18, 5123.76, 10 232.2, 10 232.4, 5123.76, 15.7352}}
```

The function “fNewSaito” solves the ODE system and gives the lowest microbial population size (this is used to calculate the Robustness). The function “fNewSaito” receives a network and a disturbance value as arguments.

In[7094]:=
```
fNewSaitoBth[Net_, Dh_] := (
```

$$dB_1 = B_1[t]\left(-B_1[t]\,\kappa_1 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4} + c_{1,5} + Dh\right) B_1[t];$$

$$dB_2 = B_2[t]\left(-B_2[t]\,\kappa_2 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4} + c_{2,5} + Dh\right) B_2[t];$$

$$dB_3 = B_3[t]\left(-B_3[t]\,\kappa_3 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4} + c_{3,5} + Dh\right) B_3[t];$$

$$dB_4 = B_4[t]\left(-B_4[t]\,\kappa_4 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} *\right.$$

$$\frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Bigg) - \left(c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4} + c_{4,5} + Dh\right) B_4[t];$$

$$dB_5 = B_5[t] \Bigg(-B_5[t]\,\kappa_5 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\Bigg) - \left(c_{5,1} + c_{5,2} + c_{5,3} + c_{5,4} + c_{5,5} + Dh\right) B_5[t];$$

$$dM_1 = -M_1[t]\,\left(Dh + q_1\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{1,1} - B_2[t]\,d_{1,2} - B_3[t]\,d_{1,3} - B_4[t]\,d_{1,4} - B_5[t]\,d_{1,5}\right) + B_1[t]\,\Omega_{1,1} + B_2[t]\,\Omega_{1,2} + B_3[t]\,\Omega_{1,3} + B_4[t]\,\Omega_{1,4} + B_5[t]\,\Omega_{1,5};$$

$$dM_2 = -M_2[t]\,\left(Dh + q_2\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{2,1} - B_2[t]\,d_{2,2} - B_3[t]\,d_{2,3} - B_4[t]\,d_{2,4} - B_5[t]\,d_{2,5}\right) + B_1[t]\,\Omega_{2,1} + B_2[t]\,\Omega_{2,2} + B_3[t]\,\Omega_{2,3} + B_4[t]\,\Omega_{2,4} + B_5[t]\,\Omega_{2,5};$$

$$dM_3 = -M_3[t]\,\left(Dh + q_3\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{3,1} - B_2[t]\,d_{3,2} - B_3[t]\,d_{3,3} - B_4[t]\,d_{3,4} - B_5[t]\,d_{3,5}\right) + B_1[t]\,\Omega_{3,1} + B_2[t]\,\Omega_{3,2} + B_3[t]\,\Omega_{3,3} + B_4[t]\,\Omega_{3,4} + B_5[t]\,\Omega_{3,5};$$

$$dM_4 = -M_4[t]\,\left(Dh + q_4\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{4,1} - B_2[t]\,d_{4,2} - B_3[t]\,d_{4,3} - B_4[t]\,d_{4,4} - B_5[t]\,d_{4,5}\right) + B_1[t]\,\Omega_{4,1} + B_2[t]\,\Omega_{4,2} + B_3[t]\,\Omega_{4,3} + B_4[t]\,\Omega_{4,4} + B_5[t]\,\Omega_{4,5};$$

$$dM_5 = -M_5[t]\,\left(Dh + q_5\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{5,1} - B_2[t]\,d_{5,2} - B_3[t]\,d_{5,3} - B_4[t]\,d_{5,4} - B_5[t]\,d_{5,5}\right) + B_1[t]\,\Omega_{5,1} + B_2[t]\,\Omega_{5,2} + B_3[t]\,\Omega_{5,3} + B_4[t]\,\Omega_{5,4} + B_5[t]\,\Omega_{5,5};$$

```
KK = 0.2;
cc = 0.05;
qq = 0.3;
dd = 0.00015;
OM = 1;
nu = 1500;
den = 2;

tmax = 1000;
par = {
  κ1 → KK, κ2 → KK, κ3 → KK, κ4 → KK, κ5 → KK,
```

```
c1,1 → cc Net[[1]][[1]], c1,2 → cc Net[[1]][[2]],
c1,3 → cc Net[[1]][[3]], c1,4 → cc Net[[1]][[4]], c1,5 → cc Net[[1]][[5]],
c2,1 → cc Net[[2]][[1]], c2,2 → cc Net[[2]][[2]], c2,3 → cc Net[[2]][[3]],
c2,4 → cc Net[[2]][[4]], c2,5 → cc Net[[2]][[5]],
c3,1 → cc Net[[3]][[1]], c3,2 → cc Net[[3]][[2]], c3,3 → cc Net[[3]][[3]],
c3,4 → cc Net[[3]][[4]], c3,5 → cc Net[[3]][[5]],
c4,1 → cc Net[[4]][[1]], c4,2 → cc Net[[4]][[2]], c4,3 → cc Net[[4]][[3]],
c4,4 → cc Net[[4]][[4]], c4,5 → cc Net[[4]][[5]],
c5,1 → cc Net[[5]][[1]], c5,2 → cc Net[[5]][[2]], c5,3 → cc Net[[5]][[3]],
c5,4 → cc Net[[5]][[4]], c5,5 → cc Net[[5]][[5]],

q1 → qq, q2 → qq, q3 → qq, q4 → qq, q5 → qq,

d1,1 → dd, d1,2 → dd, d1,3 → dd, d1,4 → dd, d1,5 → dd,
d2,1 → dd, d2,2 → dd, d2,3 → dd, d2,4 → dd, d2,5 → dd,
d3,1 → dd, d3,2 → dd, d3,3 → dd, d3,4 → dd, d3,5 → dd,
d4,1 → dd, d4,2 → dd, d4,3 → dd, d4,4 → dd, d4,5 → dd,
d5,1 → dd, d5,2 → dd, d5,3 → dd, d5,4 → dd, d5,5 → dd,

Ω1,1 → OM Net[[1]][[1]], Ω1,2 → OM Net[[1]][[2]],
Ω1,3 → OM Net[[1]][[3]], Ω1,4 → OM Net[[1]][[4]], Ω1,5 → OM Net[[1]][[5]],
Ω2,1 → OM Net[[2]][[1]], Ω2,2 → OM Net[[2]][[2]], Ω2,3 → OM Net[[2]][[3]],
Ω2,4 → OM Net[[2]][[4]], Ω2,5 → OM Net[[2]][[5]],
Ω3,1 → OM Net[[3]][[1]], Ω3,2 → OM Net[[3]][[2]], Ω3,3 → OM Net[[3]][[3]],
Ω3,4 → OM Net[[3]][[4]], Ω3,5 → OM Net[[3]][[5]],
Ω4,1 → OM Net[[4]][[1]], Ω4,2 → OM Net[[4]][[2]], Ω4,3 → OM Net[[4]][[3]],
Ω4,4 → OM Net[[4]][[4]], Ω4,5 → OM Net[[4]][[5]],
Ω5,1 → OM Net[[5]][[1]], Ω5,2 → OM Net[[5]][[2]], Ω5,3 → OM Net[[5]][[3]],
Ω5,4 → OM Net[[5]][[4]], Ω5,5 → OM Net[[5]][[5]],
nuK → nu,
denK → den


};


B10 = 1500;
B20 = 1500;
B30 = 1500;
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
```

```
M30 = 10;
M40 = 10;
M50 = 10;


sol =
 NDSolve[
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,

    M1'[t] == dM1,
    M2'[t] == dM2,
    M3'[t] == dM3,
    M4'[t] == dM4,
    M5'[t] == dM5,


    B1[0] == B10,
    B2[0] == B20,
    B3[0] == B30,
    B4[0] == B40,
    B5[0] == B50,
    M1[0] == M10,
    M2[0] == M20,
    M3[0] == M30,
    M4[0] == M40,
    M5[0] == M50

   } /. par,
  {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
  {t, 0, tmax}];

{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
   M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par;

Min[{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax]} /. sol /. par]

)
```

The function “robustnessNewSaito” uses the previous function “fNewSaito” and calculates the Robustness. The function “robustnessNewSaito” simply receives a network as an argument.

In[7095]:=
```
robustnessNewSaitoBth[NetTop_] := (
  n1 = 1;
  n2 = 5000;
  mid = (n1 + n2) / 2;

  While[(n1 ≠ mid && n2 ≠ mid),
   (If[fNewSaitoBth[NetTop, mid] < 1, n2 = mid, n1 = mid];
    mid = Floor[N[(n1 + n2) / 2]];); {n1, n2, mid}]; mid

