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A STUDY ON DOMINATION IN TWO FUZZY MODELS

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Abstract. The aim of this 9th expository article is to conclude a study on domination in two fuzzy models, including $t-$norm fuzzy graphs and fuzzy graphs. All parts are twofold even if we don't mention, directly. I.e., all results depicts some properties about fuzzy graph and $t-$norm fuzzy graph.
Keywords: fuzzy graphs, $t-$norm fuzzy graphs

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1. Introduction and Overview

In this study, author analyze the structure of domination in $t-$norm fuzzy graphs and a its special case when using $T_{\min}$, as fuzzy graphs.
In **Ref.** [?], we have a real world application concerning this concept. you can refer it if you need or are interested. Some issues in **Ref.** [?], "$\cdots$ The Global Slavery Index is an annual study of world-wide slavery conditions by country published by the Walk Free Foundation. In 2016, the study estimated a total of 45.8 million people to be in some form of modern slavery in 167 countries. The report contains data for countries concerning the estimate of the prevalence of modern slavery, vulnerability measures, and an assessment of the strength of government response$\cdots$"

2. Preliminaries

In this work, author always use v if the vertex is specific. Otherwise, author apply its indices, i.e. v_i. So v or v_i always refers to vertices and their twofold part refers to edge. The power "$'$" usually states that one edge is deleted.
At first, author introduce two types of a fuzzy models concerning $t-$norm. It is well known that T_{min} is a function (precisely a relation) which is greater than any $t-norm$.
"Basic Definition', ''Size", "Order", "Scalar Cardinality", "Path", "Fuzzy Cycle", "Isolate", "$\alpha-$strong", "$M-$strong", "Bridge", "Bipartite", "Star", "Complete", "Spanning Subgraph", "Fuzzy Tree" and "Operations" are introduced as preliminaries in what follows. Some concepts are not related to choosing any $t-$norm because they don't state any relation between two functions μ and σ which are depended on each other by definition of fuzzy model (precisely using $t-$norm). So in all fuzzy models can be the same.

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Definition 2.1 (Definitions, Size and Order, Scalar Cardinality)**.** *author introduce some elementary concepts as follows*

(i) *[Definitions] Let V be a nonempty finite set and $E \subseteq V \times V$. Then $G = (\sigma, \mu)$ is called a **Fuzzy Graph** if $\forall v_1v_2 \in E$, $\mu(v_1v_2) = \mu(v_2v_1) \leq \min\{\sigma(v_1), \sigma(v_2)\}$. And is called an **t-norm Fuzzy Graph** if $\forall v_1v_2 \in E$, $\mu(v_1v_2) = \mu(v_2v_1) \leq T(\sigma(v_1), \sigma(v_2))$, where $\sigma : V \to [0,1]$ and $\mu : E \to [0,1]$ be the fuzzy sets, μ is reflexive and T is an arbitrary $t-$norm.*

(ii) *[Size and Order] The **Order** p and the **Size** q are defined $p = \Sigma_{v\in V}\sigma(v)$ and $q = \Sigma_{v_1v_2\in E}\mu(v_1v_2)$.*

(iii) *[Scalar Cardinality] The **Scalar Cardinality** of S is defined to be $\Sigma_{v\in S}\sigma(v)$.*

Definition 2.2 (Path, Fuzzy Cycle)**.** *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph.*

(i) *[Path, its Strength] A **Path** P of length n is a sequence of distinct vertices $v_0, v_1, \cdots, v_n$ such that $\mu(v_{i-1}v_i) > 0, i = 1, 2, \cdots, n$ and $T(\mu(v_0v_1), \cdots, \mu(v_{i-1}v_i))$ is defined as its **Strength**. The **Strength of Connectedness** between two vertices v_1 and v_2 in G is defined as the maximum of the strengths of all paths between v_1 and v_2 and is denoted by $\mu^{\infty}_{G}(v_1, v_2)$.*

(ii) *[Fuzzy Cycle, its Strength] Let $v_0, v_1, \cdots, v_n$ be a path. It is called a **Fuzzy Cycle** C of length n If $v_0 = v_n, n \geq 3$ and at least the values of two edges are $T(\mu(v_0v_1, \cdots, \mu(v_{i-1}v_i)$ which is defined as **Strength** of a fuzzy cycle.*

Definition 2.3 (Types of Vertices)**.** *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph. A vertex v is said **isolated** if $\mu(vv_1) = 0$ for all $v \neq v_1$.*

Definition 2.4 (Types of Edges)**.** *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph. Let $v_1v_2 \in E$. Note that $\mu^{\infty}_{G'}(v_1, v_2)$ is the strength of connectedness between v_1 and v_2 in the fuzzy model which is obtained from G by deleting the edge v_1v_2.*
An edge v_1v_2 in G is called

(i) *$\alpha-$**strong** if $\mu(v_1v_2) > \mu^{\infty}_{G'}(v_1, v_2)$ and **strong** if $\mu(v_1v_2) \geq \mu^{\infty}_{G'}(v_1, v_2)$. The case $\mu(v_1v_2) = \mu^{\infty}_{G'}(v_1, v_2)$, is not considered in any study of domination. The case $\mu(v_1v_2) < \mu^{\infty}_{G'}(v_1, v_2)$ is not possible.*

(ii) *$M-$**strong** if both $\mu(v_1v_2) = \sigma(v_1) \wedge \sigma(v_2)$ and G is a fuzzy graph or both $\mu(v_1v_2) = T(\sigma(v_1), \sigma(v_2))$ and G is an $t-$norm fuzzy graph.*

(iii) ***bridge** if $\mu^{\infty}_{G'}(v_3, v_4) < \mu^{\infty}_{G}(v_3, v_4)$ for some $v_3, v_4 \in V$.*

Definition 2.5 (Types of Models)**.** *Let $G = (\sigma, \mu)$ and $G_1 = (\tau, \nu)$ be a fuzzy graph or an $t-$norm fuzzy graph. Then $G = (\sigma, \mu)$ is said to be*

(i) ***Bipartite** if V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(v_1v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$;*

(ii) ***Star** which is denoted by $K_{1,\sigma}$ If it is a bipartite and either $|V_1| = 1$ or $|V_2| = 1$ which imply that we call its corresponded vertex a **center**;*

(iii) ***Complete** if all edges be $M-$ strong. e.g., Complete bipartite fuzzy graph, Complete fuzzy graph, Complete bipartite $t-$norm fuzzy graph, Complete $t-$norm fuzzy graph.*

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(iv) *has a* ***Spanning Subgraph*** $G_1 = (\tau, \nu)$ *if* $\tau = \sigma$ *and* $\nu \subseteq \mu$.

(v) ***Fuzzy tree*** *if its spanning subgraph* $F = (\sigma, \tau)$ *is a tree (Ref.* [**?**]*), where for all edges* v_1v_2 *is in* G *but not* F*, we have* $\mu(v_1v_2) < \tau_F^{\infty}(v_1, v_2)$.

