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Beyond The Graph Theory

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Abstract

Graph theory has the widely ranges of applications and theoretical aspects. In this book, focus in on definitions and there's an effort to make connections about different types of graphs with using the new ideas which arise from the definitions and using examples which are the tools to get the understandable perspective about the concepts. The context is away from some texts which aren't in literature of mathematics. The author avoid to bring some texts to describe the ideas and the results before bringing them. The main idea is to write the main concepts but in examples, some explanations are found about the connections of the definitions. In this book, the goal is to present the relations between definition in the ways, the number of definitions in the results is the matter minds and it's avoided to pay attention to the degree of the results in the terms of being hard. The book is devised to make the gentle comparison between concepts and in this way, there's the priority about including the easy concept for covering the wides ranges of readers and spreading the ranges of readership. Easy connections with the priority of making connections with the most definitions as possible.

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Acknowledgements

The author is going to say his gratitude and his appreciation about the brains and their hands which are showing the importance of words in the framework of every wisdom, knowledge, arts and emotions which are streaming in the lines from the words, notions, ideas and approaches to have the material which is only the way to flourish the minds, the growing the notions, advancing ways and making the stable ways to be amid the events and storms of minds for surviving from them and making the outstanding experiences about the tools and ideas to be on the star lines of words and shining like stars, forever.

The words of mind and the minds of words, are too eligible to be in the stage of aknowledgements

Contents

CHAPTER 1

Words Of Graph Theory

Points and continuum of points as lines and curves call for a theory to call it graph theory and there are too many points about seen points and unseen points like one point is seen and have a name but points are unseen and have no names but all of them could be called lines, loops and curve. In the term of pronoun, there are two pronouns we and us when the pronoun we is used for known point but the pronoun us is used for unknown points as the elements of straight lines, curves and loops

Beyond Of Graph Theory #1

Words of graph theory could act on any notions to get new notions.

1.1 Definition And Its Consequences

There are different kinds of defining a graph. Sometimes, the couple of two distinct sets of objects with function amid them. The domain is called edges set and the image of this function is called vertices set. In other ways, there are new notions to define a graph on a set of objects instead of defining by a function but second style has open ways to connect other branch because every function could be a graph.

1.2 Different Styles Of Graph

Definition 1.2.1 (Graph: Function-Orientated Style)**.** Let $\mathcal{F}$ be a function from $\mathcal{V}$ to $\mathcal{E}$ where $\mathcal{V}$ and $\mathcal{E}$ are the sets of distinct objects and function is assigning unordered couple of objects of $\mathcal{V}$ to any of object of $\mathcal{E}$. Then the couple of $\mathcal{V}$ and $\mathcal{E}$ is called graph and it's denoted by $\mathcal{G} : (\mathcal{V}, \mathcal{E})$.

Definition 1.2.2 (Graph: Set-Orientated Style)**.** Let $\mathcal{V}$ be a set of objects and any set $\mathcal{E}$ of unordered couple objects of $\mathcal{V}$ is up. Therefore, $\mathcal{G} : (\mathcal{V}, \mathcal{E})$ is a graph.

Definition 1.2.3 (Graph: Matrix-Orientated Style)**.** Let $\mathcal{G}$ be a matrix. If all entries are zero and one, then the couple of vertical set of objects which is denoted by $\mathcal{V}$ and the couple of horizontal set of objects which is denoted by $\mathcal{E}$, is denoted by $\mathcal{G} : (\mathcal{V}, \mathcal{E})$ and it's called a graph.

Definition 1.2.4 (Graph: Matrix-Orientated Style)**.** Let $\mathcal{G}$ be a matrix. If all entries are zero and one, then the couple of vertical set of objects which is denoted by $\mathcal{V}$ and the couple of horizontal set of objects which is denoted by $\mathcal{V}$, is denoted by $\mathcal{G} : (\mathcal{V}, \mathcal{E})$ and it's called a graph. Eigenvalue of graph is the eigenvalues of this matrix and characteristic polynomial of graph is the characteristic polynomial of this matrix where the characteristic polynomial of a matrix is $det(A - xI)$ and the eigenvalues of a matrix is the roots of $det(A - xI)$.

Exercises

1. If $\mathcal{G}$ is graph, then eigenvalue is lower or equal with Δ.
2. If $\mathcal{G}$ is a connected graph and Δ is eigenvalue, then the graph is regular.
3. If $\mathcal{G}$ is a connected graph and $-\Delta$ is eigenvalue, then the graph is both regular and bipartite.

1.3 List Of The Vertices

Definition 1.3.1 (List Of One Vertex)**.** Let v be a given vertex. Then the list of vertices $v_1v_2\cdots v_n$ which have the edge with v is called list of vertex v.

Definition 1.3.2 (List Of Two Vertices)**.** Let v and u be given vertices. Then the list of vertices $vv_1v_2\cdots v_nu$ which are consecutive vertices from v to u is called list of vertex v and u.

1.4 Degree Of A Vertex

Definition 1.4.1 (Degree List Of Couple)**.** Let v and u be given vertices. Then the list of vertices $v_1v_2\cdots v_n$ which are consecutive vertices' degrees from v to u is called degree list of vertex v and u.

Definition 1.4.2 (Degree List Of Vertices)**.** The list of vertices $v_1v_2\cdots v_n$ which are vertices' degrees is called degree list of vertices.

Definition 1.4.3 (Graphic)**.** The list of numbers $v_1v_2\cdots v_n$ is called graphic if it's degree list of vertices.

The summation of degree of all vertices is forever even number. So it's the criteria to decide whether a number could make a graph or not. If a given number is odd, then it can't form any of graph when the usage of formula is up. Assigning the number to the summation when is possible that this number is even. Usage of first version of Matrix-Orientated Style as graph is selected to prove upcoming result by summation all enters twice. Once all columns and another all rows.

Theorem 1.4.4. *Let $\mathcal{G} : (\mathcal{V}, \mathcal{E})$ be a graph. Then*

$$\sum_{x \in \mathcal{V}} degree(x) = 2|E| \tag{1.1}$$

{R1}

Proof. For any given vertex x, summation of its row gives us $degree(x)$ so the summation of all rows equals $\sum_{x\in\mathcal{V}} degree(x)$ which is left hand of the statement. Every edge has two endpoints so for any given edge e, summation of its column gives us 2 thus the summation of all columns equals $2|E|$ which is right hand of the statement. By summation twice, the statement is proved.

First, summation all enters from each rows and secondly, summation all enters from each column. Summations of all enters from each column with each others are summation of all numbers in the matrix. Therefore, there are two systematically ways to summation of all enters of a matrix. Column by column or row by row but the result is the same.

Strongest tool about determine number whether is or not an even number, is module 2. 0 tells us the structure of number is even but 1 tells us the structure of the number is odd. The even number is shown the warmly welcome to the numbers which belongs to this class and characteristics. ∎

The number of odd degree is even. So this could be another characteristic for graphs in the terms of numbers.

Theorem 1.4.5. *Let $\mathcal{G} : (\mathcal{V}, \mathcal{E})$ be a graph. Then the number of vertices is even where these vertices have odd degree.*

Proof. By equation (1.1) under module 2, right-hand side is zero. Thus left-hand side has to be zero which implies the number of vertices is even where these vertices have odd degree. ∎

Degree of a vertex could spread to all vertices when all vertices have the same degree thus there's a new graph which is created by the notion of degree of a vertex.

Definition 1.4.6. Let $\mathcal{G} : (\mathcal{V}, \mathcal{E})$ be a graph such that all vertices have the same degree t. Then a graph is called $t-$regular.

1.5 Some Classes Of Graphs

Definition 1.5.1 (Connected)**.** A graph is called connected if for any given two vertices, there's a list of two vertices.

Definition 1.5.2 (Disconnected)**.** A graph is called disconnected if it isn't connected.

Definition 1.5.3 (Finite)**.** A graph is called finite if both its vertex set and edge set are finite.

Definition 1.5.4 (Null)**.** A graph is called null if its vertex set is empty set.

Definition 1.5.5 (Trivial)**.** A graph is called trivial if its vertex set has one vertex.

Definition 1.5.6 (Nontrivial)**.** A graph is called nontrivial if its vertex set has at least two vertices.

Definition 1.5.7 (Simple)**.** A graph is called simple if its edges are neither parallel nor loop.

Definition 1.5.8 (Planar)**.** A graph is called planar if its edges meet each other in vertex. This exhibition is called planar embedding.

Definition 1.5.9 (Complete)**.** A graph is called complete if every two given vertices has one edge.

Definition 1.5.10 (Empty)**.** A graph is called empty if every two given vertices has no edge.

Definition 1.5.11 (Bipartite)**.** A graph is denoted by $\mathcal{G}[\mathcal{X}, \mathcal{Y}]$ where $\mathcal{X}$ and $\mathcal{Y}$ are parts of $\mathcal{G}$ and is called bipartite if its vertex set is partitioned two parts and every part has no edge inside.

Definition 1.5.12 ($n-$Partite)**.** A graph is called $n-$partite if its vertex set is partitioned n parts and every part has no edge inside.

Definition 1.5.13 (Turan Graph)**.** An $n-$partite is denoted $\mathcal{T}_{k,n}$ by and is called Turan graph if every part has equal vertices such that $\lfloor \frac{n}{k} \rfloor = \lfloor \frac{k}{n} \rfloor = 1$.

Definition 1.5.14 ($n-$Path)**.** A graph is denoted $\mathcal{P}_n$ by and is called $n-$path if there are two vertices such that it's a list of two vertices where n is the number of its edges.

Exercises

1. Every $n-$path is bipartite.

Definition 1.5.15 ($n-$Cycle)**.** A graph is denoted $\mathcal{C}_n$ by and is called $n-$cycle if there are one vertex such that it's a list of one vertex where n is the number of its edges. 3-cycle, 4-cycle, 5-cycle and 6-cycle are often called triangle, quadrilateral, pentagon, and hexagon, resepectively.

Exercises

1. Every $n-$cycle is bipartite if and only if n is even.

With having known attributes of vertex set and edge set, there's new class of graphs.

Definition 1.5.16 ($n-$Cube)**.** A graph is denoted by $\mathcal{Q}_n$ and is called $n-$cube where the vertex set is the set of all $n - tuple$ including 0s and 1s and edge set is the set of couple of n-tuple of vertex set which have one difference in their coordinate.

Definition 1.5.17 (Boolean Lattice)**.** A graph is denoted by $\mathcal{BL}_n$ and is called boolean lattice where the vertex set is the set of all subsets of $\{1, 2, \cdots, n\}$ and edge set is the set of couple of subsets of vertex set which have one difference in their elements.

Definition 1.5.18 (Complement)**.** A graph is denoted by $\bar{\mathcal{G}}$ and is called complement if it introduces a graph which its vertex set is vertex set of $\bar{\mathcal{G}}$ but its edge set is non-edge set of $\bar{\mathcal{G}}$.

Definition 1.5.19 (Strongly Regular Graph)**.** A simple graph $\mathcal{G}$ which is neither empty nor complete is called strongly regular with parameters (v, k, λ, μ) if:

- $v(\mathcal{G}) = v$,
- $\mathcal{G}$ is $k-$regular,
- For any two given vertices which are in the list of each other, they've λ common members in list of each other.
- For any two given vertices which aren't in the list of each other, they've μ common members in list of each other.

CHAPTER 2

Connections Of The Words

In the term of pronoun, there are two pronouns we and us when the pronoun we is used for known point but the pronoun us is used for unknown points as the elements of straight lines, curves and loops

Beyond Of Graph Theory #1

2.1 Unary Operations And Graphs

Definition 2.1.1 (Identical)**.** Using function-orientated style, two graphs $\mathcal{G}$ and $\mathcal{H}$ are denoted by $\mathcal{G} = \mathcal{H}$ and are called identical if $\mathcal{V}(\mathcal{G}) = \mathcal{V}(\mathcal{H}),\ \mathcal{E}(\mathcal{G}) = \mathcal{E}(\mathcal{H})$ and $\mathcal{F}_{\mathcal{G}} = \mathcal{F}_{\mathcal{H}}$.

Definition 2.1.2 (Isomorphic)**.** Two unary operations are called isomorphism amid $\mathcal{G}$ and $\mathcal{H}$ such that using function-orientated Style, two graphs $\mathcal{G}$ and $\mathcal{H}$ are denoted by $\mathcal{G} \cong \mathcal{H}$ and are called isomorphic if there are unary operations $\theta : \mathcal{V}(\mathcal{G}) \to \mathcal{V}(\mathcal{H}),\ \phi : \mathcal{E}(\mathcal{G}) \to \mathcal{E}(\mathcal{H})$ which are bijection and $\psi_{\mathcal{G}}(e) = uv$ if and only if $\psi_{\mathcal{H}}(\phi(e)) = \theta(u)\theta(v)$. A representative of an equivalence class of isomorphic graphs, is called unlabelled graph. Up to isomorphism, it makes sense to use the notations $\mathcal{K}_n, \mathcal{K}_{n,m}, \mathcal{P}_n$ and $\mathcal{C}_n$ for complete, complete bipartite, path and cycle graphs.

Exercises

1. Boolean lattice $\mathcal{BL}_n$ and $n-$cube $\mathcal{Q}_n$ are isomorphic.

Definition 2.1.3 (Automorphism)**.** An unary operation is called automorphism if it's an isomorphism from one graph to itself. The set of all automorphisms of a graph $\mathcal{G}$ is denoted by $Aut(\mathcal{G})$ and is called automorphism group.

Exercises

1. Automorphism of a complete graph $\mathcal{K}_n$ is the symmetric group $\mathcal{S}_n$.
2. Automorphism of a simple graph $\mathcal{G}$ on n vertices $Aut(\mathcal{G})$ is a subgroup of symmetric group $\mathcal{S}_n$.
3. Automorphism of an $n-$cycle graph $\mathcal{C}_n$ is the dihedral group $\mathcal{D}_n$, i.e., $Aut(\mathcal{C}_n) = \mathcal{D}_n$ but automorphism of an $n-$path graph $\mathcal{P}_n$ is isomorphic with symmetric group $\mathcal{S}_2$, i.e., $Aut(\mathcal{P}_n) \cong \mathcal{S}_2$.
4. For any simple graph $\mathcal{G}$, $Aut(\mathcal{G}) = Aut(\bar{\mathcal{G}})$.

2.2 Related Classes Of Graphs

Definition 2.2.1 (Self-complementary). A simple graph is called self-complementary if it's isomorphic to its complement.

Exercises

1. $\mathcal{C}_5$ is self-complementary.
2. $\mathcal{P}_4$ is self-complementary.

Definition 2.2.2 (Edge-Transitive Graph). A simple graph is called edge-transitive if for any two edges uv and xy, there's an automorphism α such that $\alpha(u)\alpha(v) = xy$.

Definition 2.2.3 (Vertex-Transitive Graph). A simple graph is called vertex-transitive if for any two vertices u and x, there's an automorphism α such that $\alpha(u) = x$. Two vertices u and x are called similar. Similarity is an equivalence relation on the vertex set of a graph and its equivalence classes are called orbits.

Exercises

1. A connected graph is bipartite if it's edge-transitive and it isn't vertex-transitive.

Definition 2.2.4 (Folkman Graph). A graph is depicted by picture (2.1), and is called Folkman, is the $4-$regular graph obtained the left picture by replacing each vertex v of degree eight by two vertices of degree four, both of which have the same vertices in their lists as v.