 )
```

As an example let’s take the following Network

In[7096]:=
```
NetK = {
    {0, 1, 0, 1, 0},
    {1, 0, 1, 1, 0},
    {1, 0, 1, 0, 1},
    {0, 1, 0, 1, 0},
    {0, 0, 0, 0, 1}
  };
```

Using the function fNewSaito we can calculate the smallest value of a bacterial population in the community for a given disturbance vale. For example, let’s take Disturbance value 1 and 500:

In[7097]:=
```
fNewSaitoBth[NetK, 0]
```

Out[7097]= 6661.43

In[7098]:=
```
fNewSaitoBth[NetK, 500]
```

Out[7098]= $1.36438 \times 10^{-63}$

Using the function fNewSaito we can calculate Robustness of the Network:

In[7099]:=
```
robustnessNewSaitoBth[NetK]
```

Out[7099]= 240

We can calculate the (Relative) Entropy and the Assortativity:

In[7100]:=
```
RelatEntrop5[NetK]
```
Out[7100]=
```
0.960956
```

In[7101]:=
```
assortativity[NetK]
```
Out[7101]=
```
-0.113228
```

We can calculate the robustness of the previously generated random networks with different number of auxotrophies:

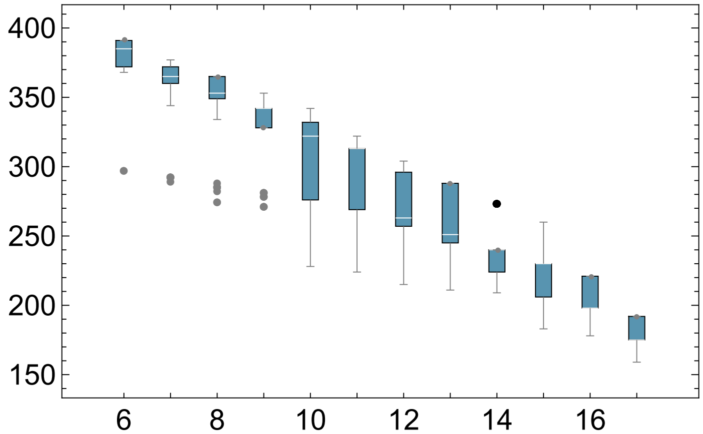
In[7102]:=
```
AuxoComm6Bth = Parallelize[robustnessNewSaitoBth /@ hk6];
AuxoComm7Bth = Parallelize[robustnessNewSaitoBth /@ hk7];
AuxoComm8Bth = Parallelize[robustnessNewSaitoBth /@ hk8];
AuxoComm9Bth = Parallelize[robustnessNewSaitoBth /@ hk9];
AuxoComm10Bth = Parallelize[robustnessNewSaitoBth /@ hk10];
AuxoComm11Bth = Parallelize[robustnessNewSaitoBth /@ hk11];
AuxoComm12Bth = Parallelize[robustnessNewSaitoBth /@ hk12];
AuxoComm13Bth = Parallelize[robustnessNewSaitoBth /@ hk13];
AuxoComm14Bth = Parallelize[robustnessNewSaitoBth /@ hk14];
AuxoComm15Bth = Parallelize[robustnessNewSaitoBth /@ hk15];
AuxoComm16Bth = Parallelize[robustnessNewSaitoBth /@ hk16];
AuxoComm17Bth = Parallelize[robustnessNewSaitoBth /@ hk17];
```

In[7114]:=
```
LikBth = {AuxoComm6Bth, AuxoComm7Bth, AuxoComm8Bth, AuxoComm9Bth,
   AuxoComm10Bth, AuxoComm11Bth, AuxoComm12Bth, AuxoComm13Bth,
   AuxoComm14Bth, AuxoComm15Bth, AuxoComm16Bth, AuxoComm17Bth};
```

In[∘]:=
```
coco = RGBColor[0.34509803921568627, 0.5803921568627451, 0.6901960784313725]
```
Out[∘]=

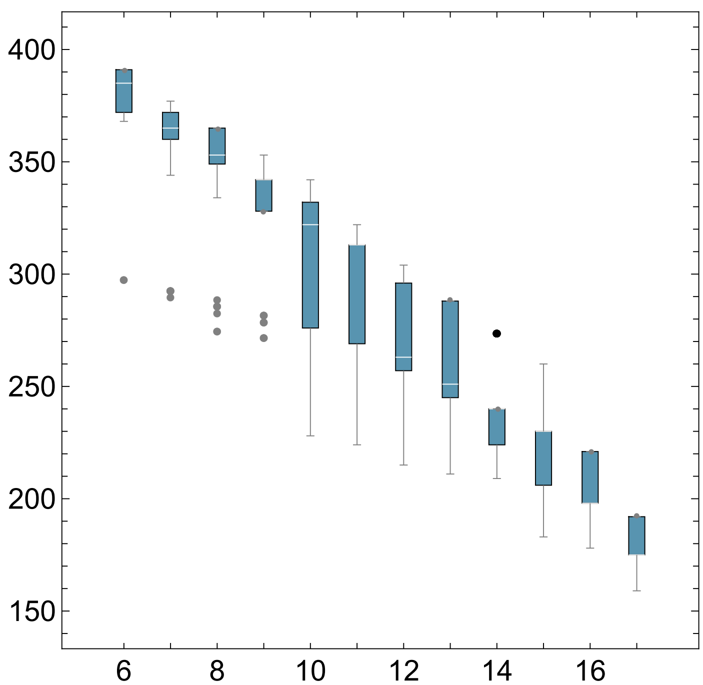
In[7115]:=
```
BoxWhiskerChart[LikBth, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{coco}}, Frame → True,
 ChartLabels → {"6", "", "8", "", "10", "", "12", "", "14", "", "16", ""},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15]]
```

Out[7115]=




In[7116]:=
```
BoxWhiskerChart[LikBth, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{coco}}, Frame → True,
 ChartLabels → {"6", "", "8", "", "10", "", "12", "", "14", "", "16", ""},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15], AspectRatio → 1]
```

Out[7116]=



In[7117]:=
```
AuxoComm7Bth
```

Out[7117]=
```
{360, 360, 377, 360, 360, 360, 290, 377, 293, 293, 377, 372, 360, 356, 372, 360, 377, 377,
 372, 360, 356, 372, 372, 356, 377, 360, 372, 344, 356, 344, 360, 372, 372, 293, 377,
 377, 377, 372, 360, 372, 377, 372, 356, 372, 372, 377, 356, 377, 372, 372, 372, 377,
 360, 377, 360, 377, 372, 360, 372, 356, 360, 365, 360, 360, 377, 360, 372, 377,
 377, 377, 360, 377, 360, 377, 356, 360, 365, 372, 360, 365, 360, 360, 293, 372,
 377, 360, 360, 360, 365, 372, 372, 344, 377, 344, 360, 377, 372, 293, 360, 360}
```

We can study the correlation between Relative entropy and assortativity with Robustness for Networks with 7 auxotrophies.

In[ ]:=
```
Entropy7 = RelatEntrop5 /@ hk7;
```

In[ ]:=
```
Assort7 = assortativity /@ hk7;
```

In[7118]:=
```
RobustNewSaito7bBth = AuxoComm7Bth;
```

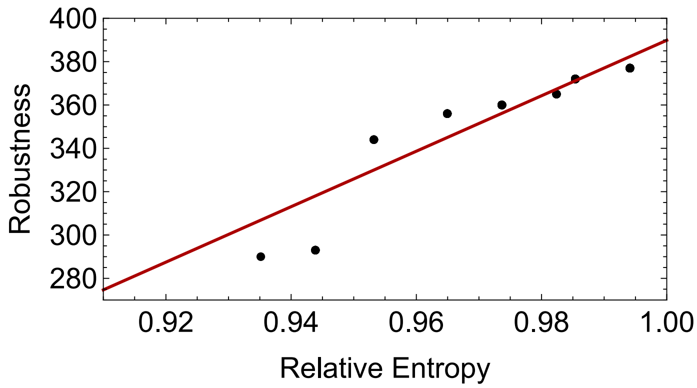
In[7119]:=
```
Length[Entropy7]
Length[Assort7]
Length[RobustNewSaito7bBth]
```

Out[7119]= 100

Out[7120]= 100

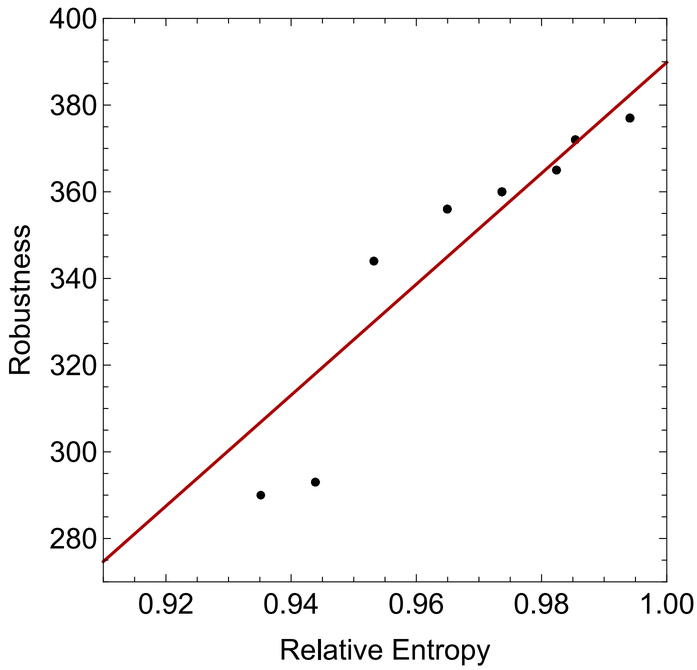
Out[7121]= 100

In[7122]:=
```
{Min[Entropy7], Max[Entropy7]}
{Min[Assort7], Max[Assort7]}
```
Out[7122]=
```
{0.935154, 0.994118}
```
Out[7123]=
```
{-0.416667, 0.25}
```
In[∘]:=
```
Position[Entropy7, Min[Entropy7]]
```
Out[∘]=
```
{{7}}
```
In[7124]:=
```
RobustNewSaito7bBth[[#]] & /@ {1, 2, 24}
```
Out[7124]=
```
{360, 360, 356}
```
In[7125]:=
```
{Min[RobustNewSaito7bBth], {Max[RobustNewSaito7bBth]}}
```
Out[7125]=
```
{290, {377}}
```
In[7126]:=
```
linerobustnessNewSaito25Bth =
  Fit[Partition[Riffle[Entropy7, RobustNewSaito7bBth], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Entropy7, RobustNewSaito7bBth], {2}],
  Frame → True, FrameLabel → {"Relative Entropy", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{0.91, 1}, {270, 400}},
  AspectRatio → 0.5], Plot[linerobustnessNewSaito25Bth,
  {x, 0.91, 1}, AspectRatio → 0.5, PlotStyle → Darker[Red]]]
```
Out[7127]=



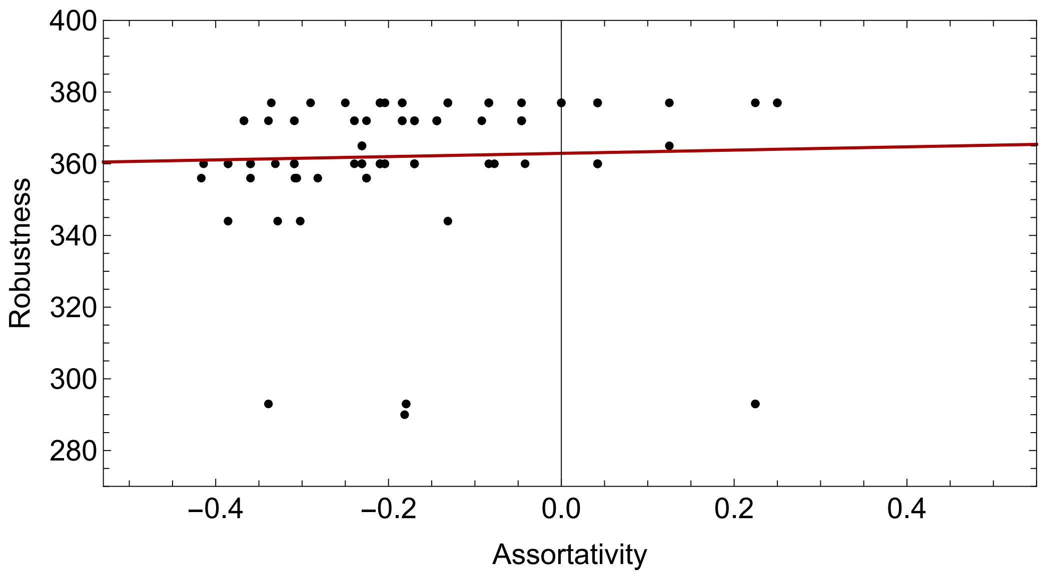
In[7128]:=
```
linerobustnessNewSaito25Bth =
  Fit[Partition[Riffle[Entropy7, RobustNewSaito7bBth], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Entropy7, RobustNewSaito7bBth], {2}],
  Frame → True, FrameLabel → {"Relative Entropy", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{0.91, 1}, {270, 400}},
  AspectRatio → 1], Plot[linerobustnessNewSaito25Bth,
  {x, 0.91, 1}, AspectRatio → 1, PlotStyle → Darker[Red]]]
```

Out[7129]=



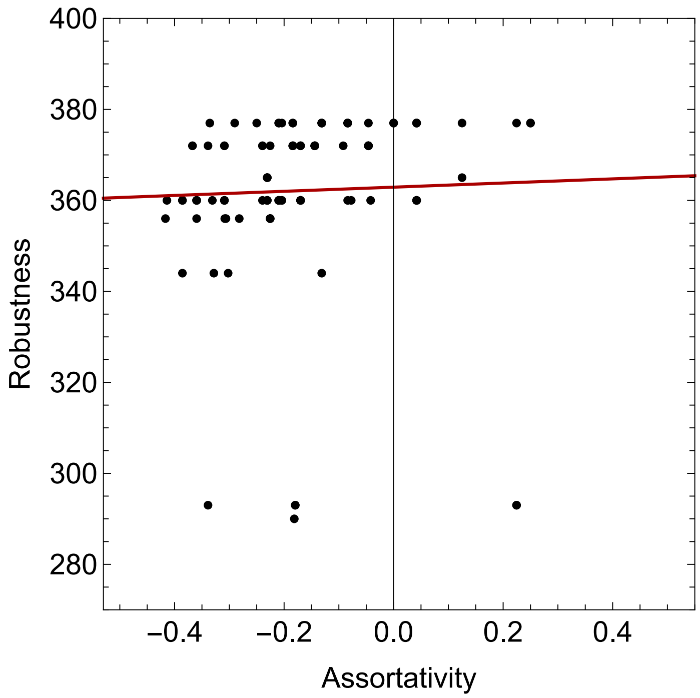
In[7130]:=
```
lineAssoRobrobustnessNewSaito25Bth =
  Fit[Partition[Riffle[Assort7, RobustNewSaito7bBth], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Assort7, RobustNewSaito7bBth], {2}],
  Frame → True, FrameLabel → {"Assortativity", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{-0.53, 0.55}, {270, 400}},
  AspectRatio → 0.5], Plot[lineAssoRobrobustnessNewSaito25Bth,
  {x, -0.53, 0.55}, AspectRatio → 0.5, PlotStyle → Darker[Red]]]
```

Out[7131]=



In[7132]:=
```
lineAssoRobrobustnessNewSaito25Bth =
  Fit[Partition[Riffle[Assort7, RobustNewSaito7bBth], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Assort7, RobustNewSaito7bBth], {2}],
  Frame → True, FrameLabel → {"Assortativity", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{-0.53, 0.55}, {270, 400}},
  AspectRatio → 1], Plot[lineAssoRobrobustnessNewSaito25Bth,
  {x, -0.53, 0.55}, AspectRatio → 1, PlotStyle → Darker[Red]]]
```

Out[7133]=




In[7134]:=
```
SpearmanRankTest[Entropy7, RobustNewSaito7bBth, "TestDataTable"]
```

Out[7134]=

| | Statistic | P-Value |
|---|---|---|
| Spearman Rank | 1. | 0. |

In[7135]:=
```
SpearmanRankTest[Assort7, RobustNewSaito7bBth, "TestDataTable"]
```

Out[7135]=

| | Statistic | P-Value |
|---|---|---|
| Spearman Rank | 0.351095 | 0.000341587 |

## Solving the system of ODE with Overproduction

In[7136]:=
```
fNewSaitoOVxBth[Net_, Dh_, coop_] := (
```

$$dB_1 = B_1[t]\left(-B_1[t]\,\kappa_1 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4} + c_{1,5} + Dh\right) B_1[t];$$

$$dB_2 = B_2[t]\left(-B_2[t]\,\kappa_2 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4} + c_{2,5} + Dh\right) B_2[t];$$

$$dB_3 = B_3[t]\left(-B_3[t]\,\kappa_3 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4} + c_{3,5} + Dh\right) B_3[t];$$

$$dB_4 = B_4[t]\left(-B_4[t]\,\kappa_4 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4} + c_{4,5} + Dh\right) B_4[t];$$

$$dB_5 = B_5[t]\left(-B_5[t]\,\kappa_5 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{5,1} + c_{5,2} + c_{5,3} + c_{5,4} + c_{5,5} + Dh\right) B_5[t];$$

$$dM_1 = -M_1[t]\left(Dh + q_1\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{1,1} - B_2[t]\,d_{1,2} - B_3[t]\,d_{1,3} - B_4[t]\,d_{1,4} - B_5[t]\,d_{1,5}\right) + B_1[t]\,\Omega_{1,1} + B_2[t]\,\Omega_{1,2} + B_3[t]\,\Omega_{1,3} + B_4[t]\,\Omega_{1,4} + B_5[t]\,\Omega_{1,5};$$

$$dM_2 = -M_2[t]\left(Dh + q_2\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{2,1} - B_2[t]\,d_{2,2} - B_3[t]\,d_{2,3} - B_4[t]\,d_{2,4} - B_5[t]\,d_{2,5}\right) + B_1[t]\,\Omega_{2,1} + B_2[t]\,\Omega_{2,2} + B_3[t]\,\Omega_{2,3} + B_4[t]\,\Omega_{2,4} + B_5[t]\,\Omega_{2,5};$$

$$dM_3 = -M_3[t]\left(Dh + q_3\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{3,1} - B_2[t]\,d_{3,2} - B_3[t]\,d_{3,3} - B_4[t]\,d_{3,4} - B_5[t]\,d_{3,5}\right) + B_1[t]\,\Omega_{3,1} + B_2[t]\,\Omega_{3,2} + B_3[t]\,\Omega_{3,3} + B_4[t]\,\Omega_{3,4} + B_5[t]\,\Omega_{3,5};$$

$$dM_4 = -M_4[t]\left(Dh + q_4\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{4,1} - B_2[t]\,d_{4,2} - B_3[t]\,d_{4,3} - B_4[t]\,d_{4,4} - B_5[t]\,d_{4,5}\right) + B_1[t]\,\Omega_{4,1} + B_2[t]\,\Omega_{4,2} + B_3[t]\,\Omega_{4,3} + B_4[t]\,\Omega_{4,4} + B_5[t]\,\Omega_{4,5};$$

