Module 1, Classes 2-3: Composing and solving the problems for a mathematical competition

Course: EDUCATIONAL INNOVATION

January 17 - 27, 2022

Stanislav Harizanov

Insitute of Mathematics and Informatics
Bulgarian Academy of Sciences

19.01.2022

ispas paths to successful innovations

Main Bulgarian math competitions 2021–2022

in cooperation with the Union of Bulgarian Mathematicians

Name of the competition	Type	Date
National	4 problems	till December 12, 2021
Mathematical	4 problems	February 12, 2022
Olympiad	2 days 3 problems	April 15-18, 2022
Fall Math Tournament	4 problems	November 5-7, 2021
Winter Math Competitions	4 problems	January 28-30, 2022
Spring Math Competitions	4 problems	March 25-27, 2022
"Chernorizets Hrabar"	Test	November 1, 2021
"Ivan Salabashev"	Test	December 4, 2021
"European Kangaroo"	Test	March 17, 2022

ispas paths to successful innovations

Test competitions: Multiple choice questions

Main qualities a student need to possess for successful participation:

Test competitions: Multiple choice questions

Main qualities a student need to possess for successful participation:

- Concentration and attention to details

Test competitions: Multiple choice questions

Main qualities a student need to possess for successful participation:

- Concentration and attention to details
- Quick reactions

Test competitions: Multiple choice questions

Main qualities a student need to possess for successful participation:

- Concentration and attention to details
- Quick reactions
- Intuition

Test competitions: Multiple choice questions

Main qualities a student need to possess for successful participation:

- Concentration and attention to details
- Quick reactions
- Intuition
- Induction and Deduction

ispas paths to successful innovations

Test competitions: Multiple choice questions

Main qualities a student need to possess for successful participation:

- Concentration and attention to details
- Quick reactions
- Intuition
- Induction and Deduction
- Logical thinking and ability to rule out wrong answers

Test competitions: Multiple choice questions

Main qualities a student need to possess for successful participation:

- Concentration and attention to details
- Quick reactions
- Intuition
- Induction and Deduction
- Logical thinking and ability to rule out wrong answers
- Risk calculation

Testing the students' concentration and attention to details

When composing multiple choice problems, the choice of the set of answers is the key point for creating distracters – correct answers for certain modifications of the original problem.

Testing the students' concentration and attention to details

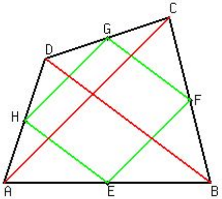
When composing multiple choice problems, the choice of the set of answers is the key point for creating distracters – correct answers for certain modifications of the original problem.

Example ("Ivan Salabashev" 2021, 8-9 Grade, Problem 6)

Let $ABCD$ be a convex quadrilateral with midpoints of the sides AB, BC, CD, and DA the points E, F, G, and H, respectively. On the diagonal AC is chosen a point S, such that the areas of $HSGD$, $AESH$, and $EBFS$ are 22, 24, and 32, respectively. The ratio $AS : SC$ is equal to

A) $\frac{4}{3}$ **B)** $\frac{3}{4}$ **C)** $\frac{5}{4}$ **D)** $\frac{4}{5}$

Sketch of solution: $AS : SC = S_{AESH} : S_{CGSF}$.



Testing the students' quick reactions

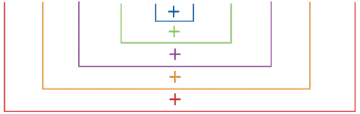
Due to the severe time limitation during test competitions, it is crucial for the competitors to possess a very good "computation technique". Therefore, some of the composed multiple choice problems should check it, meaning that they should allow straightforward but computationally heavy solutions, which exhaust plenty of time, and also tricky but computationally light solutions, which save time.

Testing the students' quick reactions

Example (Gauss Summation)

The Gauss Summation is named for Johann Karl Friedrich Gauss. He was a German mathematician. Gauss is one of history's most influential mathematical thinkers. A legend suggests that Gauss came up with a new method of summing sequences at a very young age. The legend says that his math teacher asked the class to add the numbers 1 to 100. In other words, the teacher wanted them to add $1 + 2 + 3 + 4 + 5 \ldots$ all the way up to 100!
The teacher assumed that this would take the students a very long time. Think about how long it would take you to add up all the numbers from 1 to 100 one by one. However, Gauss answered 5050 almost immediately.