Definition 2.6 (Types of New Models)**.** *If we alter* $\min, \max$ *(precisely t-norm* T_{min}, T_{max}*) with an arbitrary t-norm* T*, we have these concepts for* $t-$*norm fuzzy graphs. To avoid confusion, we only write down for fuzzy graph and the analogues concepts are supposed to be obvious and we use these names for both models, fuzzy graphs and fuzzy* $t-$*norm graphs.*
Let $G_1 = (\sigma_1, \mu_1)$ *and* $G_2 = (\sigma_2, \mu_2)$ *be fuzzy graphs on* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$*, respectively. Then*

(i) *[Unary Operation, Complement] A* ***Complement*** *of a fuzzy graph* $G_1 = (\sigma_1, \mu_1)$ *is denoted by* $\bar{G_1}$ *and is defined to* $\bar{G_1} = (\sigma_1, \bar{\mu_1})$*, where* $\bar{\mu_1}(v_1v_2) = \min\{\sigma_1(v_1), \sigma_1(v_2)\} - \mu_1(v_1v_2)$*, for all* $v_1, v_2 \in V_1$;

(ii) *[Binary Operation, Cartesian Product] A* ***Cartesian product*** $G = G_1 \times G_2$ *is defined as a fuzzy graph* $G = (\sigma_1 \times \sigma_2, \mu_1\mu_2)$ *on* $G^* = (V_1 \times V_2, E)$ *where* $E = \{(v, v_1)(v, v_2) | v \in V_1, v_1v_2 \in E_2\} \cup \{(v_1, v)(v_2, v)\} | v_1v_2 \in E_1, v \in V_2\}$*. Fuzzy sets* $\sigma_1 \times \sigma_2$ *on* $V_1 \times V_2$ *and* $\mu_1\mu_2$ *on* E*, are defined as* $(\sigma_1 \times \sigma_2)(v_1, v_2) = \min\{\sigma_1(v_1), \sigma_2(v_2)\}, \forall (v_1, v_2) \in V_1 \times V_2$ *and* $\forall v \in V_1, \forall v_1v_2 \in E_2, \mu_1\mu_2((v, v_1)(v, v_2)) = \min\{\sigma_1(v), \mu_2(v_1v_2)\}$ *and* $\forall v_1v_2 \in E_1, \forall v \in V_2, \mu_1\mu_2((v_1, v)(v_2, v)) = \min\{\mu_1(v_1v_2), \sigma_2(v)\}$;

(iii) *[Binary Operation, Union] An* ***Union*** $G = G_1 \cup G_2$ *is defined as a fuzzy graph* $G = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ *on* $G^* = (V_1 \cup V_2, E_1 \cup E_2)$*. Fuzzy sets* $\sigma_1 \cup \sigma_2$ *and* $\mu_1 \cup \mu_2$ *are defined as* $(\sigma_1 \cup \sigma_2)(v) = \sigma_1(v)$ *if* $v \in V_1 - V_2, (\sigma_1 \cup \sigma_2)(v) = \sigma_2(v)$ *if* $v \in V_2 - V_1$*, and* $(\sigma_1 \cup \sigma_2)(v) = \max\{\sigma_1(v), \sigma_2(v)\}$ *if* $v \in V_1 \cap V_2$*. Also* $(\mu_1 \cup \mu_2)(v_1v_2) = \mu_1(v_1v_2)$ *if* $v_1v_2 \in E_1 - E_2$ *and* $(\mu_1 \cup \mu_2)(v_1v_2) = \mu_2(v_1v_2)$ *if* $v_1v_2 \in E_2 - E_1$*, and* $(\mu_1 \cup \mu_2)(v_1v_2) = \max\{\mu_1(v_1v_2), \mu_2(v_1v_2)\}$ *if* $v_1v_2 \in E_1 \cap E_2$;

(iv) *[Binary Operation, Join] A* ***Join*** $G = G_1 + G_2$ *is defined as a fuzzy graph* $G = (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ *on* $G^* = (V_1 \cup V_2, E = E_1 \cup E_2 \cup E^{'})$ *where* $E^{'}$ *is the set of all edges joining vertices of* V_1 *with the vertices of* V_2 *and we assume that* $V_1 \cap V_2 = \emptyset$*. Fuzzy sets* $\sigma_1 + \sigma_2$ *and* $\mu_1 + \mu_2$ *are defined as* $(\sigma_1 + \sigma_2)(v) = (\sigma_1 \cup \sigma_2)(v)$ *and* $\forall v \in V_1 \cup V_2; (\mu_1 + \mu_2)(v_1v_2) = (\mu_1 \cup \mu_2)(v_1v_2)$ *if* $v_1v_2 \in E_1 \cup E_2$ *and* $(\mu_1 + \mu_2)(v_1v_2) = \min\{\sigma_1(v_1), \sigma_2(v_2)\}$ *if* $v_1v_2 \in E^{'}$.

3. Basic Ideas

We choose a name for our new definition as vertex domination and we refer to others with only the name domination. To avoid confusion, we bring references if it is necessary.

Definition 3.1 (Domination: Edge, Set, Number)**.** *Let* $G = (\sigma, \mu)$ *be a fuzzy graph or an* $t-$*norm fuzzy graph and* $v, v_1 \in V$*. Then*

(i) *A vertex* v α*-strongly* **dominates** *a vertex* v_1 *in* G*, if its corresponded edge* vv_1 *is an* α*-strong edge;*

(ii) D *is called an* α**-strong dominating set** *in* G*, if for every* $v_1 \in V \setminus D$*, there is* $v \in D$ *such that* v α*-strongly dominates* v_1.

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(*iii*) *The* **weight** *of* D *is defined by* $w_v(D) = \sum_{v\in D}(\sigma(v) + \frac{\sum_{vv_1\in S}\mu(vv_1)}{\sum_{vv_1\in E}\mu(vv_1)})$, *where* $S = \{v_1v_2 \in E \mid \mu(v_1v_2) > \mu^{\infty}_{G'}(v_1, v_2)\}$.

(*iv*) *A vertex domination number of* G *is defined as* $\gamma_v(G) = \min_{D\in\mathcal{D}}\{w_v(D)\}$, *where* $\mathcal{D}$ *is the set of all* α**-strong dominating sets** *in* G. *The* α*-strong dominating set that corresponds to* $\gamma_v(G)$ *is called by* **vertex dominating set.**

We give some definitions concerning domination on fuzzy graphs. It can be extended to $t-$norm fuzzy graphs. We only use them in some examples for illustrating our concepts and do a comparison between them with ours. It is worth to note that if we alter min (precisely t-norm T_{min}) with any t-norm T, we have these concepts for $t-$norm fuzzy graphs. To avoid confusion, we only write down for fuzzy graph and the analogues concepts are supposed to be obvious.

Definition 3.2. *Let* $G = (\sigma, \mu)$ *be a fuzzy graph,* $D \subseteq V$ *and* $\mathcal{D}$ *is a set of all dominating sets in* G. *Then*

(*i*) (**A. Somasundaram and S. Somasundaram (Ref.**[?])) $D \subseteq V$ *is said to be an* **dominating set** *in* G, *if for every* $v_1 \in V \setminus D$, *there exists* $v \in D$ *such that its corresponded edge* vv_1 *is an* $M-$*strong edge.*$\gamma(G) = \min_{D\in\mathcal{D}}\{\sum_{v\in D}\sigma(v)\}$ *is said to be a* **domination number** *of* G.