```
dM5 = -M5[t] (Dh + q5) + (nuK * M1[t]/(denK + M1[t]) * M2[t]/(denK + M2[t]) * M3[t]/(denK + M3[t]) *
      M4[t]/(denK + M4[t]) * M5[t]/(denK + M5[t])) (-B1[t] d5,1 - B2[t] d5,2 - B3[t] d5,3 - B4[t] d5,4 -
      B5[t] d5,5) + B1[t] Ω5,1 + B2[t] Ω5,2 + B3[t] Ω5,3 + B4[t] Ω5,4 + B5[t] Ω5,5;

KK = 0.2;
cc = 0.05;
qq = 0.3;
dd = 0.00015;
OM = 1;
nu = 1500;
den = 2;


op = coop; (*Number of links with overExpression*)
posNe = Position[Net, 1];
(*Positions in the matrix where there are links (=1)*)
RaN = RandomSample[posNe, op];
(*Random sample of op links that will be overproduced*)

costincr = 1.3; (*Term multiplying the cost link*)
overprodincr = 1.15;
(*Term multiplying the overproduction link*)


NewNetCost = Net cc;
Table[NewNetCost[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] =
  NewNetCost[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] * costincr, {i, Length[RaN]}];

NewNetOvProd = Net OM;
Table[NewNetOvProd[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] =
  NewNetOvProd[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] * overprodincr, {i,
  Length[RaN]}];


tmax = 1000;
par = {
  κ1 → KK, κ2 → KK, κ3 → KK, κ4 → KK, κ5 → KK,

  c1,1 → NewNetCost[[1]][[1]],
  c1,2 → NewNetCost[[1]][[2]], c1,3 → NewNetCost[[1]][[3]],
  c1,4 → NewNetCost[[1]][[4]], c1,5 → NewNetCost[[1]][[5]],
  c2,1 → NewNetCost[[2]][[1]], c2,2 → NewNetCost[[2]][[2]],
```

$c_{2,3} \to$ NewNetCost[[2]][[3]], $c_{2,4} \to$ NewNetCost[[2]][[4]],
$c_{2,5} \to$ NewNetCost[[2]][[5]],
$c_{3,1} \to$ NewNetCost[[3]][[1]], $c_{3,2} \to$ NewNetCost[[3]][[2]],
$c_{3,3} \to$ NewNetCost[[3]][[3]], $c_{3,4} \to$ NewNetCost[[3]][[4]],
$c_{3,5} \to$ NewNetCost[[3]][[5]],
$c_{4,1} \to$ NewNetCost[[4]][[1]], $c_{4,2} \to$ NewNetCost[[4]][[2]],
$c_{4,3} \to$ NewNetCost[[4]][[3]], $c_{4,4} \to$ NewNetCost[[4]][[4]],
$c_{4,5} \to$ NewNetCost[[4]][[5]],
$c_{5,1} \to$ NewNetCost[[5]][[1]], $c_{5,2} \to$ NewNetCost[[5]][[2]],
$c_{5,3} \to$ NewNetCost[[5]][[3]], $c_{5,4} \to$ NewNetCost[[5]][[4]],
$c_{5,5} \to$ NewNetCost[[5]][[5]],

$q_1 \to$ qq, $q_2 \to$ qq, $q_3 \to$ qq, $q_4 \to$ qq, $q_5 \to$ qq,

$d_{1,1} \to$ dd, $d_{1,2} \to$ dd, $d_{1,3} \to$ dd, $d_{1,4} \to$ dd, $d_{1,5} \to$ dd,
$d_{2,1} \to$ dd, $d_{2,2} \to$ dd, $d_{2,3} \to$ dd, $d_{2,4} \to$ dd, $d_{2,5} \to$ dd,
$d_{3,1} \to$ dd, $d_{3,2} \to$ dd, $d_{3,3} \to$ dd, $d_{3,4} \to$ dd, $d_{3,5} \to$ dd,
$d_{4,1} \to$ dd, $d_{4,2} \to$ dd, $d_{4,3} \to$ dd, $d_{4,4} \to$ dd, $d_{4,5} \to$ dd,
$d_{5,1} \to$ dd, $d_{5,2} \to$ dd, $d_{5,3} \to$ dd, $d_{5,4} \to$ dd, $d_{5,5} \to$ dd,

$\Omega_{1,1} \to$ NewNetOvProd[[1]][[1]],
$\Omega_{1,2} \to$ NewNetOvProd[[1]][[2]], $\Omega_{1,3} \to$ NewNetOvProd[[1]][[3]],
$\Omega_{1,4} \to$ NewNetOvProd[[1]][[4]], $\Omega_{1,5} \to$ NewNetOvProd[[1]][[5]],
$\Omega_{2,1} \to$ NewNetOvProd[[2]][[1]], $\Omega_{2,2} \to$ NewNetOvProd[[2]][[2]],
$\Omega_{2,3} \to$ NewNetOvProd[[2]][[3]], $\Omega_{2,4} \to$ NewNetOvProd[[2]][[4]],
$\Omega_{2,5} \to$ NewNetOvProd[[2]][[5]],
$\Omega_{3,1} \to$ NewNetOvProd[[3]][[1]], $\Omega_{3,2} \to$ NewNetOvProd[[3]][[2]],
$\Omega_{3,3} \to$ NewNetOvProd[[3]][[3]], $\Omega_{3,4} \to$ NewNetOvProd[[3]][[4]],
$\Omega_{3,5} \to$ NewNetOvProd[[3]][[5]],
$\Omega_{4,1} \to$ NewNetOvProd[[4]][[1]], $\Omega_{4,2} \to$ NewNetOvProd[[4]][[2]],
$\Omega_{4,3} \to$ NewNetOvProd[[4]][[3]], $\Omega_{4,4} \to$ NewNetOvProd[[4]][[4]],
$\Omega_{4,5} \to$ NewNetOvProd[[4]][[5]],
$\Omega_{5,1} \to$ NewNetOvProd[[5]][[1]], $\Omega_{5,2} \to$ NewNetOvProd[[5]][[2]],
$\Omega_{5,3} \to$ NewNetOvProd[[5]][[3]], $\Omega_{5,4} \to$ NewNetOvProd[[5]][[4]],
$\Omega_{5,5} \to$ NewNetOvProd[[5]][[5]],
nuK → nu,
denK → den

};

```
B10 = 1500;
B20 = 1500;
```

```
B30 = 1500;
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
M30 = 10;
M40 = 10;
M50 = 10;


sol =
 NDSolve[
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,

    M1'[t] == dM1,
    M2'[t] == dM2,
    M3'[t] == dM3,
    M4'[t] == dM4,
    M5'[t] == dM5,