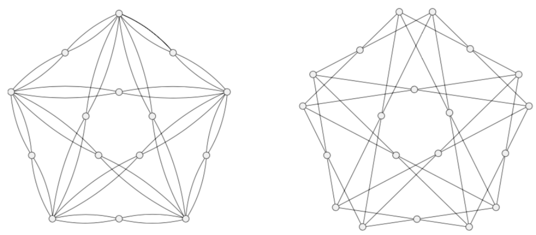


Figure 2.1: Graph And Folkman Graph

G1

Exercises

1. Consider Folkman graph. It's edge-transitive and it isn't vertex-transitive.

Definition 2.2.5 (Generalized Petersen Graph). A simple graph is denoted by $\mathcal{P}_{k,n}$ when $n, k \in \mathbb{N},\ n > 2k$ and is called generalized Petersen graph with vertices $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$, and edges $x_i x_{i+1}, y_i y_{i+k}, x_i y_i, 1 \leq i \leq n$, indices are taken under module n. $\mathcal{P}_{2,5}$ is the Petersen graph.

Definition 2.2.6 (Hypergraph)**.** A hypergraph is denoted by $\mathcal{HG} : (\mathcal{V}, \mathcal{F})$ where $\mathcal{V}$ is a set of elements which are called vertices and $\mathcal{F}$ is a family of subsets of $\mathcal{V}$ which are called hyperedges. A hypergraph is $k-$uniform if each hyperedge is $k-$set where $k-$set is a set of k elements.

Definition 2.2.7 (Geometric Configuration)**.** A Geometric configuration is denoted by $\mathcal{GC} : (\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$ is a finite set of elements which are called points and $\mathcal{F}$ is a finite family of subsets of $\mathcal{V}$ which are called lines such that at most one line contains any given pair of points.

Example 2.2.8. Fano hypergraph has seven points and seven lines and Desargues hypergraph has ten points and ten lines. They are 3-uniform hypergraphs where each lines consists of three points. They are depicted by the picture (2.2).

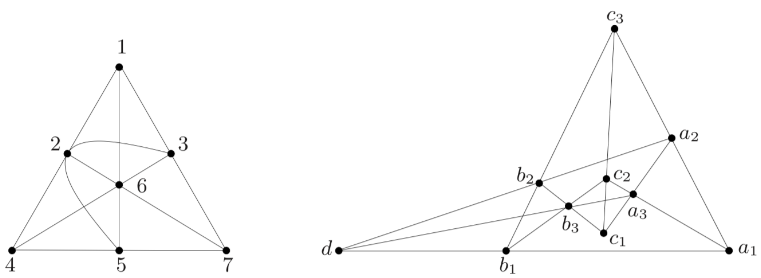


Figure 2.2: Fano Hypergraph And Desargues Hypergraph

G2

Definition 2.2.9 (Incidence Graph)**.** A graph associated with a hypergraph is denoted by $\mathcal{CG} : (\mathcal{V}, \mathcal{F})$ is bipartite graph where $v \in \mathcal{V}$ and $F \in \mathcal{F}$ have common edge if $v \in F$ and it's called incidence graph.

Example 2.2.10. The incidence graph of Fano hypergraph is depicted by picture (2.3) and is called Heawood hypergraph. Incidence graph is an unary operation on hypergraphs.

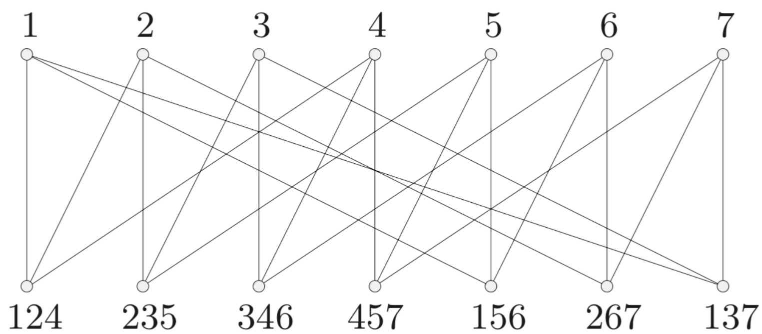


Figure 2.3: Fano Hypergraph And Heawood Hypergraph

G3

Intersection Graph as Unary Operation And Its Variants

Definition 2.2.11 (Intersection Graph)**.** A graph associated with a hypergraph is denoted by $\mathcal{SG} : (\mathcal{V}, \mathcal{F})$ where $\mathcal{F}$ is the vertex set and two sets in $\mathcal{F}$ are adjacent if their intersection isn't nonempty and it's called intersection graph. Intersection graph is an unary operation on hypergraphs.

Definition 2.2.12 (Line Graph)**.** A graph associated with a graph is denoted by $\mathcal{LG} : (\mathcal{V}, \mathcal{F})$ where $\mathcal{F}$ is the vertex set and two edges in $\mathcal{F}$ are adjacent if their intersection isn't nonempty and it's called line graph. Line graph is an unary operation on graphs. Line graph is a variant of intersection graph.

Exercises

1. Intersection graph of Desargues hypergraph is isomorphic to line graph of $\mathcal{K}_5$.
2. Line graph of $\mathcal{K}_5$ is isomorphic to complement of the Petersen graph.
3. Intersection graph of Fano hypergraph is isomorphic to line graph of $\mathcal{K}_7$.
4. Line graph of $\mathcal{K}_{3,3}$ is self-complementary.
5. $Aut(\mathcal{LK}_n \ncong Aut(\mathcal{K}_n), \ n = 2, 4$
6. $Aut(\mathcal{LK}_n \ncong Aut(\mathcal{K}_n), \ n = 3, \ n \geq 5$

Definition 2.2.13 (Interval Graph)**.** A graph associated with a graph is denoted by $\mathcal{LG} : (\mathcal{V}, \mathcal{F})$ where $\mathcal{V} = \mathbb{R}$ and $\mathcal{F}$ is a set of closed intervals of $\mathbb{R}$ where $\mathcal{F}$ is the vertex set and two intervals in $\mathcal{F}$ are adjacent if their intersection isn't nonempty and it's called interval graph. Interval graph is an unary operation on graphs. Interval graph is a variant of intersection graph.

Other Classes Of Graphs

Definition 2.2.14 (Kneser Graph)**.** A graph is denoted by $\mathcal{KG}_{m,n}$ when $n, k \in \mathbb{N}, \ n > 2m$ and is called Kneser Graph where vertices are the $m-$subsets of an $n-$set $\mathcal{S}$ and two subsets have edges if their intersection is empty.

Exercises

1. $\mathcal{KG}_{1,n} \cong \mathcal{K}_n, \ n \geq 3.$
2. $\mathcal{KG}_{2,n}$ is isomorphic to the complement of $\mathcal{LK}_n, \ n \geq 5.$

Definition 2.2.15 (Cayley Graph)**.** A graph is denoted by $\mathcal{YG}(\Gamma, \mathcal{S})$ when Γ is a group and $\mathcal{S}$ is a set of elements of Γ with the exception the identity element and $\mathcal{S}$ includes inverse of every elements of itself and it's called Cayley graph of Γ with respect to $\mathcal{S}$ where Γ is vertex set and two vertices x, y have edge if $xy^{-1} \in \mathcal{S}$.

Exercises

1. $\mathcal{Q}_n = \mathcal{YG}(\Gamma, \mathcal{S})$.
2. $\mathcal{YG}(\Gamma, \mathcal{S})$ is vertex-transitive graph.
3. Every vertex-transitive graph isn't $\mathcal{YG}(\Gamma, \mathcal{S})$.

Definition 2.2.16 (Circulant Graph)**.** A Cayley graph is denoted by $\mathcal{YG}(\mathbb{Z}_n, \mathcal{S})$ and it's called circulant graph. In special case, let p be a prime number, then $\mathcal{YG}(\mathbb{Z}_p, \mathcal{S})$ is circulant graph.

Definition 2.2.17 (Paley Graph)**.** A graph is denoted by $\mathcal{PG}_q(\mathbb{Z}_n, \mathcal{S})$ where q is a prime power such that $q \cong 1$ and it's called Paley graph where vertex set is the set of elements of the field $\mathcal{GF}(q)$ and two vertices have edge if their difference is a nonzero square in $\mathcal{GF}(q)$.

Exercises

1. $\mathcal{GF}(5), \mathcal{GF}(9)$ and $\mathcal{GF}(13)$ are self-complementary.

Product Of Graphs as Binary Operation

Definition 2.2.18 (Union Graph)**.** Two simple graphs $\mathcal{G}$ and $\mathcal{H}$ are denoted by $\mathcal{G} \cup \mathcal{H}$ where the vertex set is $\mathcal{V}(\mathcal{G}) \cup \mathcal{V}(\mathcal{H})$ and the edge set is $\mathcal{E}(\mathcal{G}) \cup \mathcal{E}(\mathcal{H})$ and it's called union graph. If $\mathcal{G}$ and $\mathcal{H}$ are disjoint, they're denoted by $\mathcal{G} + \mathcal{H}$ and it's called disjoint union graph. It's associative, commutative and extended to any number of given graphs.

Definition 2.2.19 (Intersection Graph)**.** Two simple graphs $\mathcal{G}$ and $\mathcal{H}$ are denoted by $\mathcal{G} \cap \mathcal{H}$ where the vertex set is $\mathcal{V}(\mathcal{G}) \cap \mathcal{V}(\mathcal{H})$ and the edge set is $\mathcal{E}(\mathcal{G}) \cap \mathcal{E}(\mathcal{H})$ and it's called intersection graph. If $\mathcal{G}$ and $\mathcal{H}$ are disjoint, it's called disjoint intersection graph and is null graph. It's associative, commutative and extended to any number of given graphs.

Definition 2.2.20 (Cartesian Graph)**.** Two simple graphs $\mathcal{G}$ and $\mathcal{H}$ are denoted by $\mathcal{G} \square \mathcal{H}$ where the vertex set is $\mathcal{V}(\mathcal{G}) \times \mathcal{V}(\mathcal{H})$ and the edge set is the set of all $(x_1, y_1)(x_2, y_2)$ such that either $x_1 x_2 \in \mathcal{E}(\mathcal{G})$ and $y_1 = y_2$ or $y_1 y_2 \cap \mathcal{E}(\mathcal{H})$ and $x_1 = x_2$ and it's called cartesian graph. For each edge $\mathcal{G}$ in and for each edge $\mathcal{H}$, there are four edges in $\mathcal{G} \square \mathcal{H}$ which the notation reflects this fact.

Example 2.2.21. The picture (2.4) depicts $\mathcal{P}_1 \square \mathcal{P}_1$ and $(5 \times 4)-$grid.

Example 2.2.22. The picture (2.5) depicts $\mathcal{C}_3 \square \mathcal{P}_1$ and $\mathcal{C}_5 \square \mathcal{P}_1$. $\mathcal{C}_n \square \mathcal{P}_1$ is called $n-$prism. In special case, it's called triangular prism, the cube and the pentagonal prism.

Exercises

1. $\mathcal{P}_n \square \mathcal{P}_m$ is $(m \times n)-$grid.

1. For $n \geq 3$, $\mathcal{C}_n \square \mathcal{P}_1$ is polyhedral graphs.

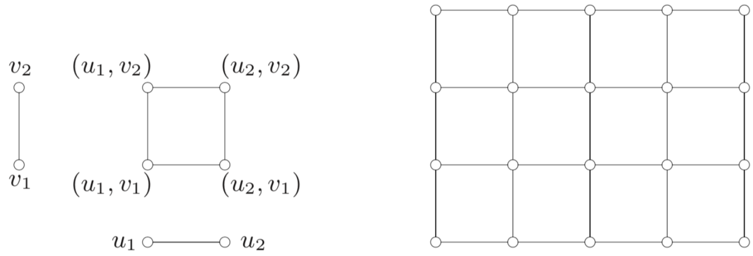


Figure 2.4: Cartesian Graphs And $(m \times n)-$grid

G4

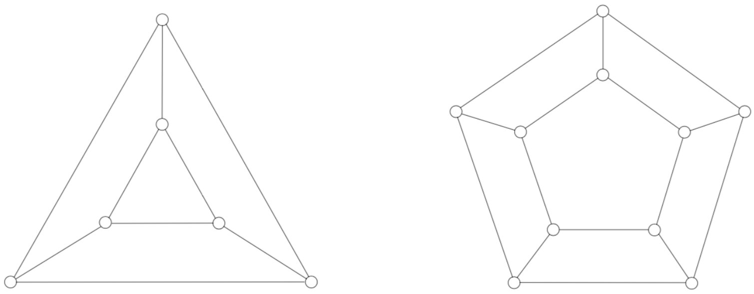


Figure 2.5: Cartesian Graphs And $n-$prism

G5

2.3 Directed Graphs And Its Variants

Definition 2.3.1 (Directed Graph: Set-Orientated Style)**.** Let $\mathcal{V}$ be a set of objects and any set $\mathcal{A}$ of ordered couple objects of $\mathcal{V}$ is up. Therefore, $\mathcal{D} : (\mathcal{V}, \mathcal{A})$ is a directed graph. All notions which are defined on graph, become twofold with labels, in and out. The term of dominating could be used when one object is on the first position in the ordered couple and the second object is on the second position thus it's called first object dominates second object. Replacing all arrows by segments, gives us the underlying graph $\mathcal{G}$ of $\mathcal{D}$ and it's called $\mathcal{G}(\mathcal{D})$. Replacing all segments by two arrows, gives us the directed graph $\mathcal{D}$ of $\mathcal{G}$ and it's called $\mathcal{D}(\mathcal{G})$.

Definition 2.3.2 (Directed Graph: Matrix-Orientated Style)**.** Let $\mathcal{D}$ be a matrix. If all enters are 0, 1 and -1, then the couple of vertical set of objects which is denoted by $\mathcal{V}$ and the couple of horizontal set of objects which is denoted by $\mathcal{A}$, is denoted by $\mathcal{D} : (\mathcal{V}, \mathcal{A})$ and it's called a digraph.

Definition 2.3.3 (Directed Graph: Matrix-Orientated Style)**.** Let $\mathcal{D}$ be a matrix. If all enters are $0, -1$ and 1, then the couple of vertical set of objects which is denoted by $\mathcal{V}$ and the couple of horizontal set of objects which is denoted by $\mathcal{V}$, is denoted by $\mathcal{D} : (\mathcal{V}, \mathcal{A})$ and it's called a digraph. Eigenvalue of digraph is the eigenvalues of this matrix and characteristic polynomial of digraph is the characteristic polynomial of this matrix where the characteristic polynomial of a matrix is $det(A - xI)$ and the eigenvalues of a matrix is the roots of $det(A - xI)$.

Definition 2.3.4 (Orientation Of The Graph)**.** In a graph $\mathcal{G}$, replacing segment by one of two arrows is denoted $\overrightarrow{\mathcal{G}}$ and it's called an orientation of $\mathcal{G}$.

Definition 2.3.5 (Orientated Graph)**.** An orientation of a simple graph is called orientated graph.

Definition 2.3.6 (Tournament)**.** An orientation of a complete graph is called tournament.

Example 2.3.7. Picture (2.6) depicts four unlabelled tournaments on four vertices. The orientation of unlabelled complete graph from the order four is called unlabelled tournaments on four vertices.