(*ii*) (**C. Natarajan and S.K. Ayyaswamy** (**Ref.**[?])) $D \subseteq V$ *is said to be a* **dominating set,** *if for every* $v_1 \in V \setminus D$, *there exists* $v \in D$ *such that its corresponded edge* vv_1 *is an* $M-$*strong edge and* $d_e(v) = \sum_{v_2\in N(v)}\sigma(v_2) \geq d_e(v_1) = \sum_{v_2\in N(v_1)}\sigma(v_2)$ *where for all* $v \in V, N(v) = \{v_1 \in V \mid \mu(vv_1) = \min\{\sigma(v), \sigma(v_1)\}\}$. $\gamma(G) = \min_{D\in\mathcal{D}}\{\sum_{v\in D}\sigma(v)\}$ *is said to be a* **domination number** *of* G.

(*iii*) (**O.T. Manjusha and M.S. Sunitha** (**Ref.**[?])) $D \subseteq V$ *is said to be* **dominating set** *if for every* $v_1 \in V \setminus D$, *there exists* $v \in D$ *such that its corresponded edge* vv_1 *is a strong edge.* $\gamma(G) = \min_{D\in\mathcal{D}}\{\sum_{v\in D}\sigma(v)\}$ *is said to be a* **domination number** *of* G.

(*iv*) (**A. Nagoor Gani and K. Prasanna Devi (Ref.**[?])) $D \subseteq V$ *is said to be* **dominating set,** *if for every* $v_1 \in V \setminus D$, *there exists two vertices like* $v \in D$ *such that their corresponded edges are strong edges.* $\gamma(G) = \min_{D\in\mathcal{D}}\{\sum_{v\in D}\sigma(v)\}$ *is said to be a* **domination number;**

(*v*) (**O.T. Manjusha and M.S. Sunitha (Ref.**[?])) $D \subseteq V$ *is said to be* **dominating set,** *if for every* $v_1 \in V \setminus D$, *there exists* $v \in D$ *such that its corresponded edge* vv_1 *is an strong edge.* **domination number** *of* G *is said to be* $\gamma(G) = \min_{D\in\mathcal{D}}\{\sum_{v\in D}\min_{vv_1 \text{is a strong edge.}}\{\mu(v, v_1)\}\}$.

In two upcoming examples, we illustrates the concept of our definition.

Example 3.3 ($\alpha-$strong edge)**.** *Let* $G = (\sigma, \mu)$ *be a fuzzy graph as Figure* **??**. *Then the edges* $\{v_2v_5, v_2v_4, v_3v_4, v_1v_3\}$ *are* α*-strong and the edges* $\{v_1v_4, v_1v_2, v_4v_5\}$ *are not* α*-strong.*

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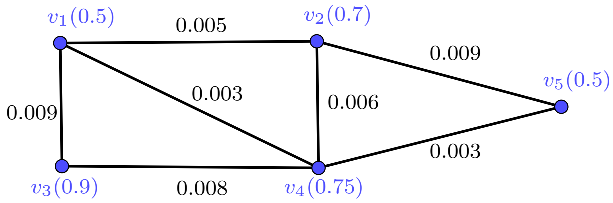

FIGURE 1. vertex domination

Example 3.4 (Domination)**.** *Let $G = (\sigma, \mu)$ be a fuzzy graph as Figure* **??***. The set $S = \{v_2, v_3\}$ is an α-strong dominating set. This set is also vertex dominating set in fuzzy graph G. Hence $\gamma_v(G) = 1.75 + 0.9 + 0.7 = 3.35$. So $\gamma_v(G) = 3.35$.*

In two upcoming examples, we compare our definition with others as theoretic and practical aspects.

Example 3.5 (Theoretic Aspect)**.** *The following is a table consist of a brief fundamental comparison between types of domination in fuzzy graphs. There are two different types of the complete bipartite fuzzy graphs as Figures* **??** *and* **??***, which compare types of domination in fuzzy graphs.*

Types of Edges	*Types of Numbers*	*Figure* **??**	*Figure* **??**
$M-$*strong*	*Scalar cardinality*	0.9	0.9
$M-$*strong and* $d_e(u) \geq d_e(v)$	*Scalar cardinality*	1.9	1.3
Strong	*Scalar cardinality*	0.9	0.9
$\beta-$*strong*	*Scalar cardinality*	0.9	1.5
Strong	$\Sigma_{u\in D} t(u, v)$	0.8	0.4
Our new definition	*vertex weight*	1.9	2.4

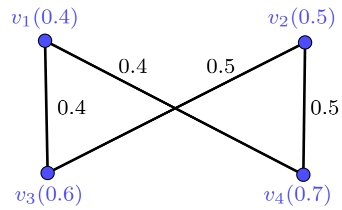

FIGURE 2. Comparison of Dominations

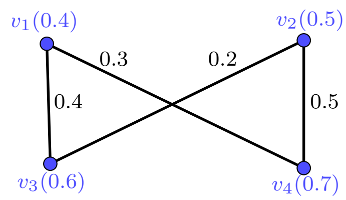

FIGURE 3. Comparison of Dominations with Different Values

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Example 3.6 (Practical Aspect: A Comparison in Real-World Problem)**.** *In this section, we introduce one practical application in related to this concept. In the following, we will try to solve this problem by previous definitions, too.*
Suppose the Figure **??**, *the fuzzy graph model of the hypothetical condition of cities and the paths between them in a region.*
Problem*[reducing waste of time in transport planning] Consider a set of cities connected by communication paths. Which cities have these properties? Having low traffic levels and other cities associating with at least ones by low-cost roads.*
The terms "low traffic" and "low-cost" are vague in nature. So we are faced with a fuzzy graph model. In other words, Let G be a graph which represents the roads between cities. Let the vertices denote the cities and the edges denote the roads connecting the cities. From the statistical data that represents the high traffic flow of cities and high-cost roads, the functions σ and μ on the vertex set and edge set of G can be constructed by using the standard techniques. In this fuzzy graph, a dominating set D can be interpreted as a set of cities which have low traffic and every city not in D is connected to a member in D by a low-cost road. We now look at the answer to the problem raised by using the old

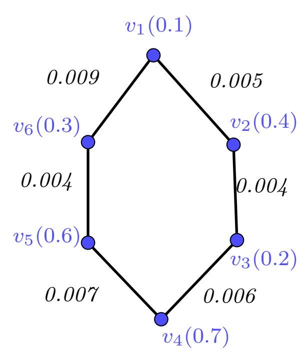

Figure 4. The exemplary scheme of road infrastructure

and the new definitions. As you can see in this model, finding the desirable cities is more important than finding the domination number. Because the numbers given for the set and each situation are compared with each others in the context of the same definition, and this number is merely to compare the different sets of cities in the context of the same definition. Therefore, speaking of the magnitude of this number is meaningless. The table below illustrates the solutions presented for this problem.