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = ?



(1+10) + (2+9) + (3+8) + (4+7) + (5+6) = ?

Testing the students' intuition

In test competitions only the correct answer matters, not the process of deriving it. Thus, it is not necessary to completely solve the problem mathematically (e.g., to clearly prove the validity of each of the argument you are applying), but rather to "trust" your intuition.

Testing the students' intuition

In test competitions only the correct answer matters, not the process of deriving it. Thus, it is not necessary to completely solve the problem mathematically (e.g., to clearly prove the validity of each of the argument you are applying), but rather to "trust" your intuition.

Example ("Ivan Salabashev" 2019, 8-9 Grade, Problem 8)

The positive integers a and b satisfy the equality $4a - 7b + 28ab = 2020$. Their product ab equals to:

A) 96 **B)** 102 **C)** 106 **D)** 108

Sketch of solution: Consider the small cases, e.g., $a = 1$. Observe that $b = 96$ is a solution. Therefore, $ab = 96$.

ispas paths to successful innovations

Testing the students' ability to induct and deduct

In test competitions it is not necessary to attack a given problem in its full generality, so if you find a partial case that satisfies all the assumptions and is easier to be solved – you can deduce the general answer by solving the partial case problem. And vice versa.

Testing the students' ability to induct and deduct

Example (Gauss Summation)

Compute

$$1 + 2 + 3 + \cdots + 2021.$$

Sketch of solution: By induction

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Therefore, replacing n by 2021 we obtain

$$1 + 2 + 3 + \cdots + 2021 = \frac{2021 \cdot 2022}{2} = 2021 \cdot 1011 = 2043231.$$

Testing the students' ability to induct and deduct

Example ("Ivan Salabashev" 2021, 8-9 Grade, Problem 14)

Cards with numbers 1 to 52 are in a deck. In a game, a player draws a card randomly from the deck, does not replace it, and repeats this step as long as the numbers drawn continue to increase. The player's score is the number of cards drawn in increasing order from the first draw. For example the sequence $3, 5, 9, 8$ would score three points. If the probability for the player scoring more than 5 points is written as an irreducible fraction $\frac{p}{q}$, than the expression $p + q$ equals to?

Sketch of solution: $P_{>5pts} = 1 - P_{\leq 5pts}$. To compute $P_{\leq 5pts}$ one needs to consider the game with only 6 cards, not 52. There, the only sequence that leads to a higher score than 5 is $1, 2, 3, 4, 5, 6$, thus

$$P_{>5pts} = 1 - P_{\leq 5pts} = \frac{1}{6!} = \frac{1}{720} \quad \Rightarrow \quad p = 1,\ q = 720 \ \text{ and } p + q = 721.$$

Testing the students' logical thinking and ability to rule out wrong answers

Sometimes, even when the student does not have an idea how to solve the problem, part of the answers could be eliminated .

Testing the students' logical thinking and ability to rule out wrong answers

Sometimes, even when the student does not have an idea how to solve the problem, part of the answers could be eliminated .

Example ("Ivan Salabashev" 2020, 6 Grade, Problem 6)

The closest positive integer to 2020, such that when subtracting 5 from it the remainder is divisible by 5, when subtracting 6 from it the remainder is divisible by 6, when subtracting 7 from it the remainder is divisible by 7, and when subtracting 8 from it the remainder is divisible by 8 is:

A) 1680 **B)** 1890 **C)** 2100 **D)** 2520

Sketch of solution: The answer should be a number, divisible by 5,6,7,and 8. Answers B) and C) are not divisible by 8, thus can be eliminated. Answers A) and D) satisfy the above restriction, so one of them is the correct answer. $|2020 - 1680| < |2020 - 2520|$, thus the correct answer is A).

Risk calculation

Let us consider a multiple-answer test problem that gives X points for the correct answer, 1 point for not choosing an answer (i.e., bonus for a fair play), and 0 points for a wrong answer. After logical thinking the student eliminates some of the wrong answers and is left with k choices. Should they risk picking an answer or should they play fair and leave it blank in the answer sheet?