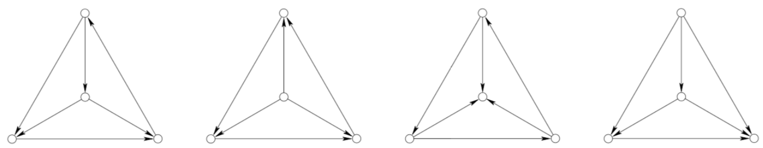
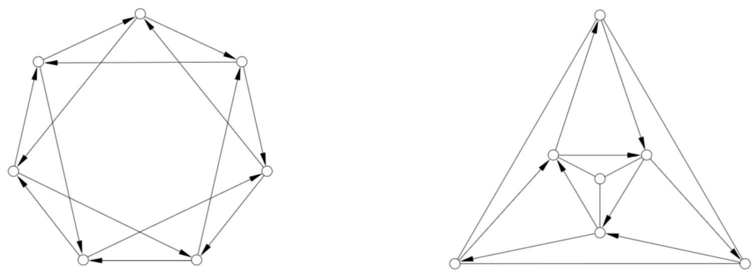


Figure 2.6: Tournaments And Complete Graph

G6

Example 2.3.8. Picture (2.7) depicts 2-diregular digraph and 3-diregular digraph. These digraphs can both be constructed from Fano hypergraph.



(a) (b)

Figure 2.7: Fano Hypergraph And Directed Graph

G7

Definition 2.3.9 (Cayley Digraph)**.** A digraph is denoted by $\mathcal{YD}(\Gamma, \mathcal{S})$ when Γ is a group and $\mathcal{S}$ is a set of elements of Γ with the exception the identity element and $\mathcal{S}$ includes inverse of every elements of itself and it's called Cayley graph of Γ with respect to $\mathcal{S}$ where Γ is vertex set and x dominates y if $xy^{-1} \in \mathcal{S}$.

Definition 2.3.10 (Circulant Digraph)**.** A Cayley digraph is denoted by $\mathcal{YD}(\mathbb{Z}_n, \mathcal{S})$ and it's called circulant graph. In special case, let p be a prime number, then $\mathcal{YG}(\mathbb{Z}_p, \mathcal{S})$ is circulant graph.

Definition 2.3.11 (Converse Of A Digraph)**.** Let $\mathcal{D} : (\mathcal{V}, \mathcal{A})$ be a directed graph, the converse of digraph $\mathcal{D}$ is denoted by $\overleftarrow{\mathcal{D}}$ where it's obtained reversing every arrow of $\mathcal{D}$ and it's called converse of $\mathcal{D}$.

Definition 2.3.12 (Balanced Digraph)**.** A digraph $\mathcal{D} : (\mathcal{V}, \mathcal{A})$ is called balanced if for all $v \in \mathcal{V}$, $|d^+(v) - d^-(v)| \leq 1$.

Definition 2.3.13 (Paley Tournament)**.** A tournament is denoted by $\mathcal{PT}_q$ where q is a prime power, $q \equiv 3 \pmod 4$ and it's called Paley tournament where the vertex set is the set of elements of the field $\mathcal{GF}(q)$, vertex i dominates vertex j if $j - i$ is a nonzero square in $\mathcal{GF}(q)$.

Definition 2.3.14 (Stockmeyer Tournament)**.** A tournament is denoted by $\mathcal{ST}_n$ where $n \geq 1$ and it's called Stockmeyer tournament where the vertex set is $\{1, 2, 3, \cdots, 2^n\}$ and vertex i dominates vertex j if $\mathrm{odd}(j - i) \equiv 1$ (mode 4) where $\mathrm{pow}(k)$ denote the greatest integer p such that 2^p divides k, and set $\mathrm{odd}(k) = \frac{k}{2^p}$ where k is nonzero integer.

Definition 2.3.15 (Arc-transitive Graph)**.** An undirected graph $\mathcal{G}$ is called arc-transitive if $\mathcal{D}(\mathcal{G})$ is arc-transitive. Equivalently, $\mathcal{G}$ is called arc-transitive if given any two ordered couple (x, y) and (u, v) of adjacent vertices, there exits an automorphism of $\mathcal{G}$ which maps (x, y) to (u, v).

2.4 Infinite Graphs And Its Variants

Definition 2.4.1 (Infinte Graph)**.** A graph $\mathcal{G}$ is called infinite if its vertex set and/or its edge set is infinite.

Definition 2.4.2 (Countable Graph)**.** A graph $\mathcal{G}$ is called countable if both its vertex set and its edge set are countable.

Example 2.4.3. Picture (2.8), depicts three countable graphs and infinite graphs which are well-known as square lattice, triangular lattice, and hexagonal lattice.

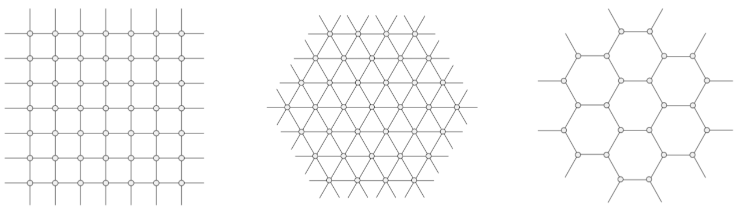


Figure 2.8: Infinite Graphs And Countable Graphs

G8

Definition 2.4.4 (Unit Distance Graph)**.** A graph $\mathcal{G}$ is called unit distance graph if its vertex set is a subset of $\mathbb{R}^2$ and two vertices (x_1, y_1) and (x_2, y_2) have one edge if their euclidean distance is 1 which it means $(x_1 - x_2)^2 + (y_1 - y_2)^2 = 1$.

Definition 2.4.5 (Rational Unit Distance Graph)**.** A graph $\mathcal{G}$ is called rational unit distance graph if its vertex set is $\mathbb{Q}^2$.

Definition 2.4.6 (Real Unit Distance Graph)**.** A graph $\mathcal{G}$ is called real unit distance graph if its vertex set is $\mathbb{R}^2$.

2.5 Vertex and Edge: Delete Or Add

Definition 2.5.1 (Edge-deleted Subgraph)**.** A graph $\mathcal{G} \setminus e$ is called edge-deleted subgraph.

Definition 2.5.2 (Vertex-deleted Subgraph)**.** A graph $\mathcal{G} \setminus v$ is called vertex-deleted subgraph.

Definition 2.5.3 (Subgraph: Function-Orientated Style)**.** A graph $\mathcal{S}$ is called subgraph of $\mathcal{G}$ if $\mathcal{V}(\mathcal{S}) \subseteq \mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{S}) \subseteq \mathcal{E}(\mathcal{G})$ and $\mathcal{F}_{\mathcal{S}}$ is the restriction of $\mathcal{F}_{\mathcal{G}}$ to $\mathcal{E}(\mathcal{S})$.

Theorem 2.5.4. *If all vertices have degree at least two in a graph, then graph has a cycle.*

Proof. If the graph isn't simple, then it has either loop making cycle from order one or parallel edges making a cycle from order two.

If the graph is simple, then let $\mathcal{P}$ is a longest path from one vertex to another vertex. By all vertices have the degree at least two, the ending vertex of $\mathcal{P}$ has the edge either with a vertex in $\mathcal{P}$ making a cycle or a vertex out of $\mathcal{P}$ which is contradiction with choosing $\mathcal{P}$ as longest path in graph. ■

Definition 2.5.5 (Maximal)**.** $S \in \mathcal{S}$ is called maximal if no member of $\mathcal{S}$ properly contains S where $\mathcal{S}$ is a family of subgraphs of a graph $\mathcal{G}$.

Definition 2.5.6 (Minimal)**.** $S \in \mathcal{S}$ is called minimal if no member of $\mathcal{S}$ is properly contained in S where $\mathcal{S}$ is a family of subgraphs of a graph $\mathcal{G}$.

Related Classes Of Graphs

Definition 2.5.7 (Acyclic)**.** A graph $\mathcal{G}$ is called acyclic if it doesn't contain a cycle.

Definition 2.5.8 (Digraph From A Poset)**.** A digraph is denoted by $\mathcal{D}(\mathcal{P})$ where $\mathcal{P} = (\mathcal{X}, <)$ is a poset. The vertex set is $\mathcal{X}$ and xy is edge if $x < y$.

Exercises

1. A digraph $\mathcal{D}(\mathcal{P})$ from poset $\mathcal{P}$ is acyclic and transitive where transitive means xz is an edge if both xy and yz are the edges.
2. Let $\mathcal{D}(\mathcal{P})$ be a digraph from poset $\mathcal{P}$. Then acyclic tournament is transitive tournament.
3. Let $\mathcal{D}(\mathcal{P})$ be a digraph from poset $\mathcal{P}$. Then chains in $\mathcal{P}$ are transitive subtournament.

Definition 2.5.9 (Topological Sort)**.** A digraph is called topological sort if there is an linear ordering of its vertices such that for every edge, its arrow precedes its starting point in the ordering.

Definition 2.5.10 (Triangle-Free Graph)**.** A graph is called triangle-free graph if it contains no triangles.

Definition 2.5.11 (Monochromatic)**.** A complete graph is called monochromatic if all of its edges have same color so all are red or all are blue.

Definition 2.5.12 (Bichromatic)**.** A complete graph is called bichromatic if all edges are either red or blue.

Definition 2.5.13 (Spanning Subgraph)**.** A graph is denoted by $\mathcal{G} \setminus \mathcal{S}$ and is called spanning subgraph if it's obtained from a graph by edge deletion only where $\mathcal{S}$ is the set of deleted edges.

Definition 2.5.14 (Spanning Supergraph)**.** A graph is denoted by $\mathcal{G} + \mathcal{S}$ and is called spanning supergraph if it's obtained from a graph by edge addition only where $\mathcal{S}$ is the set of additive edges.

Definition 2.5.15 (Joint Graph)**.** A graph is denoted by $\mathcal{G} \vee \mathcal{S}$ and is called join graph if it's obtained from an union graph of $\mathcal{G}$ and $\mathcal{S}$ and adding all possible edges amid these two graphs.

Exercises

1. $\mathcal{C}_n \vee \mathcal{K}_1 = \mathcal{W}_n$

Definition 2.5.16 (Hamilton Path)**.** Spanning path is called Hamilton path.

Definition 2.5.17 (Hamilton Cycle)**.** Spanning cycle is called Hamilton cycle.

Definition 2.5.18 ($k-$factor)**.** Spanning $k-$regular subgraph is called $k-$factor.

Definition 2.5.19 (Symmetric Difference Graph)**.** A spanning subgraph is denoted by $\mathcal{S}_1 \Delta \mathcal{S}_2$ and is called symmetric difference graph if it's obtained from two spanning subgraphs of $\mathcal{S}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{S}_2 = (\mathcal{V}, \mathcal{E}_2)$ where edge set is $\mathcal{E}_1 \Delta \mathcal{E}_2$.

Definition 2.5.20 (Induced Subgraph)**.** A graph is denoted by $\mathcal{G} - \mathcal{S}$ and is called induced subgraph if it's obtained from a graph by vertex deletion only where $\mathcal{S}$ is the set of deleted vertices.

Definition 2.5.21 (Induced Subgraph)**.** A graph is denoted by $\mathcal{G}[\mathcal{Y}]$ and is called induced subgraph by $\mathcal{Y}$ if it's obtained from graph $\mathcal{G}$ where vertex set is $\mathcal{Y}$ and edge set is all edges of $\mathcal{G}$ which have both ends in $\mathcal{Y}$.

Exercises

1. Every graph with average degree at least $2k$, where k is a positive integer, has an induced subgraph with minimum degree at least $k+1$.

Definition 2.5.22 (Weighted Graph)**.** A graph is denoted by $(\mathcal{G}, \omega)$ and is called weighted graph if each edge is corresponded to a real number $\omega(e)$ which is called its weight and $\mathcal{G}$ is with these weights on its edges where $\omega : \mathcal{E} \to \mathbb{R}$ and its denoted by $\mathbb{R}^{\mathcal{E}}$. When the weights are rational numbers, it's denoted by $\mathbb{Q}^{\mathcal{E}}$.

Definition 2.5.23 (Graph By Vertex Identification)**.** A graph is denoted by $\mathcal{G} \setminus \{x, y\}$ if the adjacent vertices x and y is to replace by a single vertex.

Definition 2.5.24 (Graph By Edge Contraction)**.** A graph is denoted by $\mathcal{G} \setminus e$ the adjacent edges e and e' is to replace by a single edge.

Definition 2.5.25 (Graph By Vertex Splitting)**.** A graph is made by vertex splitting v if v is to replace by two adjacent vertices.

Definition 2.5.26 (Graph By Edge Subdivision)**.** A graph is made by edge subdivision e if e is to delete and add new vertex joining to ends of e.

Definition 2.5.27 (Decomposition)**.** A family of graphs $\mathcal{F}$ is called a decomposition of a graph $\mathcal{G}$ if it only has edge-disjoint subgraphs of $\mathcal{G}$ such that $\cup_{F\in\mathcal{F}}\mathcal{E}(F) = \mathcal{E}(\mathcal{G})$.

Definition 2.5.28 (Path Decomposition)**.** A decomposition of a graph $\mathcal{G}$ is called path decomposition if it contains entirely of paths.

Definition 2.5.29 (Cycle Decomposition)**.** A decomposition of a graph $\mathcal{G}$ is called cycle decomposition if it contains entirely of cycles.

Definition 2.5.30 (Even Graph)**.** A graph in which each vertex has even degree is called an even graph.

Exercises

1. Every loopless graph has a trivial path decomposition, into paths of length one.
2. A graph which admits a cycle decomposition is necessarily even.
3. A graph admits a cycle if and only if it's even.

Definition 2.5.31 (Cover)**.** A family of subgraphs $\mathcal{F}$ is called covering or cover of a graph $\mathcal{G}$ if it isn't necessarily edge-disjoint and $\cup_{F\in\mathcal{F}}\mathcal{E}(F) = \mathcal{E}(\mathcal{G})$.

Definition 2.5.32 (Uniform)**.** A covering is called uniform of a graph $\mathcal{G}$ if it covers each edge of $\mathcal{G}$ the same number of times.

Definition 2.5.33 (k-cover)**.** A covering is called k-cover of a graph $\mathcal{G}$ if it covers each edge of $\mathcal{G}$ k-times. 2-cover is called a double cover. If family of subgraphs $\mathcal{F}$ only consists paths, it's called path covering. If family of subgraphs $\mathcal{F}$ only consists cycles, it's called cycle covering.

Exercises

1. 1-cover is a decomposition.

Definition 2.5.34 (Even Digraph)**.** A digraph $\mathcal{D}$ is called even if $d^-(v) = d^+(v)$ for each vertex $v \in \mathcal{V}$.

Definition 2.5.35 (Hypomorphic)**.** Two graphs $\mathcal{G}$ and $\mathcal{H}$ are called hypomorphic if for all $v \in \mathcal{V}$, their vertex-deleted subgraphs $\mathcal{G} - v$ and $\mathcal{H} - v$ are isomorphic.

Definition 2.5.36 (Reconstruction)**.** Any graph which is hypomorphic to $\mathcal{G}$ is called a reconstruction of $\mathcal{G}$.

Definition 2.5.37 (Reconstructible)**.** A graph is called reconstructible if any reconstruction of $\mathcal{G}$ is isomorphic to $\mathcal{G}$.

Exercises

1. Regular graphs are reconstructible.

2. Disconnected graphs are reconstructible.

Definition 2.5.38 (Recognizable)**.** A class $\mathcal{C}$ of graphs is called recognizable if for each graph $\mathcal{G} \in \mathcal{C}$, every reconstruction of $\mathcal{G}$ belongs to $\mathcal{C}$.