Definitions	*Given desirable set*
A. Somasundaram and S. Somasundaram (Ref. [?])	V
C. Natarajan and S.K. Ayyaswamy (Ref. [?])	V
O.T. Manjusha and M.S. Sunitha (Ref. [?])	$\{v_3, v_6\}$
A. Nagoor Gani and K. Prasanna Devi (Ref. [?])	V
O.T. Manjusha and M.S. Sunitha (Ref. [?])	$\{v_3, v_6\}$
Our new definition	$\{v_1, v_4\}$

It is obvious from the above table and Figure ?? that the desirable cities given by previous definitions, are not appropriate due to the lack of simultaneous attention to cities and roads.
We are now presenting the dynamic status of the problem. The dynamic state is the situation in which the fuzzy graph model is found over time. Since over time, changes in the values of roads are more than changes in the values of cities in the fuzzy graph model of the hypothetical condition of cities and the paths between them in a region. So values of the roads increases. Values of cities (their traffics) do not change significantly over time. Because the traffic problem is an infrastructure problem. The Figure ?? depicts the dynamic case of a fuzzy graph model. Over time, the values of the roads increases equally.

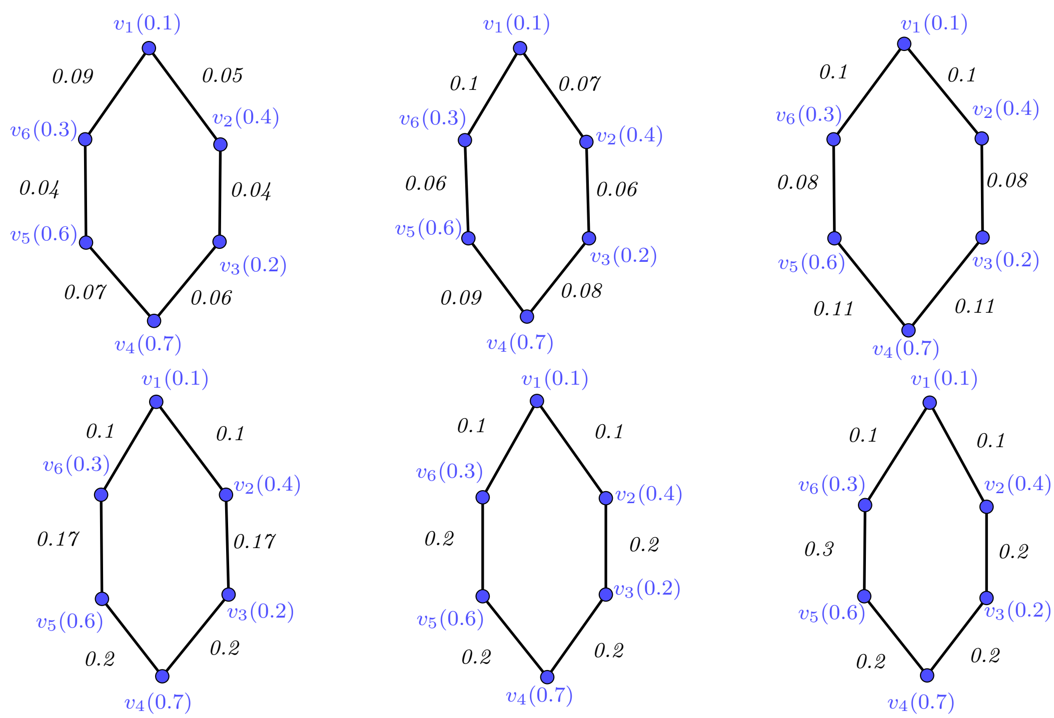

FIGURE 5. The dynamic scheme of road infrastructure

In this situation, the answer are given by the previous definitions reflects the wrong perspectives while the our new definition adapts itself well to the new situation. Previous definitions didn't use simultaneous attentions to cities and roads.
Dynamic analysis of networks in the first row of Figure ?? are the following table.

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Definitions	*Given desirable set*
A. Somasundaram and S. Somasundaram (Ref. [?])	V, $V-\{v_6\}$, $V-\{v_2,v_6\}$
C. Natarajan and S.K. Ayyaswamy (Ref. [?])	V, $V-\{v_6\}$, $V-\{v_2,v_6\}$
O.T. Manjusha and M.S. Sunitha (Ref. [?])	$\{v_3,v_6\}$, $\{v_3,v_6\}$, $\{v_3,v_6\}$
A. Nagoor Gani and K. Prasanna Devi (Ref. [?])	$\{v_1,v_3,v_5\}$, $\{v_1,v_3,v_5\}$, $\{v_1,v_3,v_5\}$
O.T. Manjusha and M.S. Sunitha (Ref. [?])	$\{v_3,v_6\}$, $\{v_3,v_6\}$, $\{v_3,v_6\}$
Our new definition	$\{v_1,v_4\}$, $\{v_1,v_4\}$, $\{v_1,v_4\}$

Dynamic analysis of networks in the second row of Figure **??** *are the following table.*

Definitions	*Given desirable set*
A. Somasundaram and S. Somasundaram (Ref. [?])	$\{v_1,v_3,v_5\}$, $\{v_1,v_3,v_5\}$, $\{v_3,v_5\}$
C. Natarajan and S.K. Ayyaswamy (Ref. [?])	$\{v_2,v_4,v_6\}$, $\{v_3,v_5,v_6\}$, $\{v_2,v_3,v_5\}$
O.T. Manjusha and M.S. Sunitha (Ref. [?])	$\{v_3,v_6\}$, $\{v_3,v_6\}$, $\{v_3,v_6\}$
A. Nagoor Gani and K. Prasanna Devi (Ref. [?])	$\{v_1,v_3,v_5\}$, $\{v_1,v_3,v_5\}$, $\{v_1,v_3,v_5\}$
O.T. Manjusha and M.S. Sunitha (Ref. [?])	$\{v_3,v_6\}$, $\{v_3,v_6\}$, $\{v_3,v_6\}$
Our new definition	$\{v_1,v_3,v_6\}$, $\{v_1,v_3,v_6\}$, $\{v_1,v_3,v_6\}$

4. **General Ideas**

All parts are twofold even if we don't mention, directly. I.e., all results depicts some properties about fuzzy graph and $t-$norm fuzzy graph.
It is well known and generally accepted that the problem of determining the domination number of an arbitrary fuzzy model is a difficult one. Because of this, researchers have turned their attention to the study of classes of fuzzy models for which the domination problem can be solved in polynomial time.

Proposition 4.1 (**Ref.** [?], Proposition 3.24. , pp. 135, 136)**.** *Let $G=(\sigma,\mu)$ be a complete $t-$norm fuzzy graph. Then*

(1) $\mu_G^\infty(v_1,v_2)=\mu(v_1v_2), \forall v_1,v_2\in V$
(2) *G has no cut vertices.*

Corollary 4.2. *Every edges in complete t-norm fuzzy graph are α-strong if $\forall v_1,v_2\in V$, there is exactly one path with strength of $\mu^\infty(v_1,v_2)$.*

Proof. Let G be complete. For all $v_1,v_2\in V$, $\mu_G^\infty(v_1,v_2)=\mu(v_1v_2)$ by Proposition (**??**). So for all $v_1v_2\in V$, $\mu_G'^\infty(v_1,v_2)<\mu(v_1,v_2)$. Hence uv is α-strong edge. The result follows. □

Proposition 4.3. *Let $G=(\sigma,\mu)$ be complete such that $\forall v_1,v_2\in V$, there is exactly one path with strength of $\mu^\infty(v_1,v_2)$. Then, every edges are α-strong.*

Proof. We prove it in two cases.