    B1[0] == B10,
    B2[0] == B20,
    B3[0] == B30,
    B4[0] == B40,
    B5[0] == B50,
    M1[0] == M10,
    M2[0] == M20,
    M3[0] == M30,
    M4[0] == M40,
    M5[0] == M50

   } /. par,
  {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
  {t, 0, tmax}];

{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
   M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par;
```

```
  Min[{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax]} /. sol /. par]

 )
```

In[7137]:=
```
robustnessNewSaitoOVxBth[NetTop_, coop_] := (
  n1 = 1;
  n2 = 5000;
  mid = (n1 + n2) / 2;

  While[(n1 ≠ mid && n2 ≠ mid),
   (If[fNewSaitoOVxBth[NetTop, mid, coop] < 1, n2 = mid, n1 = mid];
    mid = Floor[N[(n1 + n2) / 2]];); {n1, n2, mid}]; mid

 )
```

In[7138]:=
```
NetK = {
   {0, 1, 0, 1, 0},
   {1, 0, 1, 1, 0},
   {1, 0, 1, 0, 1},
   {0, 1, 0, 1, 0},
   {0, 0, 0, 0, 1}
  };
```

In[7139]:=
```
fNewSaitoBth[NetK, 0]
```

Out[7139]= 6661.43

In[7140]:=
```
fNewSaitoOVxBth[NetK, 0, 10]
```

Out[7140]= 7476.38

In[7141]:=
```
robustnessNewSaitoBth[NetK]
```

Out[7141]= 240

In[7142]:=
```
robustnessNewSaitoOVxBth[NetK, 10]
```
Out[7142]=
```
269
```

In[7143]:=
```
AuxoComm8Bth
```
Out[7143]=
```
{349, 338, 349, 365, 334, 338, 286, 365, 365, 349, 353, 353, 334, 353, 360, 349, 349, 338,
 353, 289, 349, 349, 353, 349, 349, 365, 353, 286, 365, 353, 353, 286, 353, 365, 365,
 365, 349, 360, 365, 365, 349, 365, 360, 365, 365, 353, 349, 349, 365, 349, 349, 353,
 349, 275, 365, 353, 283, 365, 365, 353, 349, 349, 365, 365, 353, 349, 349, 365,
 365, 360, 353, 349, 365, 365, 365, 365, 353, 338, 349, 365, 338, 365, 353, 349,
 349, 365, 360, 349, 349, 349, 275, 365, 365, 360, 365, 353, 349, 349, 353, 365}
```

In[7144]:=
```
coop5to15Bth = {Table[robustnessNewSaitoOVxBth[#, 5], {20}],
    Table[robustnessNewSaitoOVxBth[#, 10], {20}],
    Table[robustnessNewSaitoOVxBth[#, 15], {20}]} &;
```

In[7145]:=
```
wf8Bth = Parallelize[coop5to15Bth /@ hk8];
```

In[7146]:=
```
wf8NormalizedBth = N[wf8Bth[[#]] / AuxoComm8Bth[[#]]] & /@ Range[100]
```

In[7147]:=
```
wf8NormalizedWith5CoopBth = wf8NormalizedBth[[#]][[1]] & /@ Range[100]
```

In[7148]:=
```
wf8NormalizedWith10CoopBth = wf8NormalizedBth[[#]][[2]] & /@ Range[100]
```

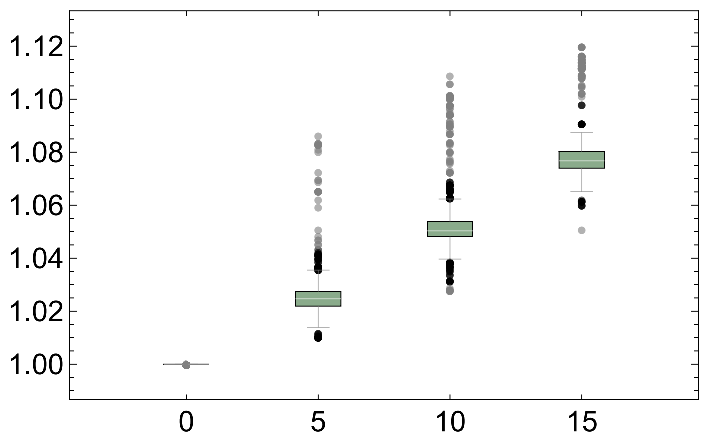
In[7149]:=
```
wf8NormalizedWith15CoopBth = wf8NormalizedBth[[#]][[3]] & /@ Range[100]
```

In[7150]:=
```
allcoopWith8AuxoBth = {Flatten[wf8NormalizedWith5CoopBth],
  Flatten[wf8NormalizedWith10CoopBth], Flatten[wf8NormalizedWith15CoopBth]}
```

In[7151]:=
```
allcoopWith8AuxoPlusAuxoBth =
 Join[{ConstantArray[1, {2000}]}, allcoopWith8AuxoBth]
```

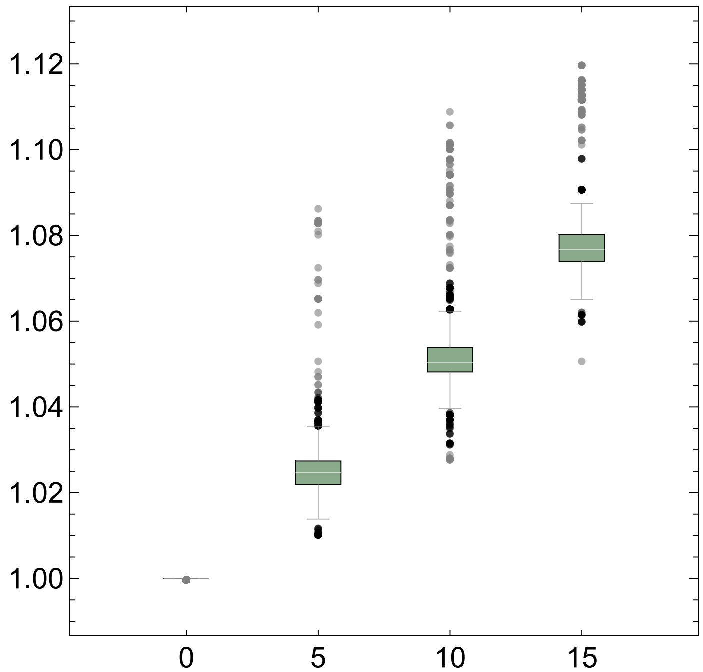
In[7152]:=
```
BoxWhiskerChart[allcoopWith8AuxoPlusAuxoBth, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{gree1}},
 Frame → True, ChartLabels → {"0", "5", "10", "15"},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15]]
```

Out[7152]=



In[7153]:=
```
BoxWhiskerChart[allcoopWith8AuxoPlusAuxoBth, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{gree1}},
 Frame → True, ChartLabels → {"0", "5", "10", "15"}, BarSpacing → 1.9,
 FrameStyle → Directive[Black, FontSize → 15], AspectRatio → 1]
```

Out[7153]=




In[7154]:=
```
allcoopWith8AuxoPlusAuxoBth // Length
```

Out[7154]= 4

In[7155]:=
```
SignedRankTest[allcoopWith8AuxoPlusAuxoBth[[2]], 1]
SignedRankTest[allcoopWith8AuxoPlusAuxoBth[[3]], 1]
SignedRankTest[allcoopWith8AuxoPlusAuxoBth[[4]], 1]
```

Out[7155]= 0.

Out[7156]= 0.

Out[7157]= 0.

## Solving the system of ODE Random parametrization

In[7158]:=
```
Knum = 0.2;
ccrnum = 0.05;
qqrnum = 0.3;
ddrnum = 0.00015;
OMrnum = 1;
nurum = 1500;
den2rum = 2;

corrpar0 = 10^3;
corrpar1 = 10^4;
corrpar2 = 10^6;

KKr := RandomVariate[
    GammaDistribution[corrpar0 Sqrt[Knum], (1/corrpar0) Sqrt[Knum]], 1][[1]];
ccr := RandomVariate[GammaDistribution[corrpar1 Sqrt[ccrnum],
     (1/corrpar1) Sqrt[ccrnum]], 1][[1]];
qqr := RandomVariate[GammaDistribution[corrpar0 Sqrt[qqrnum],
     (1/corrpar0) Sqrt[qqrnum]], 1][[1]];
ddr := RandomVariate[GammaDistribution[corrpar1 Sqrt[ddrnum],
     (1/corrpar1) Sqrt[ddrnum]], 1][[1]];
OMr := RandomVariate[GammaDistribution[corrpar0 Sqrt[OMrnum],
     (1/corrpar0) Sqrt[OMrnum]], 1][[1]];
nur := (*nurum*) RandomVariate[GammaDistribution[
     corrpar2 Sqrt[nurum], (1/corrpar2) Sqrt[nurum]], 1][[1]] ;
denr2 := (*den2rum*) RandomVariate[GammaDistribution[
     corrpar2 Sqrt[den2rum], (1/corrpar2) Sqrt[den2rum]], 1][[1]] ;