Definition 2.5.39 (Weakly Reconstructible)**.** A class $\mathcal{C}$ of graphs is called weakly reconstructible if for each graph $\mathcal{G} \in \mathcal{C}$, every reconstruction of $\mathcal{G}$ belongs to $\mathcal{C}$ is isomorphic to $\mathcal{G}$.

Definition 2.5.40 (Switching Of The Graph)**.** A graph is called a switching of the graph if it's obtained by switching a vertex where switching a vertex means to exchange its set of neighbors and non-neighbors. A collection of switchings of a graph is called a deck of graph.

Definition 2.5.41 (Deck Of The Graph)**.** A collection of switchings of a graph is called a deck of graph.

Definition 2.5.42 (kth power)**.** A graph is denoted by $\mathcal{G}^k$ and is called kth power of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where its vertex set is $\mathcal{V}$ and two distinct vertices are adjacent in $\mathcal{G}^k$ if their distance in $\mathcal{G}$ is at most k.

Definition 2.5.43 (Cage)**.** A $k-$regular graph of girth g with at least possible number of vertices is called a $(k, g)-$cage. A $(3, g)-$cage is referred to $g-$cage.

Definition 2.5.44 (Tutte-Coxeter Graph)**.** A $8-$cage is called Tutte-Coxeter Graph.

Definition 2.5.45 (t-arc-transitive)**.** A simple connected graph $\mathcal{G}$ is called t-arc-transitive if given any two t-arcs $v_0v_1 \cdots v_t$ and $w_0w_1 \cdots w_t$, there is an automorphism of $\mathcal{G}$ which maps v_i to w_i, for $0 \leq i \leq t$.

Exercises

1. $\mathcal{K}_{3,3}$ is 2-arc-transitive.

2. Petersen graph is 3-arc-transitive.

3. Heawood graph is 4-arc-transitive.

4. Tutte-Coxeter graph is 5-arc-transitive.

5. There are no t-arc-transitive cubic graphs when $t > 5$.

Definition 2.5.46 (t-arc-transitive)**.** A simple connected graph $\mathcal{G}$ is called t-arc-transitive if given any two t-arcs $v_0v_1 \cdots v_t$ and $w_0w_1 \cdots w_t$, there is an automorphism of $\mathcal{G}$ which maps v_i to w_i, for $0 \leq i \leq t$.

Definition 2.5.47 (Eulerian)**.** A connected graph is called eulerian if there's a closed walk which have all edges of $\mathcal{G}$.

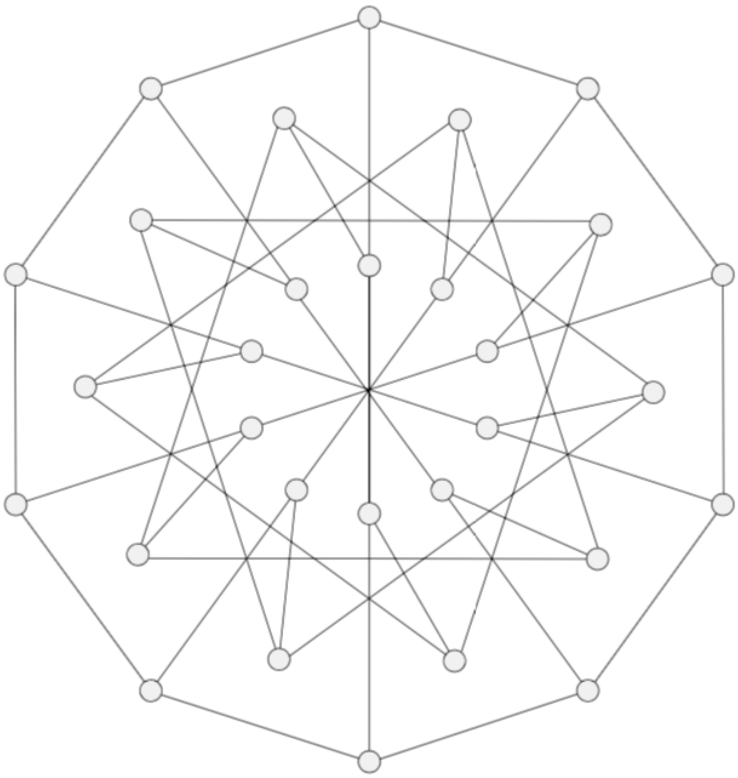


Figure 2.9: 8−cage: Tutte-Coxeter Graph

G9

Exercises

1. A connected graph is eulerian if and only if it's even.

Definition 2.5.48 (Dominating Subgraph)**.** A subgraph $\mathcal{F}$ of a graph $\mathcal{G}$ is called dominating if every edge of $\mathcal{G}$ has at least one end in $\mathcal{F}$.

Exercises

1. $\mathcal{L}(\mathcal{G})$ is hamiltonian if and only if $\mathcal{G}$ has a dominating eulerian subgraph.

Definition 2.5.49 (De Bruijn-Good Digraph)**.** A digraph is denoted by $\mathcal{BG}_n$ and it's called De Bruijn-Good digraph if the vertex set is the set of all binary sequences of length n, vertex $a_1a_2\cdots a_n$ being joined to vertex $b_1b_2\cdots b_n$ if and only if $a_{i+1} = b_i$ for $1 \leq i \leq n-1$.

Exercises

1. $\mathcal{BG}_n$ is an eulerian digraph of order 2^n and directed diameter n.

Definition 2.5.50 (Acyclic Graph)**.** A graph is called acyclic graph if it contains no cycles.

Definition 2.5.51 (Tree Graph)**.** A graph is called tree graph if it's connected acyclic graph.

Definition 2.5.52 (Distance Tree)**.** A graph is called distance tree if it's spanning $x-$tree $\mathcal{T}$ of a graph $\mathcal{G}$ and $d_{\mathcal{T}}(x,v) = d_{\mathcal{G}}(x,v)$ for all $v \in \mathcal{V}$.

Definition 2.5.53 (Fan)**.** A graph is called fan if it's $\mathcal{P} \vee \mathcal{K}_1$.

Definition 2.5.54 (Cotree)**.** A graph is denoted by $\bar{\mathcal{T}}$ and it's called cotree if it's the complement of a spanning tree $\mathcal{T}$.

Definition 2.5.55 (Matroid)**.** A graph is denoted by $(\mathcal{E},\mathcal{B})$ and it's called matroid if $\mathcal{E}$ is a finite set and $\mathcal{B}$ is a nonempty family of subsets of $\mathcal{E}$ with exchange property, which is

$$\text{If } B_1, B_2 \in \mathcal{B} \text{ and } e \in B_1 \setminus B_2$$
$$\text{then there exists } f \in B_2 \setminus B_1 \text{ such that } (B_1 \setminus \{e\}) \cup \{f\} \in \mathcal{B}$$

Definition 2.5.56 (Separable Graph)**.** A graph is called nonseparable if it's connected and has no separating vertices; otherwise, it's separable. Where a separation of a connected graph is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common which is called a separating vertex.

Definition 2.5.57 (Block Graph)**.** A subgraph is called block if it's nonseparable and maximal with respect to this property.

Definition 2.5.58 (Block Tree)**.** A bipartite graph $(\mathcal{B},\mathcal{S})$ obtained from a graph $\mathcal{G}$ is called block tree where $\mathcal{B}$ is the set of blocks of $\mathcal{G}$ and $\mathcal{S}$ is the set of separating vertices of $\mathcal{G}$. A block $\mathcal{B}$ and a separating vertex v have edge if $\mathcal{B}$ contains v. Blocks corresponding to leaves are called end blocks and the vertex which isn't separating vertex is called internal vertex.

2.6 Various Types Of Numbers

Definition 2.6.1 (Connectivity)**.** A number is denoted by κ is called connectivity of $\mathcal{G}$ if it's the maximum number k for which $\mathcal{G}$ is $k-$connected.

Definition 2.6.2 (Local Connectivity)**.** A number is denoted by $p(x,y)$ is called is local connectivity amid distinct vertices x and y which is the maximum number of edges amid all $xy-$paths.

Graphs Based On Numbers

Definition 2.6.3 ($k-$connected)**.** A nontrivial graph $\mathcal{G}$ is called $k-$connected if $p(x,y) \geq k$ for any two distinct vertices x and y.

Some Classes Of Graphs

Definition 2.6.4 (Planar)**.** A graph is called planar if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is called a planar embedding of the graph.

Definition 2.6.5 (Face-regular)**.** A planar graph is called face-regular if all of its faces have the same degree.

Sets And Numbers

Definition 2.6.6 (Stability Number)**.** A set of vertices is called stable set if no couple of vertices are adjacent. The cardinality of a maximum stable set in a graph $\mathcal{G}$ is called the stability number of $\mathcal{G}$ and is denoted by $\alpha(\mathcal{G})$.

Definition 2.6.7 (Covering Number)**.** A set of vertices is called covering set if they're incident to all edges of graph. The cardinality of a minimum covering set in a graph $\mathcal{G}$ is called the covering number of $\mathcal{G}$ and is denoted by $\beta(\mathcal{G})$.

Definition 2.6.8 (Clique Number)**.** A set of vertices is called clique set if they're mutually adjacent. The cardinality of a maximum covering set in a graph $\mathcal{G}$ is called the covering number of $\mathcal{G}$ and is denoted by $\omega(\mathcal{G})$.

CHAPTER 3

Ideas And The Approaches

Words are related to each other by different types of tools

Example 3.0.1. Consider $f : \{\alpha, \beta, \eta\} \to \{a, b, c\}$. Thus it's a graph. In the terms of simple graph, it represents one graph. One edge from one vertex to it, is called loop. Thus there's one graph with three loops and it's 3-connected graph which has three components. In Listing (3), the python code is used to engage this concept. And in Listing (3), the TeX code is used where the code is about the Figures.

lst:code_direct

Listing 3.1: Python Code

```
i=3
print (i*i)
if i==3:
    def Ver(i):
        print ("One simple graph with number of edges:", i)

x=input('i=')
n=7;
x=8;
if n%3==1:
   print ("The number of simple graphs is", 2^n)
elif x%3==2:
   print ("The number of simple graphs is", 2^x)

elif i%3==0:
    print ("The number of simple graphs is", 2^i)
else:
        print (x, ("is odd"))

print ("Number is", x)
```

List1
lst:code_direct

Listing 3.2: TeX Code

```
\begin{figure}
\subfigure[Simple Graph]{
\definecolor{uduff}{rgb}{0.30196078431372547,0.30196078431372547,1.}
\begin{tikzpicture}[line cap=round,line join=round,>=triangle 45,
x=1.0cm,y=1.0cm, scale=0.2]
\clip(-4.3,-3.08) rectangle (7.3,6.3);
\draw [line width=2.pt] (-1.92,3.12)-- (0.24,0.4);
\draw [line width=2.pt] (0.24,0.4)-- (4.18,2.94);
\draw [line width=2.pt] (4.18,2.94)-- (-1.92,3.12);
\draw (0.72,3.86) node[anchor=north west] {$\alpha$};
\draw (2.1,1.84) node[anchor=north west] {$\beta$};
\draw (-1.4,2.18) node[anchor=north west] {$\eta$};
```

```
\draw (-2.14,3.92) node[anchor=north west] {$a$};
\draw (4.32,3.48) node[anchor=north west] {$b$};
\draw (0.02,0.58) node[anchor=north west] {$c$};
\begin{scriptsize}
\draw [fill=uududff] (-1.92,3.12) circle (2.5pt);
\draw [fill=uududff] (0.24,0.4) circle (2.5pt);
\draw [fill=uududff] (4.18,2.94) circle (2.5pt);
\end{scriptsize}
\end{tikzpicture}
\subfigure[Graph With Three Loops]{
\definecolor{uududff}{rgb}{0.30196078431372547,0.30196078431372547,1.}
\begin{tikzpicture}[line cap=round,line join=round,>=triangle 45,
x=1.0cm,y=1.0cm,scale=0.3]
\clip(-4.3,-3.08) rectangle (7.3,6.3);
\draw (-2.12,4.84) node[anchor=north west] {$\alpha$};
\draw (5.14,4.2) node[anchor=north west] {$\beta$};
\draw (-1.1,0.26) node[anchor=north west] {$\eta$};
\draw (-2.14,3.92) node[anchor=north west] {$a$};
\draw (4.32,3.48) node[anchor=north west] {$b$};
\draw (0.02,0.58) node[anchor=north west] {$c$};
\draw [line width=2.pt] (-1.92,3.12) circle (1.0283968105745955cm);
\draw [line width=2.pt] (4.18,2.94) circle (1.0662082348209463cm);
\draw [line width=2.pt] (0.24,0.4) circle (0.9372299611087977cm);
\begin{scriptsize}
\draw [fill=uududff] (-1.92,3.12) circle (2.5pt);
\draw [fill=uududff] (0.24,0.4) circle (2.5pt);
\draw [fill=uududff] (4.18,2.94) circle (2.5pt);
\end{scriptsize}
\end{tikzpicture}
\caption{Two Different Graphs}
\end{figure}
```

List2

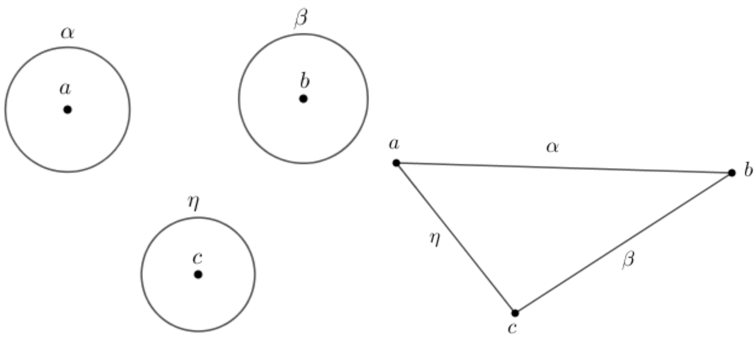


Figure 3.1: Simple Graphs And A Graph With Three Loops

G1011

Theorem 3.0.2. *Let n be a positive integer. Then there are only n numbers of labelled simple graph where n is both the number of vertices and the number of edges.*

Theorem 3.0.3. *Let $\mathcal{I}$ be an identity function. Then it represents one labelled simple graph.*

Theorem 3.0.4. *Let f be a map. If the cardinality of domain is greater than the cardinality of image, then it doesn't represent any labelled simple graph.*

Theorem 3.0.5. *Let f be a map. If there's a labelled simple graph, then the cardinality of domain is greater than the cardinality of image.*

Example 3.0.6. Let $\mathcal{V}$ be a set $\{1, 2, 3\}$. Thus the complete graph could be like functions summation, minus, production, and division as Figures (3.2). There are infinite graphs on a given set $\{1, 2, 3\}$ but $2^3 = 8$ are simple graphs.

