REPRESENTATIONS FOR $\zeta(11)$

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ABSTRACT. Here two novel expressions of the Euler's Zeta function at 11 are provided on this commemorative day.

The Euler's Zeta function is defined as

$$\zeta(m) = \sum_{n=1}^{\infty} \frac{1}{n^m}, \qquad \left.\begin{matrix} n = 1, 2, 3, \cdots \\ m = 2, 3, \cdots \end{matrix}\right\} \in \mathbb{N}. \tag{1}$$

The embarrassing fact is that everyone has failed to find exact values for (1) when *m* are odd positive integers.

As a typical example of ζ(odd), ζ(3) or ζ(5), commonly appears in the form of infinite series and definite integrals, while other ζ(7), ζ(9), and ζ(11), etc. can rarely be seen.

Clearly, ζ(11) is the fifth one of the Euler's Zeta function at odd (although the smallest odd power is *m* =1, it has been known that the harmonic series is divergent). The study has shown that a lot of representations with infinite series for ζ(11) can be obtained systematically. Now two novel expressions for ζ(11) derived last year are selected and first given in Appendix (the formulas are a little long, see Appendix for more details).

Comparison of Calculation Results

№	Source (Come from)	Formulas	Value with first 10 terms	Value with first 20 terms
0	A013669 in the OEIS (18 exact decimal digits)		ζ(11) = 1.000,494,188,604,119,464	ζ(11) = 1.000,494,188,604,119,464
1	Definition	(1)	≈1.0004941885982255	≈1.0004941886041119
2	in this paper	(2)	≈1.0004859810691056	≈1.0004938163606814
3		(3)	≈1.0004317418492734	≈1.0004912936978256

Remark:

(a) The definite expression of ζ(11) converges rapidly;

(b) Formulae (2) and (3) are for research purposes, not primarily for calculations. Their accuracy increases as the number of series terms increases.

(c) The red digits in the table above are the exact digits of the every computed value for ζ(11) by comparing to its exact decimal digits.

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Appendix

Formula (2) for ζ(11)

$$\zeta(11) = \frac{2^{31}\cdot\pi^{10}}{2098175}\left[\frac{15892911238073}{179895156432961536000} - \frac{2178514831\ln(2)}{33313917857955840} - \frac{2215103519\ln(\pi)}{71386966838476800} + \cdots + \sum_{n=1}^{\infty}\frac{\left[(3\cdot 5\cdot 7\cdot 9\cdot 18761)\cdot 2^{4} + \frac{\left[294797252^{4n+14} + 2^{2n}(2n+6)\cdot(2n+7)\cdot(2n+8)\cdot\left(560139732^{2n+4} - 1225747429\right) - 3\cdot 5\cdot 9\cdot 2^{20}\right]}{[(2n+9)\cdot(2n+10)]^{-1}}\right]\cdot\left|B_{2n}\right|\cdot\pi^{2n}}{(3\cdot 5\cdot 7\cdot 9\cdot 18761)\cdot 2^{2n+25}\cdot n\cdot(2n+10)!}\right]$$

Formula (3) for ζ(11)

$$\zeta(11) = \frac{2^{31}\cdot\pi^{10}}{2098175}\cdot\left[\frac{2404589519}{352528231301120000} + \frac{234008007\ln(2)}{616924404776960 0} - \frac{729\ln(3)}{46976204800} - \frac{59179671\ln(\pi)}{88132057825280 0} + \cdots + \sum_{n=1}^{\infty}\frac{\left[(10506160)\cdot 3^{2n+10} - \frac{\left[45\cdot 1318372^{4n+14} - 2^{2n}\cdot(2n+6)\cdot(2n+7)\cdot(2n+8)\cdot\left(162198872^{2n+4} + 98164641\right) + 45\cdot 2^{20}\right]}{[(2n+9)\cdot(2n+10)]^{-1}}\right]\cdot\left|B_{2n}\right|\cdot(\pi)^{2n}}{(5\cdot 7\cdot 18761)\cdot 2^{2n+25}\cdot n\cdot(2n+10)!}\right]$$

where, B_{2n} are Bernoulli numbers.