parR = Join[Table[KKr, {5}], Table[ccr, {25}],
  Table[qqr, {5}], Table[ddr, {25}], Table[OMr, {25}], {nur}, {denr2}]
```

Out[7175]= {0.203888, 0.193114, 0.191583, 0.205432, 0.194855, 0.0515009, 0.0501345, 0.0480727, 0.0507158, 0.0499914, 0.0517093, 0.0500052, 0.0515087, 0.0503726, 0.0500974, 0.0507375, 0.0488789, 0.0509691, 0.048728, 0.0498409, 0.0499885, 0.0501604, 0.0505392, 0.0497311, 0.0518306, 0.0476381, 0.0480647, 0.0478169, 0.0501175, 0.0503996, 0.305375, 0.271058, 0.311014, 0.309073, 0.279432, 0.000160969, 0.000182668, 0.000163317, 0.000135746, 0.000146965, 0.000134492, 0.00015405, 0.000183033, 0.00013727, 0.000150244, 0.000165478, 0.0001429, 0.000137086, 0.000159886, 0.000138627, 0.000170925, 0.000148015, 0.000144537, 0.000166234, 0.000154126, 0.000137185, 0.000158521, 0.000166449, 0.000142216, 0.000159263, 1.01037, 0.929316, 1.00016, 0.990019, 1.02044, 0.984676, 0.943614, 0.996293, 1.05574, 1.01548, 1.03203, 0.962227, 0.971079, 1.01179, 0.992651, 1.06495, 0.982743, 0.961245, 1.03151, 0.993706, 0.961125, 1.02171, 0.990215, 1.0663, 1.02996, 1500.35, 2.00021}

In[7176]:= **parR = %**

Out[7176]= {0.203888, 0.193114, 0.191583, 0.205432, 0.194855, 0.0515009, 0.0501345, 0.0480727, 0.0507158, 0.0499914, 0.0517093, 0.0500052, 0.0515087, 0.0503726, 0.0500974, 0.0507375, 0.0488789, 0.0509691, 0.048728, 0.0498409, 0.0499885, 0.0501604, 0.0505392, 0.0497311, 0.0518306, 0.0476381, 0.0480647, 0.0478169, 0.0501175, 0.0503996, 0.305375, 0.271058, 0.311014, 0.309073, 0.279432, 0.000160969, 0.000182668, 0.000163317, 0.000135746, 0.000146965, 0.000134492, 0.00015405, 0.000183033, 0.00013727, 0.000150244, 0.000165478, 0.0001429, 0.000137086, 0.000159886, 0.000138627, 0.000170925, 0.000148015, 0.000144537, 0.000166234, 0.000154126, 0.000137185, 0.000158521, 0.000166449, 0.000142216, 0.000159263, 1.01037, 0.929316, 1.00016, 0.990019, 1.02044, 0.984676, 0.943614, 0.996293, 1.05574, 1.01548, 1.03203, 0.962227, 0.971079, 1.01179, 0.992651, 1.06495, 0.982743, 0.961245, 1.03151, 0.993706, 0.961125, 1.02171, 0.990215, 1.0663, 1.02996, 1500.35, 2.00021}

In[7177]:=

fNewSaitoRBth[Net_, Dh_] := (

$$dB_1 = B_1[t]\left(-B_1[t]\,\kappa_1 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4} + c_{1,5} + Dh\right) B_1[t];$$

$$dB_2 = B_2[t]\left(-B_2[t]\,\kappa_2 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4} + c_{2,5} + Dh\right) B_2[t];$$

$$dB_3 = B_3[t]\left(-B_3[t]\,\kappa_3 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4} + c_{3,5} + Dh\right) B_3[t];$$

$$dB_4 = B_4[t]\left(-B_4[t]\,\varkappa_4 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4} + c_{4,5} + Dh\right) B_4[t];$$

$$dB_5 = B_5[t]\left(-B_5[t]\,\varkappa_5 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{5,1} + c_{5,2} + c_{5,3} + c_{5,4} + c_{5,5} + Dh\right) B_5[t];$$

$$dM_1 = -M_1[t]\left(Dh + q_1\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{1,1} - B_2[t]\,d_{1,2} - B_3[t]\,d_{1,3} - B_4[t]\,d_{1,4} - B_5[t]\,d_{1,5}\right) + B_1[t]\,\Omega_{1,1} + B_2[t]\,\Omega_{1,2} + B_3[t]\,\Omega_{1,3} + B_4[t]\,\Omega_{1,4} + B_5[t]\,\Omega_{1,5};$$

$$dM_2 = -M_2[t]\left(Dh + q_2\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{2,1} - B_2[t]\,d_{2,2} - B_3[t]\,d_{2,3} - B_4[t]\,d_{2,4} - B_5[t]\,d_{2,5}\right) + B_1[t]\,\Omega_{2,1} + B_2[t]\,\Omega_{2,2} + B_3[t]\,\Omega_{2,3} + B_4[t]\,\Omega_{2,4} + B_5[t]\,\Omega_{2,5};$$

$$dM_3 = -M_3[t]\left(Dh + q_3\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{3,1} - B_2[t]\,d_{3,2} - B_3[t]\,d_{3,3} - B_4[t]\,d_{3,4} - B_5[t]\,d_{3,5}\right) + B_1[t]\,\Omega_{3,1} + B_2[t]\,\Omega_{3,2} + B_3[t]\,\Omega_{3,3} + B_4[t]\,\Omega_{3,4} + B_5[t]\,\Omega_{3,5};$$

$$dM_4 = -M_4[t]\left(Dh + q_4\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{4,1} - B_2[t]\,d_{4,2} - B_3[t]\,d_{4,3} - B_4[t]\,d_{4,4} - B_5[t]\,d_{4,5}\right) + B_1[t]\,\Omega_{4,1} + B_2[t]\,\Omega_{4,2} + B_3[t]\,\Omega_{4,3} + B_4[t]\,\Omega_{4,4} + B_5[t]\,\Omega_{4,5};$$

$$dM_5 = -M_5[t]\left(Dh + q_5\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{5,1} - B_2[t]\,d_{5,2} - B_3[t]\,d_{5,3} - B_4[t]\,d_{5,4} - B_5[t]\,d_{5,5}\right) + B_1[t]\,\Omega_{5,1} + B_2[t]\,\Omega_{5,2} + B_3[t]\,\Omega_{5,3} + B_4[t]\,\Omega_{5,4} + B_5[t]\,\Omega_{5,5};$$

```
tmax = 1000;
par = {
  κ1 → parR[[1]], κ2 → parR[[2]], κ3 → parR[[3]], κ4 → parR[[4]], κ5 → parR[[5]],

  c1,1 → parR[[6]] × Net[[1]][[1]],
  c1,2 → parR[[7]] × Net[[1]][[2]], c1,3 → parR[[8]] × Net[[1]][[3]],
  c1,4 → parR[[9]] × Net[[1]][[4]], c1,5 → parR[[10]] × Net[[1]][[5]],
  c2,1 → parR[[11]] × Net[[2]][[1]], c2,2 → parR[[12]] × Net[[2]][[2]],
```

```
c2,3 → parR[[13]] × Net[[2]][[3]], c2,4 → parR[[14]] × Net[[2]][[4]],
c2,5 → parR[[15]] × Net[[2]][[5]],
c3,1 → parR[[16]] × Net[[3]][[1]], c3,2 → parR[[17]] × Net[[3]][[2]],
c3,3 → parR[[18]] × Net[[3]][[3]], c3,4 → parR[[19]] × Net[[3]][[4]],
c3,5 → parR[[20]] × Net[[3]][[5]],
c4,1 → parR[[21]] × Net[[4]][[1]], c4,2 → parR[[22]] × Net[[4]][[2]],
c4,3 → parR[[23]] × Net[[4]][[3]], c4,4 → parR[[24]] × Net[[4]][[4]],
c4,5 → parR[[25]] × Net[[4]][[5]],
c5,1 → parR[[26]] × Net[[5]][[1]], c5,2 → parR[[27]] × Net[[5]][[2]],
c5,3 → parR[[28]] × Net[[5]][[3]], c5,4 → parR[[29]] × Net[[5]][[4]],
c5,5 → parR[[30]] × Net[[5]][[5]],

q1 → parR[[31]], q2 → parR[[32]],
q3 → parR[[33]], q4 → parR[[34]], q5 → parR[[35]],

d1,1 → parR[[36]], d1,2 → parR[[37]],
d1,3 → parR[[38]], d1,4 → parR[[39]], d1,5 → parR[[40]],
d2,1 → parR[[41]], d2,2 → parR[[42]], d2,3 → parR[[43]],
d2,4 → parR[[44]], d2,5 → parR[[45]],
d3,1 → parR[[46]], d3,2 → parR[[47]], d3,3 → parR[[48]],
d3,4 → parR[[49]], d3,5 → parR[[50]],
d4,1 → parR[[51]], d4,2 → parR[[52]], d4,3 → parR[[53]],
d4,4 → parR[[54]], d4,5 → parR[[55]],
d5,1 → parR[[56]], d5,2 → parR[[57]], d5,3 → parR[[58]],
d5,4 → parR[[59]], d5,5 → parR[[60]],

Ω1,1 → parR[[61]] × Net[[1]][[1]],
Ω1,2 → parR[[62]] × Net[[1]][[2]], Ω1,3 → parR[[63]] × Net[[1]][[3]],
Ω1,4 → parR[[64]] × Net[[1]][[4]], Ω1,5 → parR[[65]] × Net[[1]][[5]],
Ω2,1 → parR[[66]] × Net[[2]][[1]], Ω2,2 → parR[[67]] × Net[[2]][[2]],
Ω2,3 → parR[[68]] × Net[[2]][[3]], Ω2,4 → parR[[69]] × Net[[2]][[4]],
Ω2,5 → parR[[70]] × Net[[2]][[5]],
Ω3,1 → parR[[71]] × Net[[3]][[1]], Ω3,2 → parR[[72]] × Net[[3]][[2]],
Ω3,3 → parR[[73]] × Net[[3]][[3]], Ω3,4 → parR[[74]] × Net[[3]][[4]],
Ω3,5 → parR[[75]] × Net[[3]][[5]],
Ω4,1 → parR[[76]] × Net[[4]][[1]], Ω4,2 → parR[[77]] × Net[[4]][[2]],
Ω4,3 → parR[[78]] × Net[[4]][[3]], Ω4,4 → parR[[79]] × Net[[4]][[4]],
Ω4,5 → parR[[80]] × Net[[4]][[5]],
Ω5,1 → parR[[81]] × Net[[5]][[1]], Ω5,2 → parR[[82]] × Net[[5]][[2]],
Ω5,3 → parR[[83]] × Net[[5]][[3]], Ω5,4 → parR[[84]] × Net[[5]][[4]],
Ω5,5 → parR[[85]] × Net[[5]][[5]],
nuK → parR[[86]],
denK → parR[[87]]
```

```
 };



B10 = 1500;
B20 = 1500;
B30 = 1500;
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
M30 = 10;
M40 = 10;
M50 = 10;


sol =
 NDSolve[
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,

    M1'[t] == dM1,
    M2'[t] == dM2,
    M3'[t] == dM3,
    M4'[t] == dM4,
    M5'[t] == dM5,


    B1[0] == B10,
    B2[0] == B20,
    B3[0] == B30,
    B4[0] == B40,
    B5[0] == B50,
    M1[0] == M10,
    M2[0] == M20,
    M3[0] == M30,
    M4[0] == M40,
```

```
      M5[0] == M50

     } /. par,
    {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
    {t, 0, tmax}];

  {B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
     M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par;

  Min[{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax]} /. sol /. par]

 )
```

In[7178]:=
```
robustnessNewSaitoRBth[NetTop_] := (
  n1 = 1;
  n2 = 5000;
  mid = (n1 + n2) / 2;

  While[(n1 ≠ mid && n2 ≠ mid),
   (If[fNewSaitoRBth[NetTop, mid] < 1, n2 = mid, n1 = mid];
    mid = Floor[N[(n1 + n2) / 2]];); {n1, n2, mid}]; mid