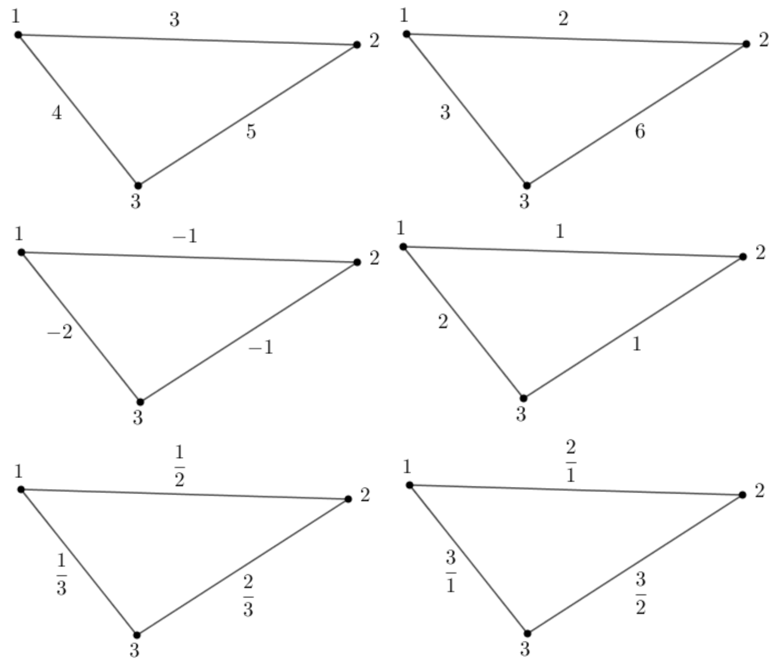


Figure 3.2: Simple Graphs: Summation, Minus, Production, And Division

G1217

Theorem 3.0.7. *Let n be a positive integer. Consider $\mathbb{N}_n = \{1, 2, \cdots, n\}$. The power set of $\mathbb{N}_n$ is all possibly simple graphs with the exception complete graph.*

Example 3.0.8. Consider $\mathbb{N}_3$. Then $\{1, 2\}$ is the graph which the edge amid vertices 1 and 2 holds but the vertex 3 is isolated. By using this notation, $\{1, 2, 3\}$ aren't complete graph because there's no edge amid the vertex 1 and the vertex 3, although, vertex 1 and 2 as has edge as vertex 2 and vertex 3.

Theorem 3.0.9. *Let n be the number of objects. The power set of $\mathbb{N}_n$ is all possibly simple graphs with the exception complete graph.*

Example 3.0.10. Let $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a matrix. It's a graph but it isn't simple graph because at third object, there's one loop if the vertical rows as are the set of vertices as horizontal columns. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a graph with three loops. $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ have the entries on upper the diagonal and downward the diagonal but both represent a complete simple graph.

Theorem 3.0.11. *For any given vertex, its list of vertices are the vertices with nonzero number in the corresponded column or corresponded row.*

Theorem 3.0.12. *For any given vertex, the cardinality of its list of vertices are the summation of all numbers belongs to its column.*

Theorem 3.0.13. *For any given vertex, summation of all numbers belongs to its column, are equal with summation of all numbers belongs to its row.*

Theorem 3.0.14. *Summation of all numbers belongs to all columns, are equal with summation of all numbers belongs to all rows. The number is double number of edges.*

Theorem 3.0.15. *For any given vertex in a simple graph, the number of its edges are equal with the number of its vertices in its list.*

Theorem 3.0.16. *For any given vertex in a simple graph, the number of its edges are equal with the number of its vertices in its list.*

R17 **Theorem 3.0.17.** *Any of zero-one square matrix represents a graph.*

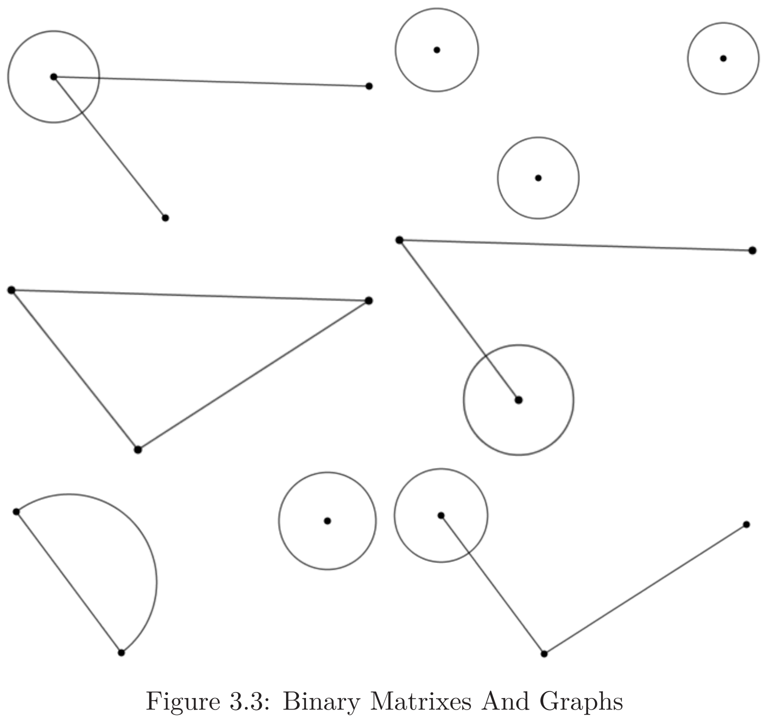


Figure 3.3: Binary Matrixes And Graphs G1823

Example 3.0.18. Consider the matrixes.

$$\begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&0\\1&0&1\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\begin{bmatrix}1&0&1\\0&0&0\\0&1&0\end{bmatrix}$$

By Theorem (3.0.17), these matrixes are graphs. Thus we want to draw the figures of these graphs as figures (3.3). Parallel edges give us the opportunity to have infinite numbers of edges amid two vertices as the 22nd item of Figures (3.3).

Theorem 3.0.19. *In matrix, the ijth place is as nonzero as jith place. Then there are at least two parallel edges.*

Theorem 3.0.20. *In the directed graphs, it's possible to get the edge in its simple graph as two edges with different directions.*

Theorem 3.0.21. *Any simple graphs could be a directed graphs where the edges are double but with different directions.*

R22 **Theorem 3.0.22.** *Any of zero-one-minus-one square matrix represents a directed graph.*

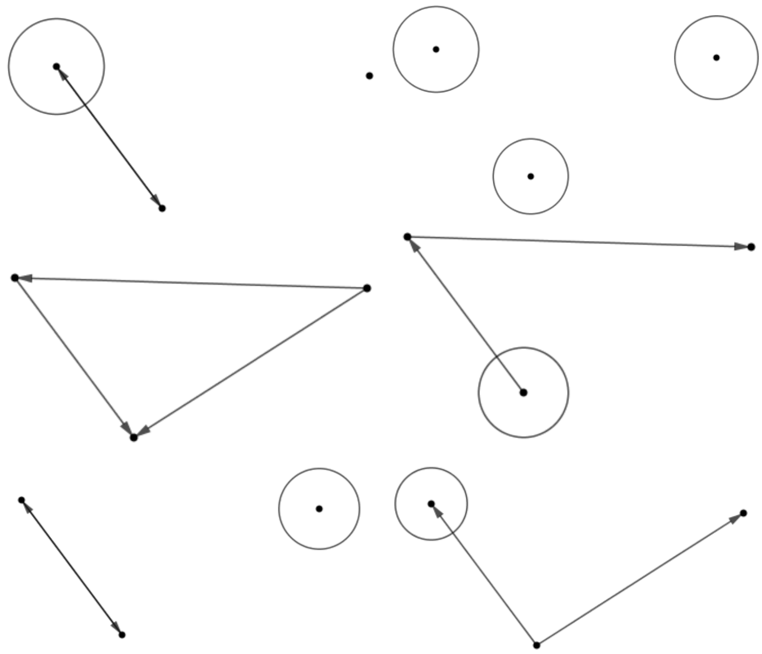


Figure 3.4: Matrixes And Directed Graphs G2429

Example 3.0.23.

$$\begin{bmatrix}-1&0&-1\\0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&1&-1\\0&0&-1\\0&0&0\end{bmatrix},\begin{bmatrix}0&-1&0\\0&0&0\\-1&0&-1\end{bmatrix},\begin{bmatrix}0&0&-1\\0&1&0\\-1&0&0\end{bmatrix},\begin{bmatrix}1&0&1\\0&0&0\\0&-1&0\end{bmatrix}$$

By Theorem (3.0.22), these matrixes are directed graphs. Thus we want to draw the figures of these directed graphs as Figures (3.4). Parallel edges give us the opportunity to have infinite numbers of edges amid two vertices as the 24th item and the 28th item of Figures (3.4).

There's one notion to have two graphs with one graph by using matrix version of a graph. Two matrixes could get summation, subtraction, production and division. This notion introduces graphs which have multiple loops and multiple parallel edges. In reverse ways, finding decomposition and packing notions when one graph is decomposition to at least two graphs or at least multiple graphs. Thus the matrixes can get numbers without restriction as its member where the numbers denotes the number of the edges amid two vertices. Thus

Theorem 3.0.24. *Every square matrix represents a graph and a directed graph.*

Example 3.0.25. Figure (3.5), shows the equation amid three matrixes which are denoted by

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 0 \\ -3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -4 & 0 & -1 \end{bmatrix}.$$

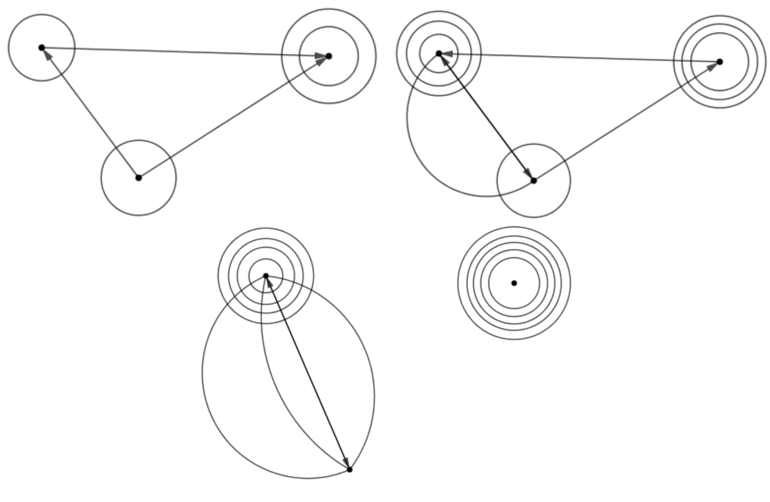


Figure 3.5: Extended Matrixes And Graphs

G3032

Theorem 3.0.26. *Consider a matrix involving $\mathcal{V}$ and $\mathcal{E}$. Then a number greater than two denotes that the edge isn't simple and it's either parallel edges or loops.*

Theorem 3.0.27. *Consider a matrix involving $\mathcal{V}$ and $\mathcal{E}$. Then summation of all numbers are equal to double number of edges*

Theorem 3.0.28. *Consider a matrix involving $\mathcal{V}$ and $\mathcal{E}$ as simple graph. Then summation of all numbers belong to one column are equal to two.*

Example 3.0.29. Another vision of a matrix is that, it represents a graph when the couple of vertical set of objects which is denoted by $\mathcal{V}$ and the couple of horizontal set of objects which is denoted by $\mathcal{E}$. Thus the number two in a matrix denotes either parallel edges or loops. Consider

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The first vertex has one outer first edge and the third vertex has one inner outer edge. The first vertex has one outer third edge but there's no destination vertex to accept one inner third edge. Thus,

Theorem 3.0.30. *Consider a matrix involving $\mathcal{V}$ and $\mathcal{E}$. Then it forever represents neither a simple graph nor directed simple graph.*

R31 **Theorem 3.0.31.** *Consider a matrix involving $\mathcal{V}$ and $\mathcal{E}$. Then it represents either a graph or directed graph.*

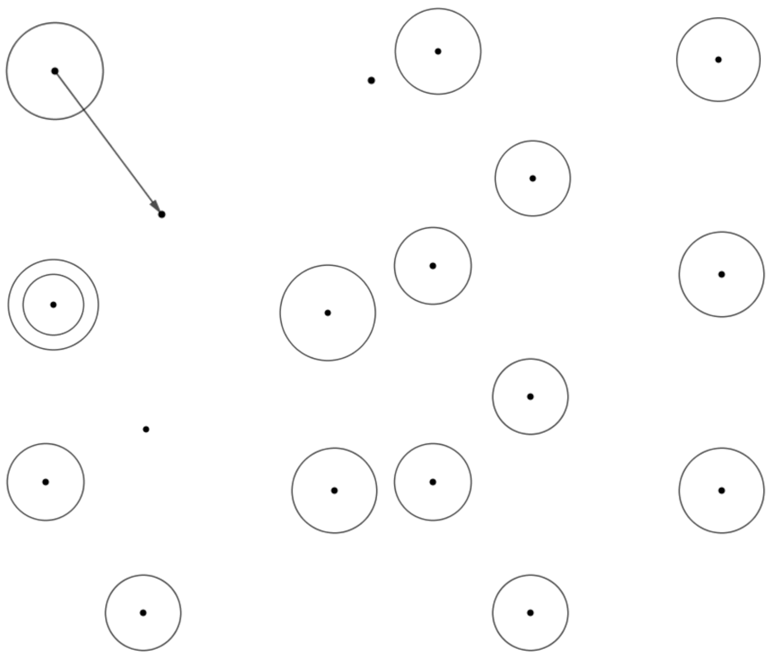


Figure 3.6: Matrixes And Directed Graphs G3338

Example 3.0.32.

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

By Theorem (3.0.22), these matrixes are directed graphs. Thus we want to draw the figures of these directed graphs as Figures (3.6). Loops give us the opportunity to have infinite numbers of edges for one vertex as the 34th item and the 36th-38th items of Figures (3.6).

Theorem 3.0.33. *The list of given vertex has the length at most $n-2$.*

Theorem 3.0.34. *The list of given vertex has the length $n-2$. Then the graph is a star graph.*

Theorem 3.0.35. *The list of given vertex has the length $n-2$ and the list has consecutive vertices. Then the graph is a wheel graph.*

Theorem 3.0.36. *For two given vertices u, v, the list of u and v has the length at most $n-1$.*

Theorem 3.0.37. *For two given vertices u, v, the list of u and v has the length $n-1$. Then the graph is a path graph.*

Theorem 3.0.38. *For two given vertices u, v, the degree list of u and v has degree at most Δ and the length at most $n-1$.*

Theorem 3.0.39. *The degree list of a given vertex has degree at most Δ and the length at most $n-2$.*

Theorem 3.0.40. *The list of numbers is graphic if and only if it's degree list.*

Theorem 3.0.41. *$\delta = \Delta$ if and only if it's $\Delta-$regular graph.*

Theorem 3.0.42. *Any cycle graph from any given order is $2-$regular graph.*

Theorem 3.0.43. *A complete graph from any given order is $(n-1)-$regular graph.*

Example 3.0.44. In Figure (3.7), the graph isn't a complete graph but it's $3-$regular graph. The number list 333333 is degree list. Thus it's graphic. It isn't a cycle graph because it has more than one cycle. Precisely, there are two cycles which are $v_1v_4v_5v_6v_1$ and $v_1v_4v_2v_1$. It isn't path graph because there're two vertices such that there is more than one path amid them. Precisely, there are two paths $v_1v_4v_5v_6$ and $v_1v_2v_3v_6$ from the vertex v_1 to the vertex v_6 and they're two lists from the vertex v_1 to the vertex v_6. $v_2v_4v_6$ is list of v_1.