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Fuzzy Graphs: Let G be a complete fuzzy graph. The strength of path P from v_1 to v_2 is of the form $\min\{\sigma(v_1), \cdots \sigma(v_2)\} \leq \min\{\sigma(v_1), \sigma(v_2)\} = \mu(v_1v_2)$. So $\mu_G^{\infty}(v_1, v_2) \leq \mu(v_1v_2)$. v_1v_2 is a path from v_1 to v_2 such that $\mu(v_1v_2) = \min\{\sigma(v_1), \sigma(v_2)\}$. Therefore $\mu_G^{\infty}(v_1, v_2) \geq \mu(v_1v_2)$. Hence $\mu_G^{\infty}(v_1, v_2) = \mu(v_1v_2)$. Then $\mu(v_1v_2) > \mu_{G'}^{\infty}(v_1, v_2)$. It means that the edge v_1v_2 is α-strong. All edges are α-strong, as we wished to show. Its proof works equally well for the latter.

$t-$**norm Fuzzy Graphs:** The strength of path P from v_1 to v_2 is of the form $T(\sigma(v_1), \cdots \sigma(v_2)) \leq T(\sigma(v_1), \sigma(v_2))$. G is complete. By regarding this point, we have $T(\sigma(v_1), \sigma(v_2)) = \mu(v_1v_2)$. Therefore, $T(\sigma(v_1), \cdots \sigma(v_2)) \leq \mu(v_1v_2)$. It means that $\mu_G^{\infty}(v_1, v_2) \leq \mu(v_1v_2)$. v_1, v_2 is a path from v_1 to v_2 such that $\mu(v_1v_2) = T(\sigma(v_1), \sigma(v_2))$. Therefore $\mu_G^{\infty}(v_1, v_2) \geq \mu(v_1v_2)$. Hence $\mu_G^{\infty}(v_1, v_2) = \mu(v_1v_2)$. Then $\mu(v_1v_2) > \mu_{G'}^{\infty}(v_1, v_2)$. It means that the edge v_1v_2 is α-strong. All edges are α-strong.

□

Corollary 4.4 (Complete)**.** *Let $G = (\sigma, \mu)$ be complete such that $\forall v_1, v_2 \in V$, there is exactly one path with strength of $\mu^{\infty}(v_1, v_2)$. Then, $\gamma_v(G) = \min_{v \in V}(\sigma(v)) + 1$.*

Proof. We prove it in two cases.

Fuzzy Graphs: All edges are α-strong and each vertex is adjacent to all other vertices. So $D = \{v\}$ is an α-strong dominating set and $\sum_{vv_1 \in S} \mu(vv_1) = \sum_{vv_1 \in E} \mu(vv_1)$ for each $v \in V$, where $S = \{v_1v_2 \in E \mid \mu(v_1v_2) > \mu_{G'}^{\infty}(v_1, v_2)\}$. The result follows.

$t-$**norm Fuzzy Graphs:** All edges are α-strong and each vertex is adjacent to all other vertices. So $D = \{v\}$ is an α-strong dominating set and $\sum_{vv_1 \in S} \mu(vv_1) = \sum_{vv_1 \in E} \mu(vv_1)$ for each $v \in V$, where $S = \{v_1v_2 \in E \mid \mu(v_1v_2) > \mu_{G'}^{\infty}(v_1, v_2)\}$. The case where equality holds is of particular interest.

□

Proposition 4.5 (Edgeless)**.** *Let $G = (\sigma, \mu)$ be an edgeless fuzzy graph. Then $\gamma_v(G) = p$, where p denotes the order of G.*

Proof. We prove it in two cases.

Fuzzy Graphs: G is edgeless. Hence V is only α-strong dominating set in G and there is no α-strong edge. So by Definition, we have $\gamma_v(G) = \Sigma_{v \in V} \sigma(v) = p$.

$t-$**norm Fuzzy Graphs:** The previous proof works equally well for this case.

□

It is interesting to note that the converse of Proposition **??**, does not hold.

Example 4.6. *We show that the converse of Proposition **??** does not hold. For this purpose, Let $G = (\sigma, \mu)$ be a fuzzy graph where $V = \{v_1, v_2, v_3, v_4, v_5\}, E = \{v_1v_2, v_1v_4, v_1v_3, v_2v_4, v_2v_5, v_3v_4, v_4v_5\}$*

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and σ, μ are fuzzy sets which are defined on V, E, respectively, as follows. For the fuzzy set σ, we have

$$\sigma(v_1) = 0.5, \sigma(v_2) = 0.7, \sigma(v_3) = 0.9, \sigma(v_4) = 0.75, \sigma(v_5) = 0.5$$

Now, for the fuzzy set μ, we have $\mu(v_1v_2) = 0.005$,

$$\mu(v_1v_4) = 0.003, \mu(v_1v_3) = 0.009, \mu(v_2v_4) = 0.006, \mu(v_2v_5) = 0.009,$$

$\mu(v_3v_4) = 0.008, \mu(v_4v_5) = 0.003$ *such that* $\forall v_1, v_2 \in V, \mu(v_1v_2) \leq \min\{\sigma(v_1), \sigma(v_2)\}$. *The edges* $\{v_2v_5, v_2v_4, v_3v_4, v_1v_3\}$ *are α-strong and the edges* $\{v_1v_4, v_1v_2, v_4v_5\}$ *are not α-strong. So the set* $\{v_2, v_3\}$ *is the α-strong dominating set. This set is also vertex dominating set in fuzzy graph G. Hence* $\gamma_v(G) = 1.75 + 0.9 + 0.7 = 3.35 = \Sigma_{v \in V} \sigma(v) = p$. *So G is not edgeless but $\gamma_v(G) = p$.*

Corollary 4.7. *Let $G = (\sigma, \mu)$ be complete bipartite such that $\forall v_1, v_2 \in V$, there is exactly one path with strength of $\mu^\infty(v_1, v_2)$. Then, every edges are α-strong.*

Proof. The proof in Proposition (**??**), works equally well for this case. □

Corollary 4.8. *Let $G = (\sigma, \mu)$ be complete star such that $\forall v_1, v_2 \in V$, there is exactly one path with strength of $\mu^\infty(v_1, v_2)$. Then, every edges are α-strong.*

Proof. The proof in Proposition (**??**), works equally well for this case. □

Corollary 4.9 (Complete Star)**.** *Let $G = (\sigma, \mu)$ be complete star such that $\forall v_1, v_2 \in V$, there is exactly one path with strength of $\mu^\infty(v_1, v_2)$. Then, $\gamma_v(G)$ is $\sigma(v) + 1$ where $v \in V$ is supposed as a center of G.*

Proof. We prove it in two cases.

- **Fuzzy Graphs:** Let $G = (\sigma, \mu)$ be a star fuzzy graph with $V = \{v, v_1, v_2, \cdots, v_n\}$ such that v is a center. Then $\{v\}$ is a vertex dominating set of G. Hence $\gamma_v(G) = \sigma(v) + 1$.
- $t-$**norm Fuzzy Graphs:** The previous proof works equally well for this case.