 )
```

As an example let's take the following Network

In[7179]:=
```
NetK = {
   {0, 1, 0, 1, 0},
   {1, 0, 1, 1, 0},
   {1, 0, 1, 0, 1},
   {0, 1, 0, 1, 0},
   {0, 0, 0, 0, 1}
  };
```

Using the function fNewSaito we can calculate the smallest value of a bacterial population in the community for a given disturbance vale. For example, let's take Disturbance value 1 and 500:

In[7180]:= `fNewSaitoBth[NetK, 0]`

Out[7180]= `6661.43`

In[7181]:= `fNewSaitoBth[NetK, 500]`

Out[7181]= $1.36438\times10^{-63}$

In[7182]:= fNewSaitoRBth[NetK, 0]

Out[7182]= 6639.57

In[7183]:= fNewSaitoRBth[NetK, 500]

Out[7183]= $1.45496 \times 10^{-63}$

Using the function fNewSaito we can calculate Robustness of the Network:

In[7184]:= robustnessNewSaitoBth[NetK]

Out[7184]= 240

In[7185]:= robustnessNewSaitoRBth[NetK]

Out[7185]= 243

We can calculate the (Relative) Entropy and the Assortativity:

In[ ]:= RelatEntrop5[NetK]

Out[ ]= 0.960956

In[ ]:= assortativity[NetK]

Out[ ]= -0.113228

We can calculate the robustness of the previously generated random networks with different number of auxotrophies:

In[7186]:=
```
AuxoComm6RBth = Parallelize[robustnessNewSaitoRBth /@ hk6];
AuxoComm7RBth = Parallelize[robustnessNewSaitoRBth /@ hk7];
AuxoComm8RBth = Parallelize[robustnessNewSaitoRBth /@ hk8];
AuxoComm9RBth = Parallelize[robustnessNewSaitoRBth /@ hk9];
AuxoComm10RBth = Parallelize[robustnessNewSaitoRBth /@ hk10];
AuxoComm11RBth = Parallelize[robustnessNewSaitoRBth /@ hk11];
AuxoComm12RBth = Parallelize[robustnessNewSaitoRBth /@ hk12];
AuxoComm13RBth = Parallelize[robustnessNewSaitoRBth /@ hk13];
AuxoComm14RBth = Parallelize[robustnessNewSaitoRBth /@ hk14];
AuxoComm15RBth = Parallelize[robustnessNewSaitoRBth /@ hk15];
AuxoComm16RBth = Parallelize[robustnessNewSaitoRBth /@ hk16];
AuxoComm17RBth = Parallelize[robustnessNewSaitoRBth /@ hk17];
```

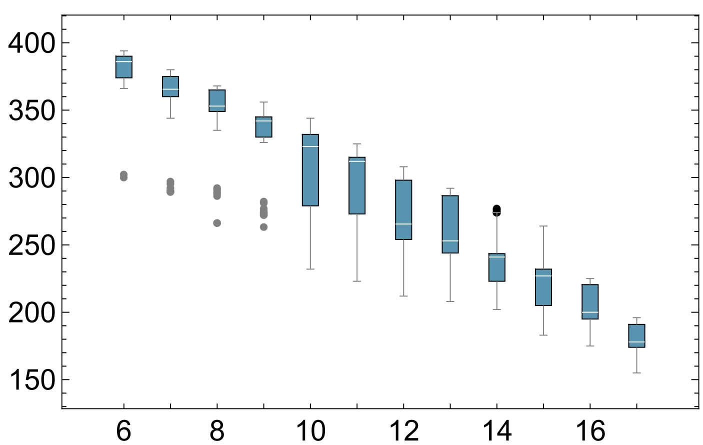
In[7198]:=
```
LikRBth = {AuxoComm6RBth, AuxoComm7RBth, AuxoComm8RBth, AuxoComm9RBth,
   AuxoComm10RBth, AuxoComm11RBth, AuxoComm12RBth, AuxoComm13RBth,
   AuxoComm14RBth, AuxoComm15RBth, AuxoComm16RBth, AuxoComm17RBth};
```

In[∘]:=
```
coco = RGBColor[0.34509803921568627, 0.5803921568627451, 0.6901960784313725]
```

Out[∘]=

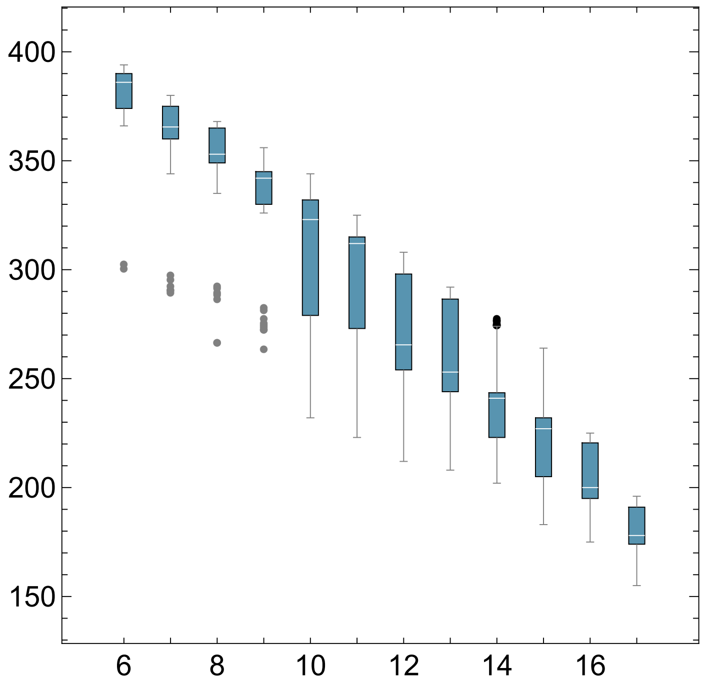
In[7199]:=
```
BoxWhiskerChart[LikRBth, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{coco}}, Frame → True,
 ChartLabels → {"6", "", "8", "", "10", "", "12", "", "14", "", "16", ""},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15]]
```

Out[7199]=



In[7200]:=
```
BoxWhiskerChart[LikRBth, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{coco}}, Frame → True,
 ChartLabels → {"6", "", "8", "", "10", "", "12", "", "14", "", "16", ""},
 BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15], AspectRatio → 1]
```

Out[7200]=



In[7201]:= AuxoComm7RBth

Out[7201]= {361, 361, 378, 358, 360, 363, 293, 379, 296, 290, 376, 373, 360, 356, 372, 359, 379, 378, 372, 361, 357, 373, 374, 358, 380, 361, 372, 345, 357, 345, 360, 372, 375, 291, 378, 378, 379, 374, 360, 372, 376, 374, 356, 373, 374, 379, 356, 379, 371, 374, 375, 376, 360, 379, 361, 378, 373, 363, 371, 357, 361, 366, 358, 358, 378, 363, 373, 377, 377, 378, 361, 379, 361, 378, 356, 362, 366, 372, 361, 364, 361, 358, 298, 374, 379, 361, 360, 362, 365, 373, 373, 344, 379, 344, 363, 378, 374, 291, 363, 363}

We can study the correlation between Relative entropy and assortativity with Robustness for Networks with 7 auxotrophies.

In[∘]:= Entropy7 = RelatEntrop5 /@ hk7;

In[∘]:= Assort7 = assortativity /@ hk7;

In[7202]:= RobustNewSaito7bRBth = AuxoComm7RBth;

In[7203]:= Length[Entropy7]
Length[Assort7]
Length[RobustNewSaito7bRBth]

Out[7203]= 100

Out[7204]= 100

Out[7205]= 100

In[∘]:= {Min[Entropy7], Max[Entropy7]}
{Min[Assort7], Max[Assort7]}

Out[∘]= {0.935154, 0.994118}

Out[∘]= {-0.416667, 0.25}

In[∘]:= Position[Entropy7, Min[Entropy7]]

Out[∘]= {{7}}

In[7206]:= RobustNewSaito7bRBth[[#]] & /@ {1, 2, 24}

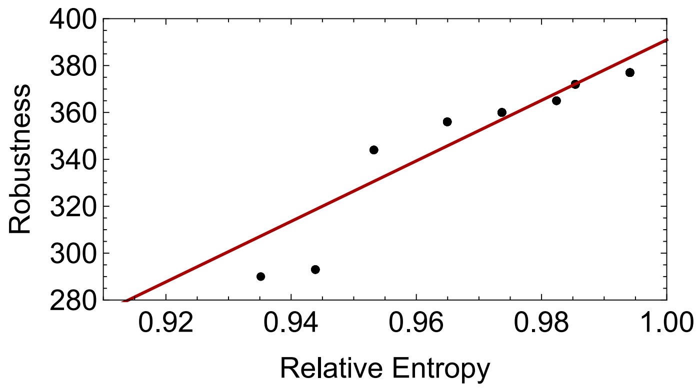
Out[7206]= {361, 361, 358}

In[7207]:= {Min[RobustNewSaito7bRBth], {Max[RobustNewSaito7bRBth]}}

Out[7207]= {290, {380}}

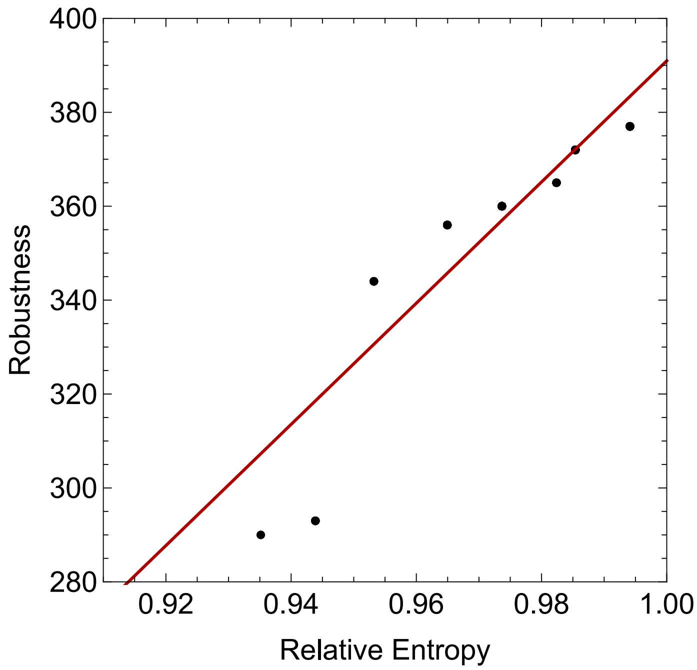
In[7208]:=
```
linerobustnessNewSaito25RBth =
  Fit[Partition[Riffle[Entropy7, RobustNewSaito7bRBth], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Entropy7, RobustNewSaito7bBth], {2}],
  Frame → True, FrameLabel → {"Relative Entropy", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{0.91, 1}, {280, 400}},
  AspectRatio → 0.5], Plot[linerobustnessNewSaito25RBth,
  {x, 0.91, 1}, AspectRatio → 0.5, PlotStyle → Darker[Red]]]
```

Out[7209]=



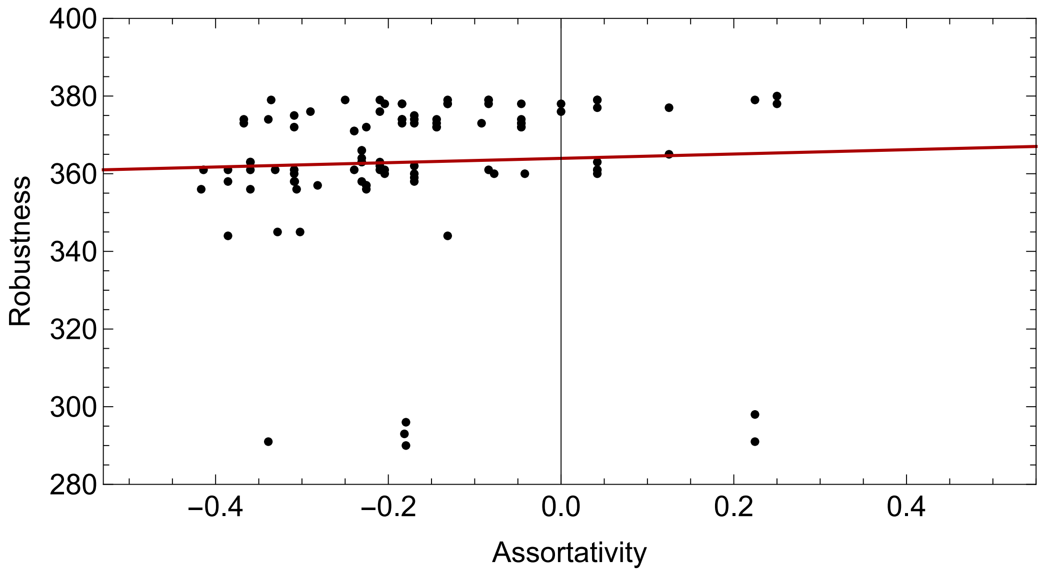
In[7210]:=
```
linerobustnessNewSaito25RBth =
  Fit[Partition[Riffle[Entropy7, RobustNewSaito7bRBth], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Entropy7, RobustNewSaito7bBth], {2}],
  Frame → True, FrameLabel → {"Relative Entropy", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{0.91, 1}, {280, 400}},
  AspectRatio → 1], Plot[linerobustnessNewSaito25RBth,
  {x, 0.91, 1}, AspectRatio → 1, PlotStyle → Darker[Red]]]
```

Out[7211]=



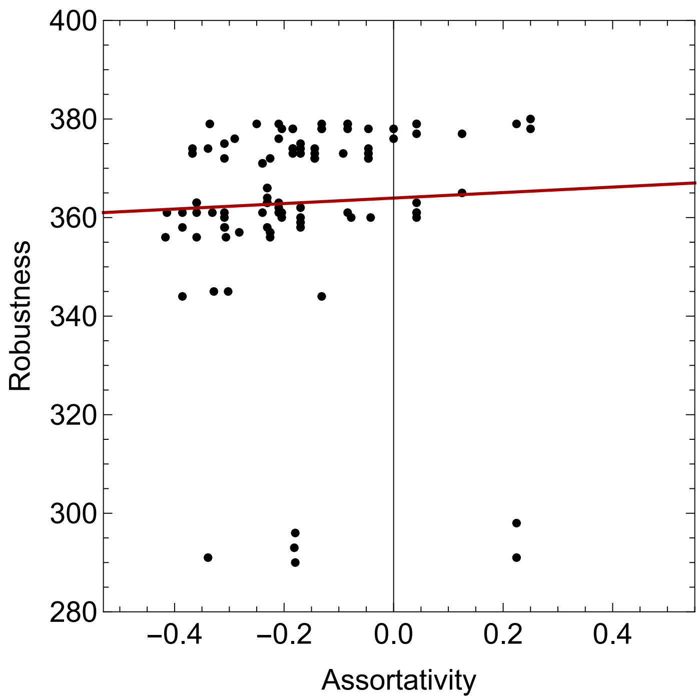
In[7212]:=
```
lineAssoRobrobustnessNewSaito25RBth =
  Fit[Partition[Riffle[Assort7, RobustNewSaito7bRBth], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Assort7, RobustNewSaito7bRBth], {2}],
  Frame → True, FrameLabel → {"Assortativity", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{-0.53, 0.55}, {280, 400}},
  AspectRatio → 0.5], Plot[lineAssoRobrobustnessNewSaito25RBth,