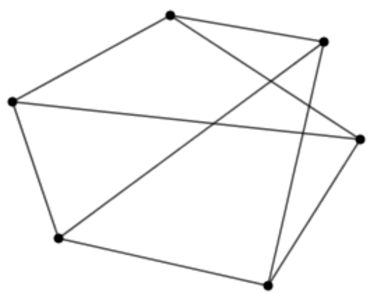


Figure 3.7: $3-$Regular Graph On 6 Vertices

G39

Theorem 3.0.45. *If n is a number in number list, then it isn't degree list in a simple graph.*

Theorem 3.0.46. *If summation of all numbers in a number list is odd, then it isn't degree list in a simple graph.*

Example 3.0.47. The number list 543258642 isn't degree list in any given simple graph with any given order. Consider 4567264. Parity of summation the all numbers is even but it isn't degree list because of attending the number 7 which is the length of number list in the number list.

Theorem 3.0.48. *If the length of number list is in the number list, then it isn't degree list in a simple graph.*

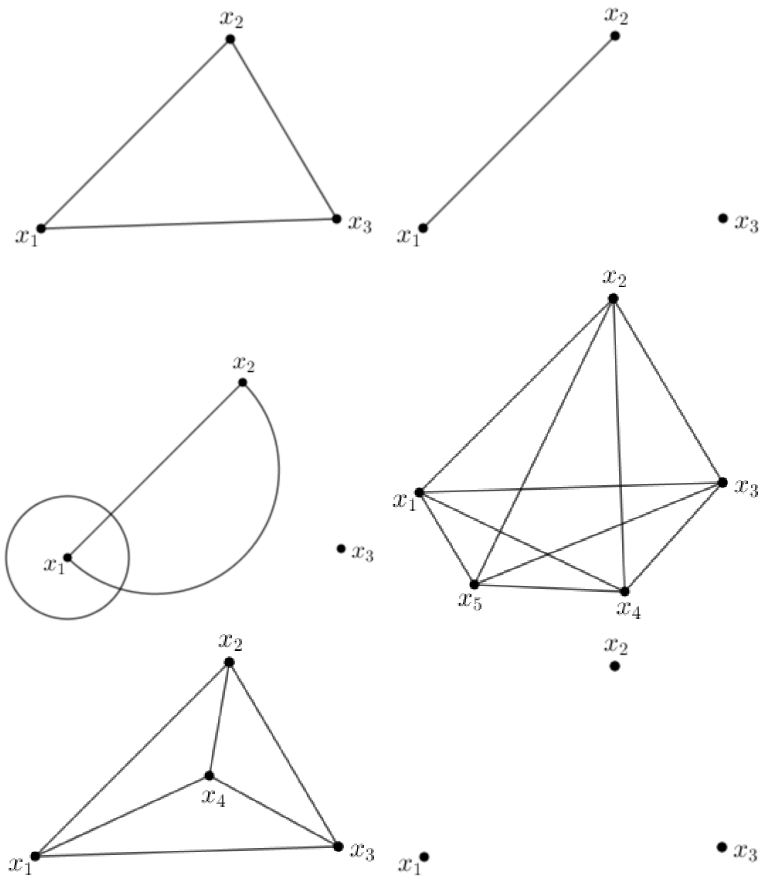


Figure 3.8: Some Classes Of Graphs

G4045

Example 3.0.49. Consider Figure (3.8). The first item is connected graph. The second item is disconnected graph which has two components. The first component has the vertex set $\{x_1, x_2\}$ and second component has the vertex set $\{x_3\}$ and it's called trivial graph. The items 1,2,4,5,6 are simple graphs but the item 3 isn't simple graph. The item 4 isn't a planar graph and it's also called complete graph from order four, K_5. The item 5 is a planar graph and it's also called K_4. The item 6 is empty graph and it isn't trivial graph and it's also called nontrivial graph. All items are nontrivial graphs but the second component from item 1, is called trivial graph. All items are finite graphs.

Theorem 3.0.50. *A graph which the second component is null graph, is called connected graph.*

Theorem 3.0.51. *A graph with vertex set,* $\mathbb{N}$ *and two vertices have edges if their parity is even, is infinite graph.*

Theorem 3.0.52. *Null graph and trivial graph are finite graph but empty graph could be either finite graph or infinite graph.*

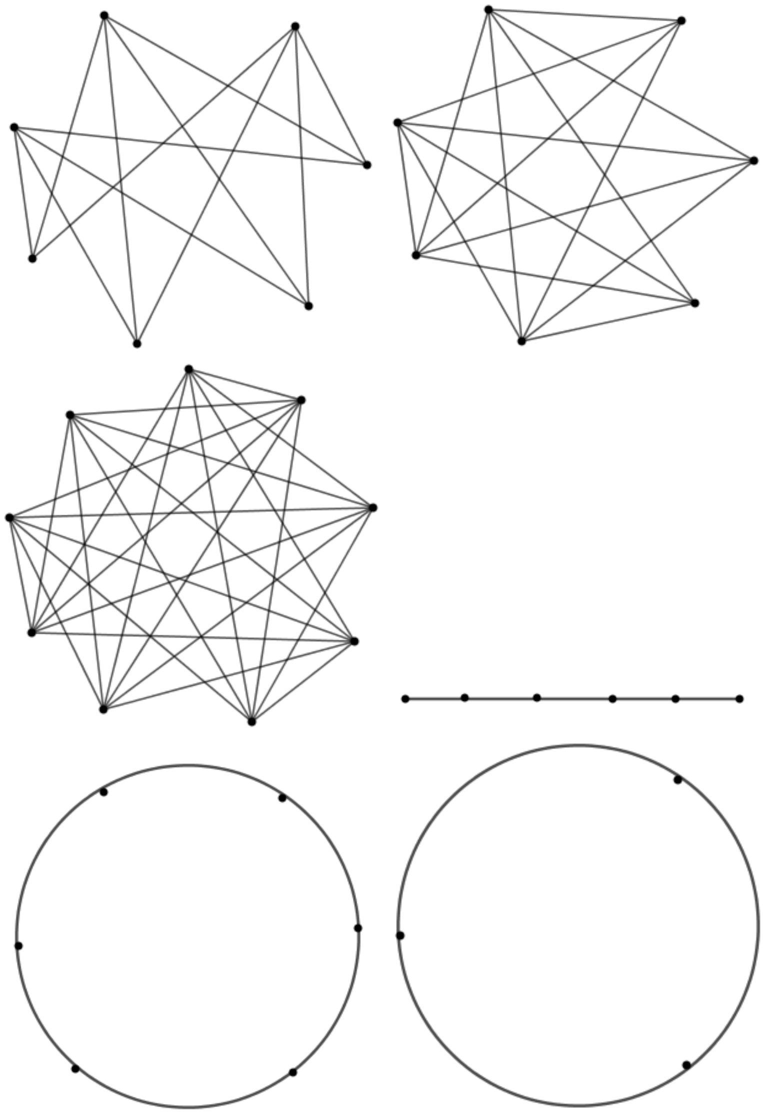


Figure 3.9: Bipartite, 3−partite, Turan, 6−path, 6−cycle, 3−cycle

G4651

Example 3.0.53. Consider Figure (3.9). The first item is bipartite because it has two parts which one part has three vertices and second part has four vertices. The second item is 3−partite because there are two parts including two vertices and one part including three vertices. Third item is Turan because every part has the three vertices and the graph is Turan from the order three. Fourth part is 6−path including seven vertices and six edges. Fifth item is 6−cycle including six vertices and six edges. The sixth item is 3−cycle including three vertices and three edges.

The fourth item is 2−partite but it isn't complete. The fifth item is 3−partite but it isn't complete. For $n \geq 3$, $n-$cycle is defined. Thus first $n-$cycle is 3−cycle. The sixth item is 3−partite but it isn't complete. The items one, two and three are complete. The sixth item is 3−partite. All items are $n-$partite but the items two, three, four, five and six aren't bipartite. The only item one is bipartite.

Theorem 3.0.54. *Turan graph is $n-$partite.*

Theorem 3.0.55. *Bipartite graph is $n-$partite.*

Theorem 3.0.56. *$m-$path is $n-$partite.*

Theorem 3.0.57. *$m-$cycle is $n-$partite.*

Theorem 3.0.58. *$m-$cycle is bipartite where m is even.*

Theorem 3.0.59. *Bipartite graph is $m-$cycle where m is even.*

Theorem 3.0.60. *Null graph and trivial graph are finite graph but empty graph could be either finite graph or infinite graph.*

The next example plays with the notions of order and the lack of it in the terms of $n-$cube and Boolean lattice.

Example 3.0.61. Consider Figure (3.10). 6−cube $\mathcal{Q}_6$, Boolean lattice $\mathcal{BL}_6$, Complement of $\mathcal{Q}_6$, Complement of $\mathcal{BL}_6$, Strongly regular graph with $(6, 4, 2, 4)$, Complement of strongly regular graph with $(6, 4, 2, 4)$ are the items, respectively. All items are defined on six vertices. One difference amid the labels, is a necessary and sufficient condition to have edges amid intended vertices.

The items of three to six are the complements of mentioned graphs. Complement is unary operation thus the input is one graph and the output is another graph which couldn't hold the property of the input. The complement of items one and two don't hold initial property. The complement of 6−cube $\mathcal{Q}_6$ isn't $n-$cube as items of one and three. The complement of Boolean lattice $\mathcal{BL}_6$ isn't Boolean lattice $\mathcal{BL}_n$ as items of two and five. The complement of strongly regular graph with $(6, 4, 2, 4)$ isn't strongly regular graph with $(6, 1, 0, 0)$ as items of three and six.

Theorem 3.0.62. *$n-$cube is a star graph if a degree one vertex is order minus one.*

Theorem 3.0.63. *$n-$cube has at most $n-1$ edges if there's one vertex whose degree is order minus one.*

Theorem 3.0.64. *There are $2^{n-1}+1$ $n-$cube if there's one vertex whose degree is order minus one.*

Theorem 3.0.65. *Boolean lattice is a star graph if there's one vertex whose degree is order minus one.*

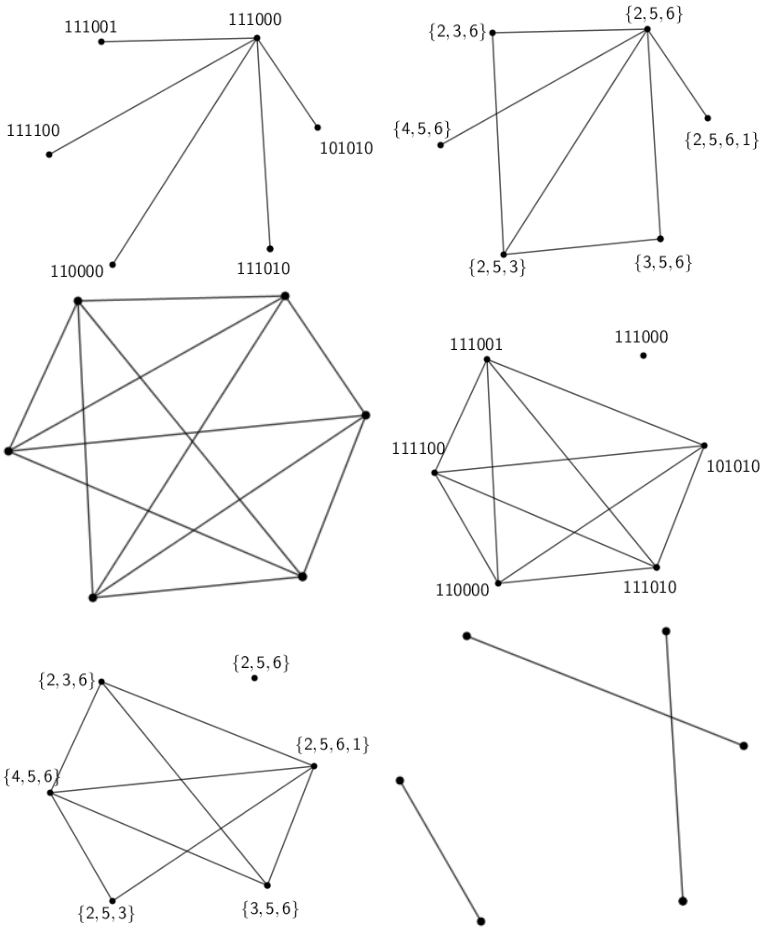


Figure 3.10: 6−cube $\mathcal{Q}_6$, Boolean lattice $\mathcal{BL}_6$, Complement of $\mathcal{Q}_6$, Complement of $\mathcal{BL}_6$, Strongly regular graph with $(6, 4, 2, 4)$, Complement of strongly regular graph with $(6, 4, 2, 4)$

G5257

Theorem 3.0.66. *Boolean lattice has at most $n-1$ edges if there's one vertex whose degree is order minus one.*

Theorem 3.0.67. *There are $2^{n-1}+1$ Boolean lattice if there's one vertex whose degree is order minus one.*

Theorem 3.0.68. *A strongly regular graph is regular.*

Theorem 3.0.69. *The complement of $n-$cube isn't forever $n-$cube.*

Theorem 3.0.70. *The complement of Boolean lattice isn't forever Boolean lattice.*

Theorem 3.0.71. *There's at least one strongly regular graph whose complement is strongly regular graph.*

Example 3.0.72. Consider Figure (3.11). 6−cycle $\mathcal{C}_6$, 6−path $\mathcal{P}_6$, 6−star $\mathcal{S}_{5,1}$,

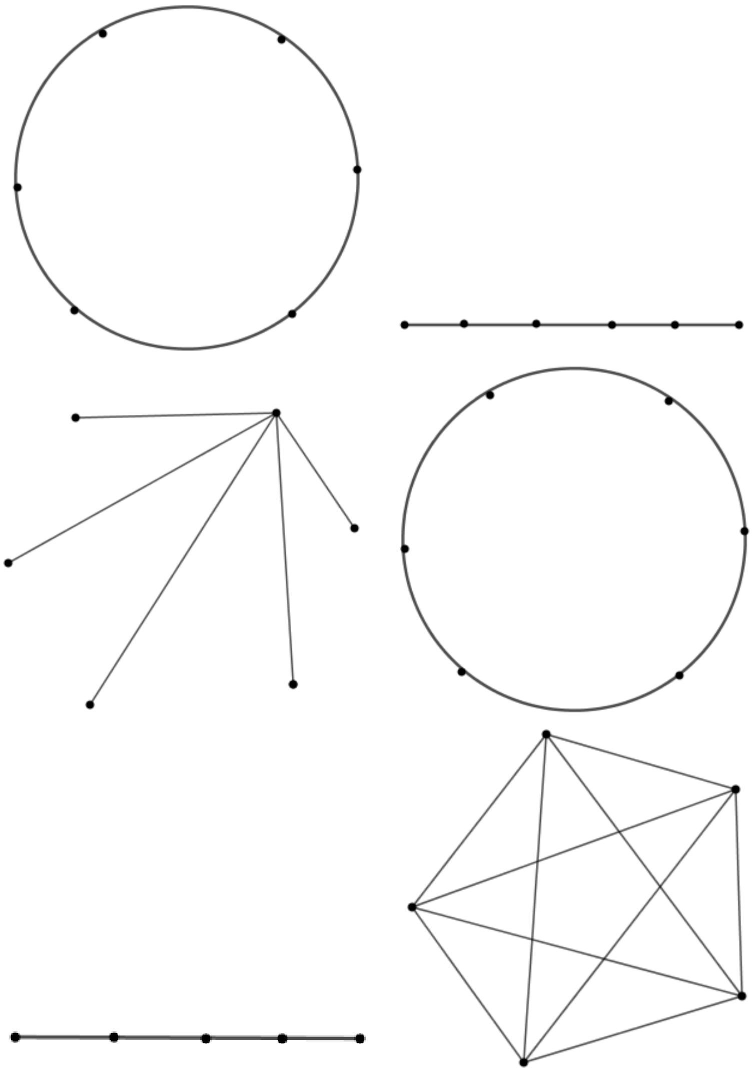


Figure 3.11: 6−cycle $\mathcal{C}_6$, 6−path $\mathcal{P}_6$, 6−star $\mathcal{S}_{5,1}$, line graph of 6−cycle $\mathcal{C}_6$, line graph of 6−path $\mathcal{P}_6$, line graph of 6−star $\mathcal{S}_{5,1}$

G5863

line graph of 6−cycle $\mathcal{C}_6$, line graph of 6−path $\mathcal{P}_6$, line graph of 6−star $\mathcal{S}_{5,1}$ are the items, respectively. All items are defined on six vertices.