- $k = 5$: Expected gain of a random pick: $\dfrac{X}{5}$
  Thus, if $X \geq 6$ it is worth guessing.
- $k = 4$: Expected gain of a random pick: $\dfrac{X}{4}$
  Thus, if $X \geq 5$ it is worth guessing.
- $k = 3$: Expected gain of a random pick: $\dfrac{X}{3}$
  Thus, if $X \geq 4$ it is worth guessing.
- $k = 2$: Expected gain of a random pick: $\dfrac{X}{2}$
  Thus, if $X \geq 3$ it is worth guessing.

ispas paths to successful innovations

Main Bulgarian math competitions 2021–2022

in cooperation with the Union of Bulgarian Mathematicians

Name of the competition	Type	Date
National	4 problems	till December 12, 2021
Mathematical	4 problems	February 12, 2022
Olympiad	2 days 3 problems	April 15-18, 2022
Fall Math Tournament	4 problems	November 5-7, 2021
Winter Math Competitions	4 problems	January 28-30, 2022
Spring Math Competitions	4 problems	March 25-27, 2022
"Chernorizets Hrabar"	Test	November 1, 2021
"Ivan Salabashev"	Test	December 4, 2021
"European Kangaroo"	Test	March 17, 2022

ispas paths to successful innovations

Problem topics

- Algebra: Similar to *Calculus* in undergraduate programs. Deals with functions, sequences, irreducibility, inequalities, optimization, etc.
- Combinatorics: Deals with game theory, graph theory, optimal strategies, counting in different ways, the pigeonhole principle, providing counterexamples, etc.
- Geometry: Deals with problems, related to the classical planar setting.
- Number Theory: Similar to *Linear Algebra* in undergraduate programs. Deals with divisibility, modular residuals, valuations, function iterations, etc.

Problem topics

- Algebra: Similar to *Calculus* in undergraduate programs. Deals with functions, sequences, irreducibility, inequalities, optimization, etc.
- Combinatorics: Deals with game theory, graph theory, optimal strategies, counting in different ways, the pigeonhole principle, providing counterexamples, etc.
- Geometry: Deals with problems, related to the classical planar setting.
- Number Theory: Similar to *Linear Algebra* in undergraduate programs. Deals with divisibility, modular residuals, valuations, function iterations, etc.

In each mathematical olympiad there should be balance among the four topics!

ispas paths to successful innovations

Competition types

All the non-test mathematical competitions within the calendar of the Union of the Bulgarian Mathematicians serve as qualifiers for a certain international competition. Therefore, the structure of their problem sets mimics the one of the corresponding international competition. We divide the competitions in two major classes:

Competition types

All the non-test mathematical competitions within the calendar of the Union of the Bulgarian Mathematicians serve as qualifiers for a certain international competition. Therefore, the structure of their problem sets mimics the one of the corresponding international competition. We divide the competitions in two major classes:

- BOM-like: Those are all seasonal competitions, which are part of the team selection for the Balkan Mathematical Olympiad. These competitions consist of 4 problems: one of each of the four topics A, C, G, NT.

Competition types

All the non-test mathematical competitions within the calendar of the Union of the Bulgarian Mathematicians serve as qualifiers for a certain international competition. Therefore, the structure of their problem sets mimics the one of the corresponding international competition. We divide the competitions in two major classes:

- BOM-like: Those are all seasonal competitions, which are part of the team selection for the Balkan Mathematical Olympiad. These competitions consist of 4 problems: one of each of the four topics A, C, G, NT.
- IMO-like: This is the final round of the National Mathematical Olympiad, which is part of the team selection for the International Mathematical Olympiad. It consists of 6 problems in total, 3 problems per day for two consecutive days. Problems 1 and 4 are considered *Easy*, Problems 2 and 5 – *Medium*, while Problems 3 and 6 – *Hard*. Problems 1,2,4,and 5 should be one per topic A, C, G, NT.