  {x, -0.53, 0.55}, AspectRatio → 0.5, PlotStyle → Darker[Red]]]
```

Out[7213]=



In[7214]:=
```
lineAssoRobrobustnessNewSaito25RBth =
  Fit[Partition[Riffle[Assort7, RobustNewSaito7bRBth], {2}], {1, x}, x];
Show[ListPlot[Partition[Riffle[Assort7, RobustNewSaito7bRBth], {2}],
  Frame → True, FrameLabel → {"Assortativity", "Robustness"},
  FrameStyle → Directive[Black, FontSize → 15],
  PlotStyle → {Black, PointSize[Medium]}, PlotRange → {{-0.53, 0.55}, {280, 400}},
  AspectRatio → 1], Plot[lineAssoRobrobustnessNewSaito25RBth,
  {x, -0.53, 0.55}, AspectRatio → 1, PlotStyle → Darker[Red]]]
```

Out[7215]=




In[7216]:=
```
SpearmanRankTest[Entropy7, RobustNewSaito7bRBth, "TestDataTable"]
```

Out[7216]=

| | Statistic | P-Value |
|---|---|---|
| Spearman Rank | 0.97348 | $1.37357 \times 10^{-64}$ |

In[7217]:=
```
SpearmanRankTest[Assort7, RobustNewSaito7bRBth, "TestDataTable"]
```

Out[7217]=

| | Statistic | P-Value |
|---|---|---|
| Spearman Rank | 0.334145 | 0.000679668 |

In[•]:= 
```
parR // Length
```
Out[•]= 87

## Solving the system of ODE with Overproduction Random parametrization

In[7218]:= 
```
parR = {0.2038876904998529`, 0.19311440359517038`, 0.19158332477378875`,
   0.20543226506020015`, 0.1948552219840135`, 0.05150093149128435`,
   0.050134479060103536`, 0.04807265259723119`, 0.05071575106783848`,
   0.049991430763200735`, 0.05170927915304682`, 0.050005155943046706`,
   0.05150867447321784`, 0.05037264233912337`, 0.05009740703744432`,
   0.050737497853994555`, 0.048878895154788646`, 0.05096914266764422`,
   0.048727953435582`, 0.04984093568809354`, 0.049988517455500536`,
   0.05016044159538885`, 0.05053916592754987 4`, 0.04973111152880456`,
   0.05183056318541718`, 0.04763808133198228`, 0.048064733856862475`,
   0.0478169233094099 2`, 0.05011750334421158`, 0.05039961955278385`,
   0.305375393984278 5`, 0.2710582126379885`, 0.31101396370475193`,
   0.3090725414680744`, 0.27943244181071314`, 0.00016096932420810 29`,
   0.00018266792567643594`, 0.00016331730093550 19`, 0.00013574573140377855`,
   0.0001469654716917689`, 0.00013449248301922057`, 0.00015405028691782593`,
   0.00018303252484807837`, 0.0001372699758115071`, 0.0001502444901937573`,
   0.00016547754538102883`, 0.00014290049530023166`, 0.0001370857780122561`,
   0.0001598858166830677`, 0.00013862660876693306`, 0.00017092504779673786`,
   0.00014801528402384555`, 0.00014453749655568817`, 0.0001662336243462559`,
   0.0001541259135694526`, 0.00013718484718252475`, 0.00015852116751359334`,
   0.00016644855119616011`, 0.00014221577702860555`, 0.00015926321273932763`,
   1.010374312723501`, 0.9293163977375384`, 1.0001635390517267`,
   0.9900194290340605`, 1.0204399616584585`, 0.9846755244546793`,
   0.9436139208805724`, 0.99629300990228`, 1.0557368155883273`,
   1.0154756283825577`, 1.0320266551504955`, 0.9622265582171438`,
   0.9710792277912424`, 1.011791349887933`, 0.992650872365134`,
   1.0649529649458906`, 0.9827427387732257`, 0.9612452525339857`,
   1.0315137658559501`, 0.9937056033603837`, 0.9611253711802232`,
   1.0217122527566802`, 0.9902147013241935`, 1.066300963531785`,
   1.0299620377061376`, 1500.3454071355402`, 2.0002144600509673`};
```

In[7219]:= 
```
fNewSaitoOVRBth[Net_, Dh_, coop_] := (
```

$$dB_1 = B_1[t]\left(-B_1[t]\,\varkappa_1 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{1,1} + c_{1,2} + c_{1,3} + c_{1,4} + c_{1,5} + Dh\right) B_1[t];$$

$$dB_2 = B_2[t]\left(-B_2[t]\,\kappa_2 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{2,1} + c_{2,2} + c_{2,3} + c_{2,4} + c_{2,5} + Dh\right) B_2[t];$$

$$dB_3 = B_3[t]\left(-B_3[t]\,\kappa_3 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{3,1} + c_{3,2} + c_{3,3} + c_{3,4} + c_{3,5} + Dh\right) B_3[t];$$

$$dB_4 = B_4[t]\left(-B_4[t]\,\kappa_4 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4} + c_{4,5} + Dh\right) B_4[t];$$

$$dB_5 = B_5[t]\left(-B_5[t]\,\kappa_5 + nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) - \left(c_{5,1} + c_{5,2} + c_{5,3} + c_{5,4} + c_{5,5} + Dh\right) B_5[t];$$

$$dM_1 = -M_1[t]\left(Dh + q_1\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{1,1} - B_2[t]\,d_{1,2} - B_3[t]\,d_{1,3} - B_4[t]\,d_{1,4} - B_5[t]\,d_{1,5}\right) + B_1[t]\,\Omega_{1,1} + B_2[t]\,\Omega_{1,2} + B_3[t]\,\Omega_{1,3} + B_4[t]\,\Omega_{1,4} + B_5[t]\,\Omega_{1,5};$$

$$dM_2 = -M_2[t]\left(Dh + q_2\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{2,1} - B_2[t]\,d_{2,2} - B_3[t]\,d_{2,3} - B_4[t]\,d_{2,4} - B_5[t]\,d_{2,5}\right) + B_1[t]\,\Omega_{2,1} + B_2[t]\,\Omega_{2,2} + B_3[t]\,\Omega_{2,3} + B_4[t]\,\Omega_{2,4} + B_5[t]\,\Omega_{2,5};$$

$$dM_3 = -M_3[t]\left(Dh + q_3\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{3,1} - B_2[t]\,d_{3,2} - B_3[t]\,d_{3,3} - B_4[t]\,d_{3,4} - B_5[t]\,d_{3,5}\right) + B_1[t]\,\Omega_{3,1} + B_2[t]\,\Omega_{3,2} + B_3[t]\,\Omega_{3,3} + B_4[t]\,\Omega_{3,4} + B_5[t]\,\Omega_{3,5};$$

$$dM_4 = -M_4[t]\left(Dh + q_4\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{4,1} - B_2[t]\,d_{4,2} - B_3[t]\,d_{4,3} - B_4[t]\,d_{4,4} - B_5[t]\,d_{4,5}\right) + B_1[t]\,\Omega_{4,1} + B_2[t]\,\Omega_{4,2} + B_3[t]\,\Omega_{4,3} + B_4[t]\,\Omega_{4,4} + B_5[t]\,\Omega_{4,5};$$

$$dM_5 = -M_5[t]\left(Dh + q_5\right) + \left(nuK * \frac{M_1[t]}{denK + M_1[t]} * \frac{M_2[t]}{denK + M_2[t]} * \frac{M_3[t]}{denK + M_3[t]} * \frac{M_4[t]}{denK + M_4[t]} * \frac{M_5[t]}{denK + M_5[t]}\right) \left(-B_1[t]\,d_{5,1} - B_2[t]\,d_{5,2} - B_3[t]\,d_{5,3} - B_4[t]\,d_{5,4} - B_5[t]\,d_{5,5}\right) + B_1[t]\,\Omega_{5,1} + B_2[t]\,\Omega_{5,2} + B_3[t]\,\Omega_{5,3} + B_4[t]\,\Omega_{5,4} + B_5[t]\,\Omega_{5,5};$$