The items of three to six are the line graph of mentioned graphs. Line graph is unary operation thus the input is one graph and the output is another graph which couldn't hold the property of the input as items two and three. The line graph of items two and three don't hold initial property but the item two holds the type of initial graph. The item three not only doesn't preserve initial graph but also it doesn't preserve the style of initial graph. The line graph of the item three, 6−star $\mathcal{S}_{5,1}$ is the item six, 5−complete $\mathcal{K}_5$. line graph of 6−cycle $\mathcal{C}_6$ is 6−cycle $\mathcal{C}_6$ and line graph of 6−path $\mathcal{P}_6$ is 5−path $\mathcal{P}_5$

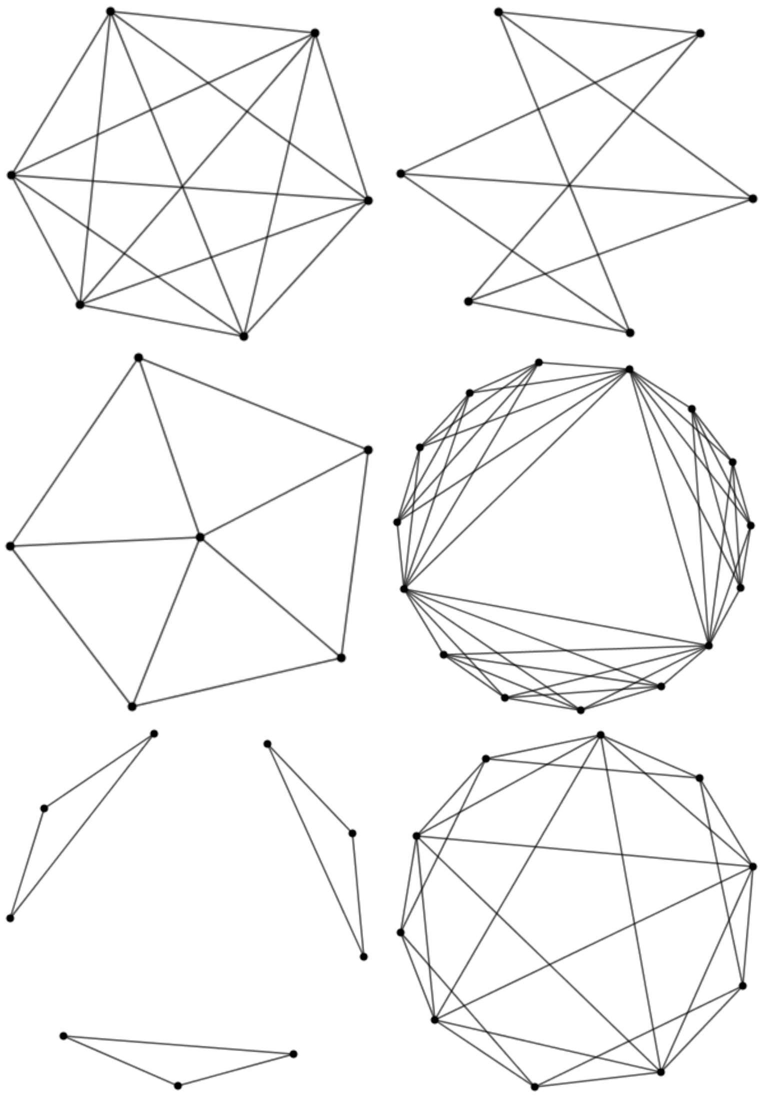


Figure 3.12: 6−complete $\mathcal{K}_6$, $(3,3)$−partite $\mathcal{K}_{3,3}$, 5−wheel $\mathcal{W}_{5,1}$, line graph of 6−complete $\mathcal{K}_6$, line graph of $(3,3)$−partite $\mathcal{K}_{3,3}$, line graph of 5−wheel $\mathcal{W}_{5,1}$

G6469

In this case, the line graph of path is path but in the case of cycle, not only a line graph of cycle is cycle but also both input and output are the same up to isomorphism. The line graph of star graph is complete.

Theorem 3.0.73. *Line graph if cycle graph is cycle graph.*

Theorem 3.0.74. *Line graph if $n-$cycle $\mathcal{C}_n$ is $n-$cycle $\mathcal{C}_{n-1}$.*

Theorem 3.0.75. *Line graph of path graph is path graph.*

Theorem 3.0.76. *Line graph of $n-$path $\mathcal{P}_n$ is $(n-1)-$path $\mathcal{P}_{n-1}$.*

Theorem 3.0.77. *Line graph of star graph is complete graph.*

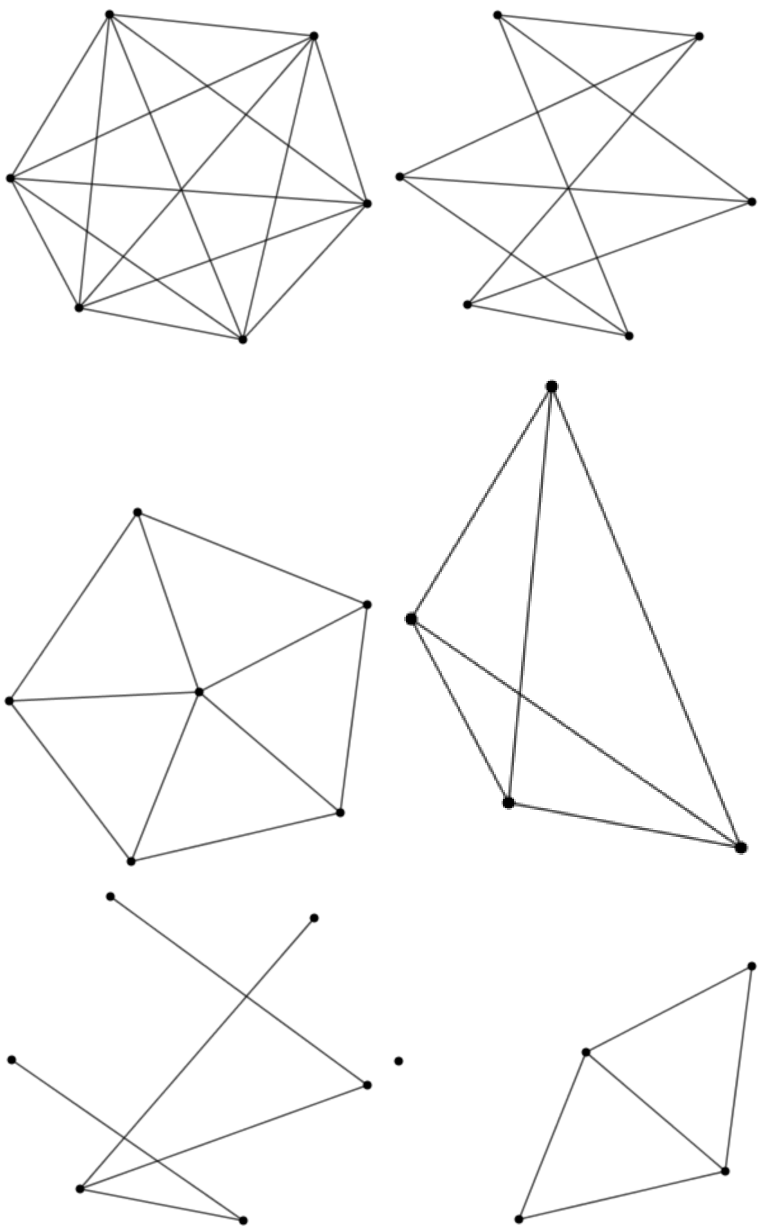


Figure 3.13: 6−complete $\mathcal{K}_6$, $(3,3)$−partite $\mathcal{K}_{3,3}$, 5−wheel $\mathcal{W}_{5,1}$, 2−vertex deletion of 6−complete $\mathcal{K}_6$, 4−edge deletion of $(3,3)$−partite $\mathcal{K}_{3,3}$, 2−vertex deletion and 5−edge deletion of 5−wheel $\mathcal{W}_{5,1}$ G7075

Theorem 3.0.78. *Line graph of $n-$star $\mathcal{S}_{n-1,1}$ is $(n-1)-$complete $\mathcal{K}_{n-1}$.*

Example 3.0.79. Consider Figure (3.12). 6−complete $\mathcal{K}_6$, $(3,3)$−partite $\mathcal{K}_{3,3}$, 5−wheel $\mathcal{W}_{5,1}$, line graph of 6−complete $\mathcal{K}_6$, line graph of $(3,3)$−partite $\mathcal{K}_{3,3}$, line graph of 5−wheel $\mathcal{W}_{5,1}$ are the items, respectively. All items are defined on six vertices.

The items of three to six are the line graph of mentioned graphs. Line graph is unary operation thus the input is one graph and the output is another graph which couldn't hold the property of the input. The line graph of items don't hold initial property. The items not only don't preserve initial graph but also they don't preserve the style of initial graph.

Example 3.0.80. Consider Figure (3.13). $6-$complete $\mathcal{K}_6$, $(3,3)-$partite $\mathcal{K}_{3,3}$, $5-$wheel $\mathcal{W}_{5,1}$, $2-$vertex deletion of $6-$complete $\mathcal{K}_6$, $4-$edge deletion of $(3,3)-$partite $\mathcal{K}_{3,3}$, $2-$vertex deletion and $5-$edge deletion of $5-$wheel $\mathcal{W}_{5,1}$ are the items, respectively. All items are defined on six vertices.

The items of three to six are the line graph of mentioned graphs. $n-$vertex deletion and $n-$edge deletion are unary operations thus the input is one graph and the output is another graph which couldn't hold the property of the input. The $n-$vertex deletion and $n-$edge deletion of items don't hold initial property. The items not only don't preserve initial graph but also they don't preserve the style of initial graph. $n-$vertex deletion and $n-$edge deletion of a graph make the graphs which are subgraphs of initial graph. All items are countable graphs. All items have cycle with the exception item five. The item five is acyclic graph. All items are triangle-free graph with the exception item five. The item five is spanning subgraph of item two.

Example 3.0.81. Consider Figure (3.14). $6-$complete $\mathcal{K}_6$, $(3,3)-$partite $\mathcal{K}_{3,3}$, $5-$wheel $\mathcal{W}_{5,1}$, Hamiltonian cycle of $6-$complete $\mathcal{K}_6$, Hamiltonian cycle of $(3,3)-$partite $\mathcal{K}_{3,3}$, Hamiltonian cycle of $5-$wheel $\mathcal{W}_{5,1}$ are the items, respectively. All items are defined on six vertices.

The items of three to six are the Hamiltonian cycle of mentioned graphs. Hamiltonian cycle is unary operations thus the input is one graph and the output is another graph which couldn't hold the property of the input as items.

Theorem 3.0.82. *$n-$complete $\mathcal{K}_n$ has Hamiltonian cycle.*

Theorem 3.0.83. *$n-$complete $\mathcal{K}_n$ has Hamiltonian path.*

Theorem 3.0.84. *$(n,m)-$partite $\mathcal{K}_{n,m}$ has Hamiltonian cycle where $n, m \neq 1$.*

Theorem 3.0.85. *$(n,m)-$partite $\mathcal{K}_{n,m}$ has Hamiltonian path where $n, m \neq 1$.*

Theorem 3.0.86. *$n-$wheel $\mathcal{W}_{n,1}$ has Hamiltonian cycle.*

Theorem 3.0.87. *$n-$wheel $\mathcal{W}_{n,1}$ has Hamiltonian path.*

Theorem 3.0.88. *$n-$star $\mathcal{S}_{n,1}$ doesn't have Hamiltonian cycle.*

Theorem 3.0.89. *$n-$star $\mathcal{S}_{n,1}$ doesn't have Hamiltonian path.*

Theorem 3.0.90. *Peterson graph has Hamiltonian cycle.*

Theorem 3.0.91. *Peterson graph has Hamiltonian path.*

Theorem 3.0.92. *$n-$regular graph has Hamiltonian cycle.*

Theorem 3.0.93. *$n-$regular graph has Hamiltonian path.*

Theorem 3.0.94. *Empty graph doesn't have Hamiltonian cycle.*

Theorem 3.0.95. *Empty graph doesn't have Hamiltonian path.*

Theorem 3.0.96. *Trivial graph has Hamiltonian cycle.*

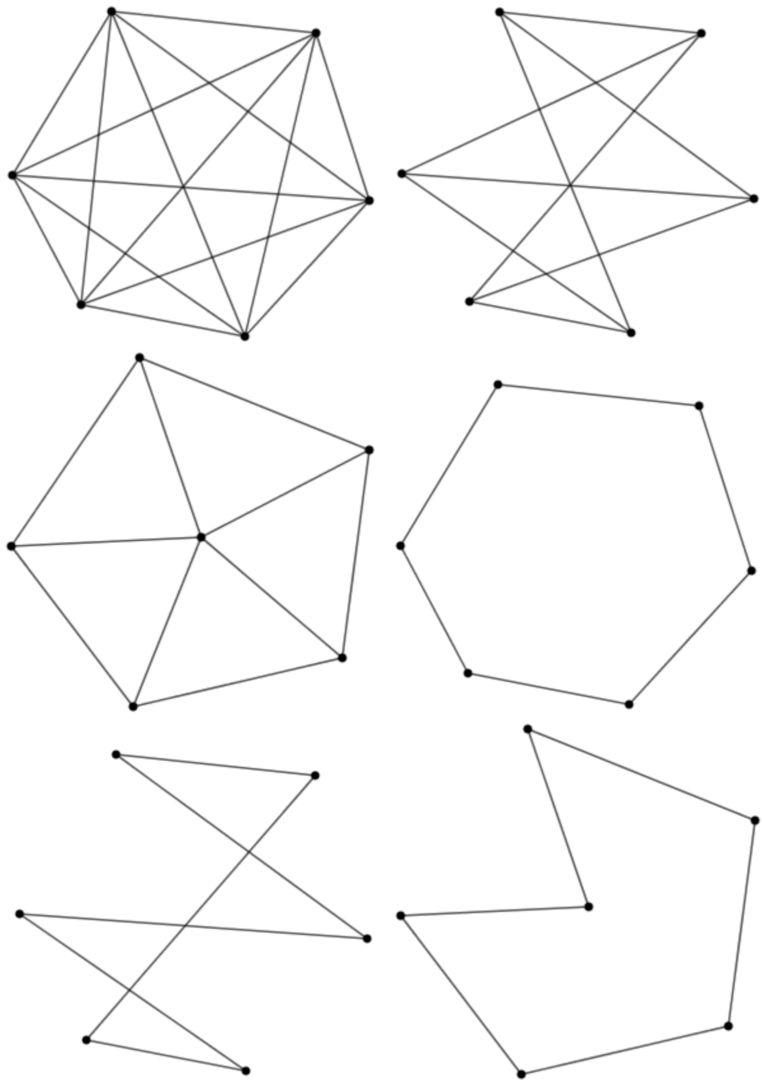


Figure 3.14: 6−complete $\mathcal{K}_6$, $(3,3)$−partite $\mathcal{K}_{3,3}$, 5−wheel $\mathcal{W}_{5,1}$, Hamiltonian cycle of 6−complete $\mathcal{K}_6$, Hamiltonian cycle of $(3,3)$−partite $\mathcal{K}_{3,3}$, Hamiltonian cycle of 5−wheel $\mathcal{W}_{5,1}$