□

Corollary 4.10 (Complete Bipartite)**.** *Let $G = (\sigma, \mu)$ be a complete bipartite such that $\forall v_1, v_2 \in V$, there is exactly one path with strength of $\mu^\infty(v_1, v_2)$. Then $\gamma_v(G)$ is either $\sigma(v) + 1$, $v \in V$ or* $\min_{v_1 \in V_1, v_2 \in V_2}(\sigma(v_1) + \sigma(v_2)) + 2$.

Proof. We prove it in two cases.

- **Fuzzy Graphs:** Let $G = (\sigma, \mu)$ be a complete bipartite fuzzy graph such that $\forall v_1, v_2 \in V$, there is exactly one path with strength of $\mu^\infty(v_1, v_2)$. By Corollary (**??**), all the edges are α-strong. If G is a complete star fuzzy graph, then by Corollary (**??**), the result follows. Otherwise, the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that both of V_1 and V_2 include more than one vertex. Every vertex in V_1 is dominated by every vertices in V_2, as α-strong and conversely. Hence in K_{σ_1,σ_2}, the α-strong dominating sets are V_1 and V_2 and any set containing 2 vertices, one in V_1 and other in V_2. So $\gamma_v(K_{\sigma_1,\sigma_2}) = \min_{v_1 \in V_1, v_2 \in V_2}(\sigma(v_1) + \sigma(v_2)) + 2$. The result follows.

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$t-$**norm Fuzzy Graphs:** Let $G = (\sigma, \mu)$ be a complete bipartite $t-$norm fuzzy graph such that $\forall v_1, v_2 \in V$, there is exactly one path with strength of $\mu^{\infty}(v_1, v_2)$. By Corollary (**??**), all the edges are α-strong.

If G is a complete star $t-$norm fuzzy graph, then by Corollary (**??**), the result follows. Otherwise, the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that both of V_1 and V_2 include more than one vertex. Every vertex in V_1 is dominated by every vertices in V_2, as α-strong and conversely. Hence in K_{σ_1,σ_2}, the α-strong dominating sets are V_1 and V_2 and any set containing 2 vertices, one in V_1 and other in V_2. So $\gamma_v(K_{\sigma_1,\sigma_2}) = \min_{v_1 \in V_1, v_2 \in V_2}(\sigma(v_1) + \sigma(v_2)) + 2$. The result follows.

□

Theorem 4.11. *Let $G = (\sigma, \mu)$ be a fuzzy graph [**Ref.**[?], Theorem 2.4., p.21] or an $t-$norm fuzzy graph [**Ref.**[?], Theorem 3.3., p.132]. Let $v_1 v_2 \in E$. Let μ' be the fuzzy subset of E such that $\mu'(xy) = 0$ and $\mu' = \mu$ otherwise. Then*

*-$t-$**norm Fuzzy Graphs:*** (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1) ***-Fuzzy Graphs:*** (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1)

(1) *$v_1 v_2$ is a bridge;*
(2) $\mu^{\infty}_{G'}(v_1, v_2) < \mu(v_1 v_2)$;
(3) *$v_1 v_2$ is not a weakest edge of any cycle.*

Corollary 4.12. *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph and $v_1 v_2 \in E$. $v_1 v_2$ is an α-strong edge if and only if $v_1 v_2$ is a bridge.*

Proof. By Theorem **??**, the result is obviously hold. □

Theorem 4.13. *[Fuzzy Graph: **Ref.**[?], Proposition 2.7, p.24] [$t-$norm Fuzzy Graph: **Ref.**[?], Theorem 3.30, p.137] Let $G = (\sigma, \mu)$ be a fuzzy tree. Then the edges of $F = (\tau, \nu)$ are just the bridges of G.*

Corollary 4.14. *Let $G = (\sigma, \mu)$ be a fuzzy tree. Then edges of $F = (\sigma, \tau)$ are just the α-strong edges of G.*

Proof. By Theorem **??** and Corollary **??**, the result follows. □

Proposition 4.15. *Let $G = (\sigma, \mu)$ be a fuzzy tree. Then $D(T) = D(F) \cup D(S)$, where $D(T), D(F)$ and $D(S)$ are vertex dominating sets of T, F and S, respectively. S is a set of edges which has no edges with connection to F.*

Proof. By Corollary **??**, the edges of $F = (\sigma, \tau)$ are just the α-strong edges of G. The result follows. □

In the following result, we will partition the edges of a fuzzy cycle to two types $\alpha-$strong and other one.

Proposition 4.16 (Fuzzy Cycle)**.** *Let $G = (\sigma, \mu)$ be a fuzzy cycle. All edges are $\alpha-$strong with the only exceptions of weakest edges.*

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Proof. We study it in two cases.

Fuzzy Graphs: By regarding the definition of a fuzzy cycle, at least two edges have minimum value between all edges. It implies two cases. The first is of weakest edges and the latter case is of $\alpha-$strong edges.

$t-$**norm Fuzzy Graphs:** We can say about the weakest edges in $t-$norm fuzzy graphs but there is no information about their relations with strength of path which they are on it. In other words, Is $T(v_1, v_2, \cdots, v_n)$ equal with strength of weakest edges?

□

Proposition 4.17. *For any fuzzy graph $G = (\mu, \sigma)$, if there is a path which an edge v_1v_2 is only weakest edge on it, then v_1v_2 is not $\alpha-$strong edge.*

Proof. We study it in two cases.

Fuzzy Graphs: There is a path which an edge v_1v_2 is only weakest edge on it. So by deleting this edge, the intended path increases the strength of connectedness between v_1 and v_2. Then v_1v_2 is not $\alpha-$strong edge.

$t-$**norm Fuzzy Graphs:** We can say about the weakest edges in $t-$norm fuzzy graphs but there is no information about their relations with strength of path which they are on it. In other words, Is $T(v_1, v_2, \cdots, v_n)$ equal with strength of weakest edges?

□

Example 4.18. *Let $G_1 = (\sigma, \mu_1)$ and $G_2 = (\sigma, \mu_2)$ be fuzzy graphs as Figures* **??** *and* **??**. *Then $G_1 = (\sigma, \mu_1)$ is a fuzzy tree, but not a tree and not a fuzzy cycle while $G_2 = (\sigma, \mu_2)$ is a fuzzy cycle, but not a fuzzy tree.*

In $G_1 = (\sigma, \mu_1)$, the set $S = \{v_1\}$ is an α-strong dominating set. This set is also vertex dominating set in fuzzy tree (but not a fuzzy cycle) G_1. Hence $\gamma_v(G_1) = 0.7 + 0.77 = 1.47$. So $\gamma_v(G_1) = 1.47$.

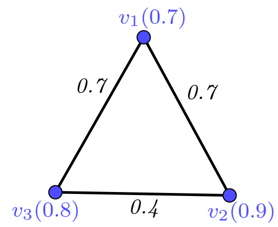

FIGURE 6. A Fuzzy Tree, but neither a Tree and nor a Fuzzy Cycle

In $G_2 = (\sigma, \mu_2)$, the set $S = \{v_1, v_3\}$ is an α-strong dominating set. This set is also vertex dominating set in fuzzy cycle (but not a fuzzy tree) G_2. Hence $\gamma_v(G_2) = 0.7 + 0.63 + 0.8 + 0 = 2.13$. So $\gamma_v(G_2) = 2.13$.

We give an upper bound for the vertex domination number, Proposition **??**.