```
op = coop; (*Number of links with overExpression*)
posNe = Position[Net, 1];
(*Positions in the matrix where there are links (=1)*)
RaN = RandomSample[posNe, op];
(*Random sample of op links that will be overproduced*)

costincr = 1.3; (*Term multiplying the cost link*)
overprodincr = 1.15;
(*Term multiplying the overproduction link*)


NewNetCost = Partition[Flatten[Net] × parR[[6 ;; 30]], {5}];
Table[NewNetCost[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] =
  NewNetCost[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] * costincr, {i, Length[RaN]}];

NewNetOvProd = Partition[Flatten[Net] × parR[[61 ;; 85]], {5}];
Table[NewNetOvProd[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] =
  NewNetOvProd[[RaN[[i]][[1]]]][[RaN[[i]][[2]]]] * overprodincr, {i,
  Length[RaN]}];


tmax = 1000;
par = {
  κ1 → parR[[1]], κ2 → parR[[2]], κ3 → parR[[3]], κ4 → parR[[4]], κ5 → parR[[5]],

  c1,1 → NewNetCost[[1]][[1]],
  c1,2 → NewNetCost[[1]][[2]], c1,3 → NewNetCost[[1]][[3]],
  c1,4 → NewNetCost[[1]][[4]], c1,5 → NewNetCost[[1]][[5]],
  c2,1 → NewNetCost[[2]][[1]], c2,2 → NewNetCost[[2]][[2]],
  c2,3 → NewNetCost[[2]][[3]], c2,4 → NewNetCost[[2]][[4]],
  c2,5 → NewNetCost[[2]][[5]],
  c3,1 → NewNetCost[[3]][[1]], c3,2 → NewNetCost[[3]][[2]],
  c3,3 → NewNetCost[[3]][[3]], c3,4 → NewNetCost[[3]][[4]],
  c3,5 → NewNetCost[[3]][[5]],
  c4,1 → NewNetCost[[4]][[1]], c4,2 → NewNetCost[[4]][[2]],
  c4,3 → NewNetCost[[4]][[3]], c4,4 → NewNetCost[[4]][[4]],
  c4,5 → NewNetCost[[4]][[5]],
  c5,1 → NewNetCost[[5]][[1]], c5,2 → NewNetCost[[5]][[2]],
  c5,3 → NewNetCost[[5]][[3]], c5,4 → NewNetCost[[5]][[4]],
  c5,5 → NewNetCost[[5]][[5]],

  r1,1 → parR[[31]], r1,2 → parR[[32]],
  r1,3 → parR[[33]], r1,4 → parR[[34]], r1,5 → parR[[35]],
  r2,1 → parR[[36]], r2,2 → parR[[37]], r2,3 → parR[[38]],
```

```
r2,4 → parR[[39]], r2,5 → parR[[40]],
r3,1 → parR[[41]], r3,2 → parR[[42]], r3,3 → parR[[43]],
r3,4 → parR[[44]], r3,5 → parR[[45]],
r4,1 → parR[[46]], r4,2 → parR[[47]], r4,3 → parR[[48]],
r4,4 → parR[[49]], r4,5 → parR[[50]],
r5,1 → parR[[51]], r5,2 → parR[[52]], r5,3 → parR[[53]],
r5,4 → parR[[54]], r5,5 → parR[[55]],

q1 → parR[[31]], q2 → parR[[32]],
q3 → parR[[33]], q4 → parR[[34]], q5 → parR[[35]],

d1,1 → parR[[36]], d1,2 → parR[[37]],
d1,3 → parR[[38]], d1,4 → parR[[39]], d1,5 → parR[[40]],
d2,1 → parR[[41]], d2,2 → parR[[42]], d2,3 → parR[[43]],
d2,4 → parR[[44]], d2,5 → parR[[45]],
d3,1 → parR[[46]], d3,2 → parR[[47]], d3,3 → parR[[48]],
d3,4 → parR[[49]], d3,5 → parR[[50]],
d4,1 → parR[[51]], d4,2 → parR[[52]], d4,3 → parR[[53]],
d4,4 → parR[[54]], d4,5 → parR[[55]],
d5,1 → parR[[56]], d5,2 → parR[[57]], d5,3 → parR[[58]],
d5,4 → parR[[59]], d5,5 → parR[[60]],

Ω1,1 → NewNetOvProd[[1]][[1]],
Ω1,2 → NewNetOvProd[[1]][[2]], Ω1,3 → NewNetOvProd[[1]][[3]],
Ω1,4 → NewNetOvProd[[1]][[4]], Ω1,5 → NewNetOvProd[[1]][[5]],
Ω2,1 → NewNetOvProd[[2]][[1]], Ω2,2 → NewNetOvProd[[2]][[2]],
Ω2,3 → NewNetOvProd[[2]][[3]], Ω2,4 → NewNetOvProd[[2]][[4]],
Ω2,5 → NewNetOvProd[[2]][[5]],
Ω3,1 → NewNetOvProd[[3]][[1]], Ω3,2 → NewNetOvProd[[3]][[2]],
Ω3,3 → NewNetOvProd[[3]][[3]], Ω3,4 → NewNetOvProd[[3]][[4]],
Ω3,5 → NewNetOvProd[[3]][[5]],
Ω4,1 → NewNetOvProd[[4]][[1]], Ω4,2 → NewNetOvProd[[4]][[2]],
Ω4,3 → NewNetOvProd[[4]][[3]], Ω4,4 → NewNetOvProd[[4]][[4]],
Ω4,5 → NewNetOvProd[[4]][[5]],
Ω5,1 → NewNetOvProd[[5]][[1]], Ω5,2 → NewNetOvProd[[5]][[2]],
Ω5,3 → NewNetOvProd[[5]][[3]], Ω5,4 → NewNetOvProd[[5]][[4]],
Ω5,5 → NewNetOvProd[[5]][[5]],
nuK → parR[[86]],
denK → parR[[87]]


};
```

```
B10 = 1500;
B20 = 1500;
B30 = 1500;
B40 = 1500;
B50 = 1500;
M10 = 10;
M20 = 10;
M30 = 10;
M40 = 10;
M50 = 10;


sol =
 NDSolve[
  {
    B1'[t] == dB1,
    B2'[t] == dB2,
    B3'[t] == dB3,
    B4'[t] == dB4,
    B5'[t] == dB5,

    M1'[t] == dM1,
    M2'[t] == dM2,
    M3'[t] == dM3,
    M4'[t] == dM4,
    M5'[t] == dM5,


    B1[0] == B10,
    B2[0] == B20,
    B3[0] == B30,
    B4[0] == B40,
    B5[0] == B50,
    M1[0] == M10,
    M2[0] == M20,
    M3[0] == M30,
    M4[0] == M40,
    M5[0] == M50

   } /. par,
  {B1, B2, B3, B4, B5, M1, M2, M3, M4, M5},
  {t, 0, tmax}];

{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax],
```

```
    M1[tmax], M2[tmax], M3[tmax], M4[tmax], M5[tmax]} /. sol /. par;

  Min[{B1[tmax], B2[tmax], B3[tmax], B4[tmax], B5[tmax]} /. sol /. par]

 )
```

In[7220]:=
```
robustnessNewSaitoOVRBth[NetTop_, coop_] := (
  n1 = 1;
  n2 = 5000;
  mid = (n1 + n2) / 2;

  While[(n1 ≠ mid && n2 ≠ mid),
   (If[fNewSaitoOVRBth[NetTop, mid, coop] < 1, n2 = mid, n1 = mid];
    mid = Floor[N[(n1 + n2) / 2]];); {n1, n2, mid}]; mid

 )
```

In[7221]:=
```
NetK = {
   {0, 1, 0, 1, 0},
   {1, 0, 1, 1, 0},
   {1, 0, 1, 0, 1},
   {0, 1, 0, 1, 0},
   {0, 0, 0, 0, 1}
  };
```

Compare the Robustness with and without (n links) overproduction (ratio cost/production = 1.3/1.15)

In[7222]:=
```
fNewSaitoRBth[NetK, 0]
```

Out[7222]=
```
6639.57
```

In[7223]:=
```
fNewSaitoOVRBth[NetK, 0, 5]
```
Out[7223]=
```
6639.57
```

In[7224]:=
```
robustnessNewSaitoRBth[NetK]
```
Out[7224]=
```
243
```

In[7225]:=
```
robustnessNewSaitoOVRBth[NetK, 5]
```
Out[7225]=
```
252
```

In[7226]:=
```
robustnessNewSaitoOVRBth[NetK, 10]
```
Out[7226]=
```
272
```

In[7227]:=
```
AuxoComm8RBth
```
Out[7227]=
```
{351, 339, 347, 365, 337, 335, 290, 365, 366, 349, 353, 355, 337, 356, 359, 348, 348, 338,
 354, 292, 349, 348, 352, 350, 349, 367, 356, 293, 366, 355, 353, 289, 354, 366, 364,
 366, 349, 361, 366, 367, 349, 366, 362, 366, 367, 355, 350, 349, 364, 351, 351, 353,
 350, 267, 368, 353, 287, 366, 367, 354, 351, 350, 365, 366, 354, 350, 351, 366,
 366, 361, 352, 349, 366, 366, 366, 366, 352, 343, 351, 365, 340, 368, 351, 350,
 348, 367, 361, 349, 351, 349, 267, 366, 366, 361, 365, 354, 349, 348, 355, 365}
```

In[7228]:=
```
coop5to15RBth = {Table[robustnessNewSaitoOVRBth[#, 5], {20}],
    Table[robustnessNewSaitoOVRBth[#, 10], {20}],
    Table[robustnessNewSaitoOVRBth[#, 15], {20}]} &;
```

In[7229]:=
```
wf8RBth = Parallelize[coop5to15RBth /@ hk8];
```

In[7230]:=
```
wf8NormalizedRBth = N[wf8RBth[[#]] / AuxoComm8RBth[[#]]] & /@ Range[100]
```

In[7231]:=
```
wf8NormalizedWith5CoopRBth = wf8NormalizedRBth[[#]][[1]] & /@ Range[100]
```

In[7232]:=
```
wf8NormalizedWith10CoopRBth = wf8NormalizedRBth[[#]][[2]] & /@ Range[100]
```

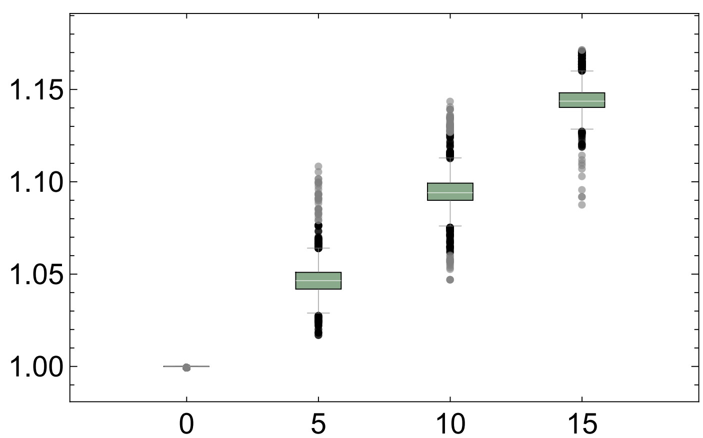
In[7233]:=
```
wf8NormalizedWith15CoopRBth = wf8NormalizedRBth[[#]][[3]] & /@ Range[100]
```

In[7234]:=
```
allcoopWith8AuxoRBth = {Flatten[wf8NormalizedWith5CoopRBth],
   Flatten[wf8NormalizedWith10CoopRBth], Flatten[wf8NormalizedWith15CoopRBth]}
```

In[7235]:=
```
allcoopWith8AuxoPlusAuxoRBth =
  Join[{ConstantArray[1, {2000}]}, allcoopWith8AuxoROM]
```

In[7236]:=
```
BoxWhiskerChart[allcoopWith8AuxoPlusAuxoRBth, "Outliers",
  ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{gree1}},
  Frame → True, ChartLabels → {"0", "5", "10", "15"},
  BarSpacing → 1.9, FrameStyle → Directive[Black, FontSize → 15]]
```

Out[7236]=



In[7237]:=

```
BoxWhiskerChart[allcoopWith8AuxoPlusAuxoRBth, "Outliers",
 ChartBaseStyle → EdgeForm[Dashing[0.99]], ChartStyle → {{gree1}},
 Frame → True, ChartLabels → {"0", "5", "10", "15"}, BarSpacing → 1.9,
 FrameStyle → Directive[Black, FontSize → 15], AspectRatio → 1]
```

Out[7237]=