Level of difficulty

How to realize if a problem is Easy, Medium, or Hard? This is a quite subjective perception and it differs from one person to another. For example, at IMO the team leader of each country ranks all the problems from the shortlist in 1 out of 5 categories: Easy, Easy to Medium, Medium, Medium to Hard, Hard. Then each problem from the shortlist goes to the category the most people voted it for.
In general, if we want to increase the difficulty of a particular problem, one of the following can be done:

- Hide the answer of the question, i.e., instead of "Prove that" do "Find" when possible.
- Invert the problem logic flow.
- Increase the number of arguments/steps, needed for solving the problem, i.e., combine several "easy" problems into a "harder" one.

ispas paths to successful innovations

Hide the answer of the question

Example (Easy version Winter Math Competitions 2021, Problem 9.2)

Let $ABCD$ be a square. The point M on the segment AB and the point N on the segment BC are chosen, such that $BM = CN$. Let DN intersect CM at point P. If $AP = AB$, then prove that $S_{AMPD} : S_{ABCD} = 11 : 20$.

Hide the answer of the question

Example (Easy version Winter Math Competitions 2021, Problem 9.2)

Let $ABCD$ be a square. The point M on the segment AB and the point N on the segment BC are chosen, such that $BM = CN$. Let DN intersect CM at point P. If $AP = AB$, then prove that $S_{AMPD} : S_{ABCD} = 11 : 20$.

Example (Hard version Winter Math Competitions 2021, Problem 9.2)

Let $ABCD$ be a square. The point M on the segment AB and the point N on the segment BC are chosen, such that $BM = CN$. Let DN intersect CM at point P. If $AP = AB$, then find the area ratio $S_{AMPD} : S_{ABCD}$.

Invert the problem logic flow

Example (Easy version)

Let ABC be an arbitrary triangle. Prove that its medians intersect in a common point G that divides each median segment in a ratio $2:1$, starting from the vertex endpoint.

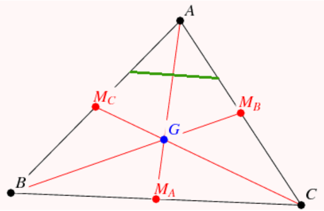
Invert the problem logic flow

Example (Easy version)

Let ABC be an arbitrary triangle. Prove that its medians intersect in a common point G that divides each median segment in a ratio $2:1$, starting from the vertex endpoint.

Example (Hard version)

In a triangle ABC, the points M_A, M_B, and M_C lie on the segments BC, CA, and AB, respectively, such that the segments AM_A, BM_B, and CM_C intersect in a common point G. If $AG : GM_A = BG : GM_B = 2:1$, prove that M_A is the midpoint of BC.



Increase the number of arguments/steps, needed for solving the problem, i.e., combine several "easy" problems into a "harder" one

Example (Greek JBMO TST 2019, Problem 4)

We consider a 8×8 chess table with all 64 unit squares white. We color 12 unit squares arbitrarily black. Prove that we can find 4 rows and 4 columns containing the 12 black unit squares.

- Proof, based on greedy argument.
- Easy to generalize with $8 \to 2n$, $12 \to 3n$, and $4 \to n$, i.e.,

Example (Generalization)

We consider a $2n \times 2n$ chess table with all $4n^2$ unit squares white. We color $3n$ unit squares arbitrarily black. Prove that we can find n rows and n columns containing the $3n$ black unit squares.

Increase the number of arguments/steps, needed for solving the problem, i.e., combine several “easy” problems into a “harder” one

Example (Turkish JBMO TST 2019, Problem 2)

Each square of an $n \times n$ chessboard either contains a rook or is empty. Suppose that for any two rooks not threatening each other there is an empty square which is threatened by both rooks. Find the maximal possible number of rooks on the chessboard.

Answer: $\lfloor 3n/2 \rfloor$. The partial case $n \to 2n$ reads as:

Example (Partial case)

Each square of an $2n \times 2n$ chessboard either contains a rook or is empty. Suppose that for any two rooks not threatening each other there is an empty square which is threatened by both rooks. Prove that the maximal possible number of rooks on the chessboard is $3n$.

Increase the number of arguments/steps, needed for solving the problem, i.e., combine several "easy" problems into a "harder" one

Example (Hard problem, combination of Greece and Turkey)

Each square of a $2n \times 2n$ chessboard either contains a rook or is empty. Suppose that for any two rooks not threatening each other there is an empty square which is threatened by both rooks. Prove that there exist n rows and n columns containing all rooks.