Proposition 4.19. *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph. Then we have $\gamma_v \leq p$.*

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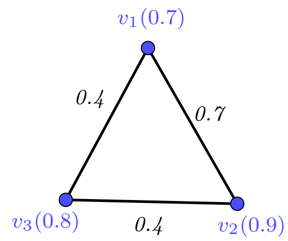

FIGURE 7. A Fuzzy Cycle, but not a Fuzzy Tree.

Proof. By Proposition **??**, the intended fuzzy graph has vertex domination number equals p. So the result follows. □

For any fuzzy graph or $t-$norm fuzzy graph, the Nordhaus-Gaddum(NG)'s result holds, (Theorem **??**).

Theorem 4.20. *For any fuzzy graph or $t-$norm fuzzy graph $G = (\sigma, \mu)$, the Nordhaus-Gaddum result holds. In other words, we have $\gamma_v + \bar{\gamma}_v \leq 2p$.*

Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph. So $\bar{G}$ is also the same type. We implement Theorem **??**, on G and $\bar{G}$. Then $\gamma_v \leq p$ and $\bar{\gamma}_v \leq p$. Hence $\gamma_v + \bar{\gamma}_v \leq 2p$. □

Definition 4.21. *An α-strong dominating set D is called a minimal α-strong dominating set if no proper subset of D is an α-strong dominating set.*

Theorem 4.22. *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph, without isolated vertices. If D is a minimal α-strong dominating set then $V \setminus D$ is a α-strong dominating set.*

Proof. By attentions to all edges between two sets, which are only α-strong, the result follows. □

A domatic partition is a partition of the vertices of a graph into disjoint dominating sets. The maximum number of disjoint dominating sets in a domatic partition of a graph is called its domatic number.
Finding a domatic partition of size 1 is trivial and finding a domatic partition of size 2 (or establishing that none exists) is easy but finding a maximum-size domatic partition (i.e., the domatic number), is computationally hard. Finding domatic partition of size two in a fuzzy graph or an $t-$norm fuzzy graph G of order $n \geq 2$ is obtained by the following.

Theorem 4.23. *Every fuzzy graph or $t-$norm fuzzy graph $G = (\sigma, \mu)$, without isolated vertices, of order $n \geq 2$ has an α-strong dominating set D such that whose complement $V \setminus D$ is also an α-strong dominating set.*

Proof. For every fuzzy graph or $t-$norm fuzzy graph $G = (\sigma, \mu)$, without isolated vertices, V is an α-strong dominating set. By analogous to the proof of Theorem **??**, we can obtain the result. □

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We improve the upper bound for the vertex domination number of fuzzy graphs and $t-$norm fuzzy graphs, without isolated vertices, (Theorem **??**).

Theorem 4.24. *For any fuzzy graph or $t-$norm fuzzy graph $G = (\sigma, \mu)$, without isolated vertices, we have $\gamma_v \leq \frac{p}{2}$.*

Proof. Let D be a minimal dominating set of G. By Theorem **??**, $V \setminus D$ is an α-strong dominating set of G. Hence $\gamma_v(G) \leq w_v(D)$ and $\gamma_v(G) \leq w_v(V \setminus D)$.
Therefore $2\gamma_v(G) \leq w_v(D) + w_v(V \setminus D) \leq p$ which implies $\gamma_v \leq \frac{p}{2}$. Hence the proof is completed. □

We also improve Nordhaus-Gaddum (NG)'s result for fuzzy graphs or $t-$norm fuzzy graphs, without isolated vertices, (Corollary **??**).

Corollary 4.25. *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph, such that both of G and $\bar{G}$ have no isolated vertices. Then $\gamma_v + \bar{\gamma_v} \leq p$, where $\bar{\gamma_v}$ is the vertex domination number of $\bar{G}$. Moreover, the equality holds if and only if $\gamma_v = \bar{\gamma_v} = \frac{p}{2}$.*

Proof. By the Implement of Theorem **??**, on G and $\bar{G}$, we have $\gamma_v(G) = \gamma_v \leq \frac{p}{2}$, and $\gamma_v(\bar{G}) = \bar{\gamma_v}(G) = \bar{\gamma_v} \leq \frac{p}{2}$. So $\gamma_v + \bar{\gamma_v} \leq \frac{p}{2} + \frac{p}{2} = p$. Hence $\gamma_v + \bar{\gamma_v} \leq p$.
Suppose $\gamma_v = \bar{\gamma_v} = \frac{p}{2}$. Then obviously, $\gamma_v + \bar{\gamma_v} = p$. Conversely, suppose $\gamma_v + \bar{\gamma_v} \leq p$. Then we have $\gamma_v \leq \frac{p}{2}$ and $\bar{\gamma_v} \leq \frac{p}{2}$. If either $\gamma_v < \frac{p}{2}$ or $\bar{\gamma_v} < \frac{p}{2}$, then $\gamma_v + \bar{\gamma_v} < p$, which is a contradiction. Hence the only possible case is $\gamma_v = \bar{\gamma_v} = \frac{p}{2}$. □

Proposition 4.26. *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph. If all edges have equal value, then G has no α-strong edge.*

Proof. By using Definition of α-strong edge, the result is hold. □

The following example illustrates this concept.

Example 4.27. *In Figure **??**, all edges have the same value but there is no α-strong edges in this fuzzy graph.*

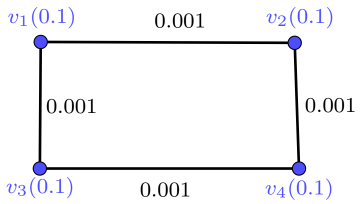

FIGURE 8. Identical edges and α-strong edges

We give the relationship between M-strong edges and α-strong edges, (Corollary **??**).

Corollary 4.28. *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph. If all edges are M-strong, then G has no α-strong edge.*

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Proof. By Proposition **??**, the result follows. □

We give a necessary and sufficient condition for vertex domination number which is half of order, under some specific conditions. In fact, the fuzzy graphs and $t-$norm fuzzy graphs, which their vertex domination number is half of order, are characterized under some specific conditions, (Theorem **??**).

Theorem 4.29. *In any fuzzy graph or any $t-$norm fuzzy graph $G = (\sigma, \mu)$, such that values of vertices are equal and all edges have same values, i.e. $\forall v_1, v_2 \in V, \sigma(v_1) = \sigma(v_2)$ and $\forall v_1v_2, v_3v_4 \in E, \mu(v_1v_2) = \mu(v_3v_4)$. $\gamma_v = \frac{p}{2}$ if and only if for any vertex dominating set D in G, we have $|D| = \frac{n}{2}$.*