G7681

Theorem 3.0.97. *Trivial graph has Hamiltonian path.*

Theorem 3.0.98. *Null graph has Hamiltonian cycle.*

Theorem 3.0.99. *Null graph has Hamiltonian path.*

Theorem 3.0.100. *Disconnected graph doesn't have Hamiltonian cycle.*

Theorem 3.0.101. *Disconnected graph doesn't have Hamiltonian path.*

Theorem 3.0.102. *Turan graph $\mathcal{T}_{n,m}$ has Hamiltonian cycle where $n, m \neq 1$.*

Theorem 3.0.103. *Turan graph $\mathcal{T}_{n,m}$ has Hamiltonian path where $n, m \neq 1$.*

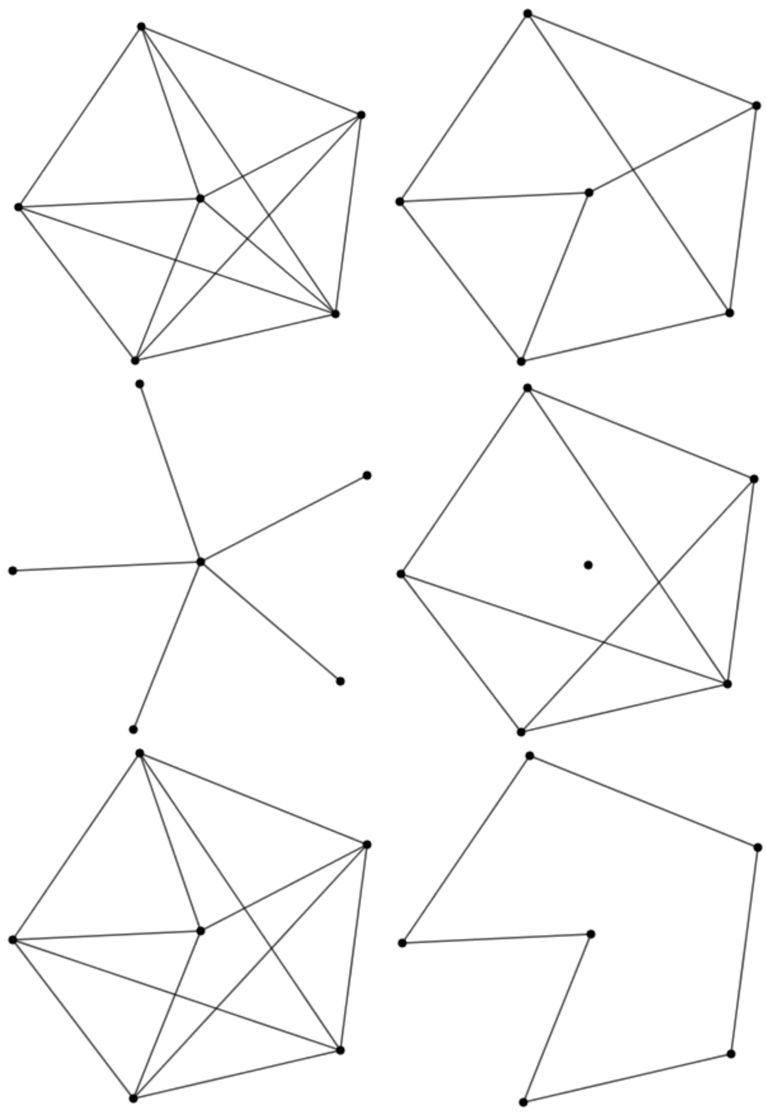


Figure 3.15: Initial graph, 2−factor of initial graph, First element of symmetric difference graph, Second element of symmetric difference graph, 4−factor of initial graph.

G8287

Example 3.0.104. Consider Figure (3.15). Initial graph, 3−factor of initial graph, First element of symmetric difference graph, Second element of symmetric difference graph, 4−factor of initial graph are the items, respectively. All items are defined on six vertices.

The items of two to six are the operations of mentioned graph as item one. $n-$factor is unary operation thus the input is one graph and the output is another graph which couldn't hold the property of the input as items. The items two and three are spanning subgraphs form the initial graph with new title symmetric difference graph.

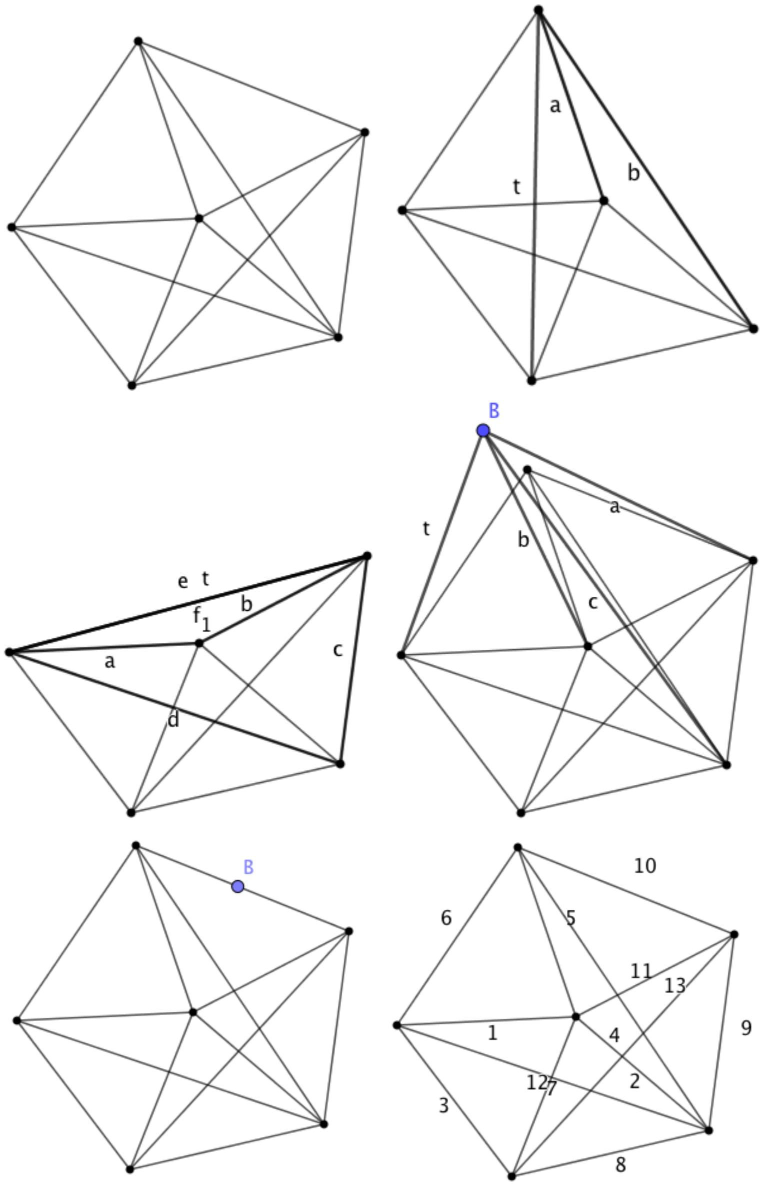


Figure 3.16: Initial graph, vertex identification of initial graph, edge contraction of initial graph, vertex splitting of initial graph, edge subdivision of initial graph, weighted graph of initial graph.

G8893

Theorem 3.0.105. *A complete graph is $(n-1)$-factor.*

Theorem 3.0.106. *A complete graph is 0-factor, 1-factor, $\cdots$, $(n-1)$-factor.*

Theorem 3.0.107. *Every graph is symmetric difference graph.*

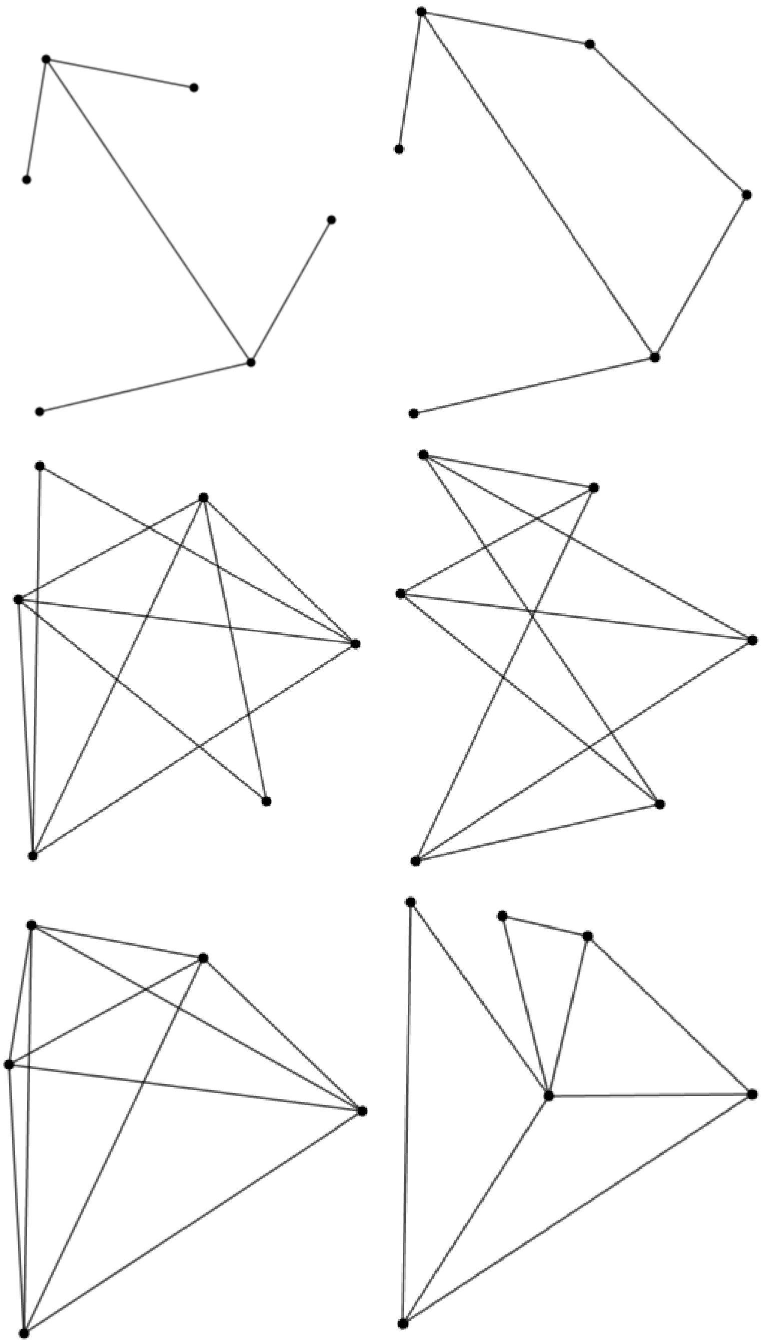


Figure 3.17: Tree graph, cyclic graph, cotree, nonplanar bipartite graph $\mathcal{K}_{3,3}$, nonplanar complete graph $\mathcal{K}_5$, fan graph.

Theorem 3.0.108. *A star graph only has* 0*-factor.*

Theorem 3.0.109. *A wheel graph only has* 0*-factor,* 1*-factor and* 2*-factor.*

Theorem 3.0.110. *A graph only has* 0*-factor, is disconnected graph.*

Theorem 3.0.111. *A graph has* 1*-factor, is connected graph.*

Example 3.0.112. Consider Figure (3.16). Initial graph, vertex identification of initial graph, edge contraction of initial graph, vertex splitting of initial graph, edge subdivision of initial graph, weighted graph of initial graph, respectively. All items are defined on six vertices.

Labeled edges and vertices are the changes in comparison to the initial graph. The items of two to six are the operations of mentioned graph as item one. all operations, vertex identification, edge contraction, vertex splitting, edge subdivision, weighted graph are unary operations thus the input is one graph and the output is another graph which couldn't hold the property of the input as items with exception item six.

Example 3.0.113. Consider Figure (3.17). Tree graph, cyclic graph, cotree, nonplanar bipartite graph $\mathcal{K}_{3,3}$, nonplanar complete graph $\mathcal{K}_5$, fan graph, respectively. All items are defined on six vertices with exception item five.

Item one and item two are planar graphs. Item four and item five aren't planar. Item two is tree graph and item three is cotree of item two. Item six is a fan graph.

Theorem 3.0.114. *A bipartite graph $\mathcal{K}_{3,3}$ is nonplanar.*

Theorem 3.0.115. *A complete graph $\mathcal{K}_5$ is nonplanar.*

Theorem 3.0.116. *A fan graph is cyclic where the order is greater than two.*

Theorem 3.0.117. *A fan graph is planar.*

Theorem 3.0.118. *A fan graph isn't tree where the order is greater than two.*

Theorem 3.0.119. *A bipartite graph $\mathcal{K}_{3,3}$ is cyclic.*

Theorem 3.0.120. *A complete graph $\mathcal{K}_5$ is cyclic.*

Theorem 3.0.121. *A bipartite graph $\mathcal{K}_{3,3}$ isn't tree.*

Theorem 3.0.122. *A complete graph $\mathcal{K}_5$ isn't tree.*

Theorem 3.0.123. *A complete graph $\mathcal{K}_n$ is cyclic where the order is greater than two.*

Theorem 3.0.124. *A bipartite graph $\mathcal{K}_{n,m}$ is cyclic where the order is greater than three.*

Theorem 3.0.125. *A complete graph $\mathcal{K}_n$ isn't tree where the order is greater than two.*

Theorem 3.0.126. *A bipartite graph $\mathcal{K}_{n,m}$ isn't tree where the order is greater than three.*

Theorem 3.0.127. *A tree graph is acyclic.*

Theorem 3.0.128. *A path graph $\mathcal{P}_n$ is acyclic.*

Theorem 3.0.129. *A complete graph $\mathcal{K}_n$ isn't tree where the order is greater than two.*

Theorem 3.0.130. *A bipartite graph $\mathcal{K}_{n,m}$ isn't tree where the order is greater than three.*

Theorem 3.0.131. *An $n-$star $\mathcal{S}_{n,1}$ is acyclic.*

Theorem 3.0.132. *An $n-$star $\mathcal{S}_{n,1}$ is tree.*

Theorem 3.0.133. *An $n-$star $\mathcal{S}_{n,1}$ is planar.*

Theorem 3.0.134. *A path graph $\mathcal{P}_n$ is tree.*

Theorem 3.0.135. *A path graph $\mathcal{P}_n$ is planar.*

Theorem 3.0.136. *A cycle graph $\mathcal{C}_n$ is planar.*

Theorem 3.0.137. *A cycle graph $\mathcal{C}_n$ isn't tree.*