Proof. Suppose D has the conditions. By Proposition **??**, $\forall v \in D, \sum_{vv_1 \in S} \mu(vv_1) = 0$ where $S = \{v_1v_2 \in E \mid \mu(v_1v_2) > \mu^{\infty}_{G'}(v_1, v_2)\}$; so by using Definition, $\gamma_v(G) = \Sigma_{v\in D}\sigma(v)$. Since values of vertices are equal and $|D| = \frac{n}{2}$, we have $\gamma_v(G) = \Sigma_{v\in D}\sigma(v) = \frac{n}{2}\sigma(v) = \frac{1}{2}(n\sigma(v)) = \frac{1}{2}(\Sigma_{v\in V}\sigma(v)) = \frac{1}{2}(p) = \frac{p}{2}$. Hence the result is hold in this case.
Conversely, suppose $\gamma_v = \frac{p}{2}$. Let $D = \{v_1, v_2, \cdots, v_n\}$ be a vertex dominating set. By Proposition **??**, $\forall v \in D, \sum_{vv_1 \in S} \mu(vv_1) = 0$ where $S = \{v_1v_2 \in E \mid \mu(v_1v_2) > \mu^{\infty}_{G'}(v_1, v_2)\}$; so by using Definition, $\gamma_v(G) = \Sigma_{v\in D}\sigma(v)$. Since $\gamma_v(G) = W_v(D)$, we have $\gamma_v = \frac{p}{2} = \frac{1}{2}(\Sigma_{v\in V}\sigma(v)) = \Sigma_{v\in D}\sigma(v)$. Suppose $n' \neq \frac{n}{2}$. So $\Sigma_{i=1}^{n''}\sigma(v_i)) = 0$ which is a contradiction with $\forall v_i \in V, \sigma(v_i) > 0$. Hence $n' = \frac{n}{2}$, i.e. $|D| = n' = \frac{n}{2}$. The result is hold in this case. □

The goal of upcoming texts is to prove some results concerning operations and study some conjectures arising from it.

Proposition 4.30. *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be fuzzy graphs or $t-$norm fuzzy graphs. A vertex dominating set in $G_1 \cup G_2$ is $D = D_1 \cup D_2$ such that D_1 and D_2 are vertex dominating sets of G_1 and G_2, respectively. Moreover, $\gamma_v(G_1 \cup G_2) = \gamma_v(G_1) + \gamma_v(G_2)$.*

Proof. By using Definition of union, the result is obviously hold. □

Corollary 4.31. *Let $G_i = (\sigma_i, \mu_i)$ be fuzzy graphs or $t-$norm fuzzy graphs, for $i = 1, \cdots, n$. A vertex dominating set in $\cup_{i=1}^{n} G_i$ is $D = \cup_{i=1}^{n} D_i$ such that D_i are vertex dominating sets in $G_i, i = 1, \cdots, n$. Moreover, $\gamma_v(\cup_{i=1}^{n} G_i) = \Sigma_{i=1}^{n}\gamma_v(G_i)$.*

Proof. By Proposition **??**, the result is hold. □

The concept of monotone decreasing, (Definition **??**), are introduced.

Definition 4.32. *Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph. A property is monotone decreasing if removing an edge, does not destroy the property.*

Conjecture (Vizing). For all G and H, $\gamma(G)\gamma(H) \leq \gamma(G \times H)$.
By using α-strong edge and monotone decreasing, the result in relation with Vizing's conjecture is determined, (Theorem **??**).

Theorem 4.33. *The Vizing's conjecture is monotone decreasing property if removed edges are $\alpha-$strong.*

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Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph and $G^{'}$ be a new one which is obtained from G by removing an edge. For every $G_1 = (\sigma_1, \mu_1)$, a $G^{'} \times G_1$ is a spanning subgraph of $G \times G_1$. So $\gamma_v(G^{'} \times G_1) \geq \gamma_v(G \times G_1) \geq \gamma_v(G)\gamma_v(G_1) = \gamma_v(G^{'})\gamma_v(G_1)$. Hence Vizing's conjecture is also hold for $G^{'}$. Then the result follows. □

Corollary 4.34. *Suppose the Vizing's conjecture is hold. Let G_1 be a spanning subgraph of G such that $\gamma_v(G_1) = \gamma_v(G)$. Then the Vizing's conjecture is also hold for G_1.*

Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph or an $t-$norm fuzzy graph and G_1 be a spanning subgraph of G such that $\gamma_v(G_1) = \gamma_v(G)$. For every $G_2 = (\sigma_2, \mu_2)$, a $G_1 \times G_2$ is a spanning subgraph of $G \times G_2$. So $\gamma_v(G_1 \times G_2) \geq \gamma_v(G \times G_2) \geq \gamma_v(G)\gamma_v(G_2) = \gamma_v(G_1)\gamma_v(G_2)$. Hence the Vizing's conjecture is also hold for G_1. So the result follows. □

Proposition 4.35. *Let $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ be fuzzy graphs or $t-$norm fuzzy graphs. A vertex dominating set of $G_1 + G_2$ is $D = D_1 \cup D_2$ such that D_1 and D_2 are vertex dominating sets of G_1 and G_2, respectively. Moreover, $\gamma_v(G_1 + G_2) = \gamma_v(G_1) + \gamma_v(G_2)$.*

Proof. By using Definition of join, M-strong edges between two models are not α-strong which is a weak edge changing strength of connectedness of G. □

Corollary 4.36. *Let $G_i = (\sigma_i, \mu_i)$ be fuzzy graphs or $t-$norm fuzzy graphs, for $i = 1, \cdots, n$, respectively. A vertex dominating set of $+_{i=1}^{n} G_i$ is $D = +_{i=1}^{n} D_i$ such that D_i are vertex dominating sets of G_i. Moreover, $\gamma_v(+_{i=1}^{n} G_i) = \Sigma_{i=1}^{n} \gamma_v(G_i)$.*

Proof. By Proposition **??**, the result is hold. □

Conjecture (Gravier and Khelladi). For all G and H,

$$\gamma(G)\gamma(H) \leq 2\gamma(G + H).$$

By using α-strong edge and monotone decreasing, the result in relation with the Gravier and Khelladi's conjecture is determined, (Theorem **??**).

Theorem 4.37. *The Gravier and Khelladi's conjecture is monotone decreasing property if removed edges are $\alpha-$strong.*

Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph or $t-$norm fuzzy graph, and $G^{'}$ be a new one which is obtained from G by removing an edge. For every $G_1 = (\sigma_1, \mu_1)$, a $G^{'} + G_1$ is a spanning subgraph of $G + G_1$. So $2\gamma_v(G^{'}) + G_1) \geq 2\gamma_v(G + G_1) \geq \gamma_v(G)\gamma_v(G_1) = \gamma_v(G^{'})\gamma_v(G_1)$. Hence the Gravier and Khelladi's conjecture is also hold for $G^{'}$. Then the result follows. □

We conclude this section with some result in relation with the Gravier and Khelladi's conjecture, (Corollary **??**).

Corollary 4.38. *Suppose the Gravier and Khelladi's conjecture is hold. Let G_1 be a spanning subgraph of G such that $\gamma_v(G_1) = \gamma_v(G)$. Then the Gravier and Khelladi's conjecture is hold for G_1*

Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph or $t-$norm fuzzy graph, and G_1 be a spanning subgraph of G such that $\gamma_v(G_1) = \gamma_v(G)$. For every $G_2 = (\sigma_2, \mu_2)$, a $G_1 \times G_2$ is a spanning subgraph of $G \times G_2$. So $2\gamma_v(G_1 + G_2) \geq 2\gamma_v(G + G_2) \geq \gamma_v(G)\gamma_v(G_2) = \gamma_v(G_1)\gamma_v(G_2)$. Hence the Gravier and Khelladi's conjecture is also hold for G_1. The result follows. $\square$

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