Template-Dependent Lifts for Path-Complete Stability Criteria and Application to Positive Switching Systems

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Abstract

In the framework of discrete-time switching systems, we analyze and compare various stability certificates relying on graph constructions. To this aim, we define several abstract expansions of graphs (so-called *lifts*), which depend on the chosen family of candidate Lyapunov functions (the *template*). We show that the validity of a given lift is linked with the analytical properties of the template. This allows us to generate new lifts, and as a by-product, to obtain comparison criteria that go beyond the concept of simulation recently introduced in the literature. We apply our constructions to the case of copositive linear norms for positive switching systems, leading to novel stability criteria that outperform the state of the art. We provide further results relying on convex duality and we demonstrate via numerical examples how the comparison among different stability criteria is affected by the properties of the copositive norms template.

Keywords: path-complete methods, switching systems, positive systems

1 Introduction

In this paper, we study discrete-time switching systems that evolve according to the following rule:

$$x(k+1) \;=\; f_{\sigma(k)}(x(k)), \tag{1}$$

where the state $x(k) \in \mathbb{R}^n$, and the mode $\sigma(k) \;\in \langle M\rangle := \{1, \ldots, M\}$ for an integer M. These systems are commonly used to model various engineering/physical phenomena, in which the state is possibly driven by several dynamic laws.

One of the major issues that researchers have tackled is the stability analysis of switching systems. The most popular approach to assess the stability is by searching for a *common Lyapunov function*, that is, a positive definite function that decreases along any trajectory of the system. Even though the existence of such a function is a sufficient and necessary condition for stability (see for example [1] and [2] for the non-linear case), it is not clear how to compute such a function in practice, see the discussion provided, for example, in [3]. *Multiple Lyapunov functions*, as defined for example in [4], [5] and [6], appear as a promising alternative to the common Lyapunov function method. Indeed, this approach consists in looking for a set of Lyapunov functions (instead of one) whose *joint behaviour* guarantees the stability. As unifying framework, [7] introduced the notion of *path-complete Lyapunov functions* (PCLF) for which the Lyapunov inequalities are encoded in a directed and labeled graph. More precisely, a PCLF is defined by two structural components: a combinatorial component that is a *path-complete* graph whose labels enable to produce any word on the alphabet $\langle M\rangle$, and an algebraic component, the *template*, that is, the set in which the candidate functions are selected.

The *path-complete Lyapunov functions* approach, despite its appealing flexibility, opens new questions and challenges, both from a theoretical and computational point of view. Indeed, the stability of system (1) can be established with different graphs structures and different candidate functions templates. In recent years, an increasing attention has been devoted to the comparison of different path-complete graphs. This leads to the introduction of order relations between graphs, where a graph is usually said "better than another one" if it provides a more accurate approximation of the decay rate of the system (1). In this context, [8] formalized two combinatorial operations on graphs called *lifts* in order to provide arbitrarily accurate estimates of the *constrained joint spectral radius* (CJSR). In parallel, other comparison techniques between path-complete graphs were proposed but only valid for some particular settings, for example in [7]. In this paper, we unify these two efforts in a more general framework, introducing formal operations on graphs, so-called *lifts*, which directly exploit the properties of the template.

This formalism allows us to recover, in a general framework, the comparison techniques proposed ad hoc for some specific situations, as the ones in [9] for instance. Moreover, we show how the lifting approach allows to reduce the size of the graph according to the template properties while keeping the same (or a better) stability criterion.

We apply our results in the framework of positive switching systems, i.e. the case where $\{f_i\}_{i\in\langle M\rangle}$ in (1) are defined by positive matrices $\{A_1,\dots,A_M\}$, see [10] for a thorough discussion. We consider the template of copositive linear norms, as introduced in [11], defined as the scalar products with respect to generic positive vectors. After having studied the analytical properties of this particular template, we describe a specific lifting procedure for this family of functions, and show that it leads to a better estimation and performance of the stability problem for positive switching systems, with the aid of a numerical example.

The structure of this paper is the following: in Section 2 we recall the main results, and the concept of comparison between path-complete criteria. In Section 3, we introduce our more general concept of *lift*, and provide a particular case for templates closed under the addition -and the minimum- operation. Finally in Section 4 we apply these lifts to study the stability of positive switching systems, showing the effectiveness of our approach for the template of copositive norms.

Notation: We define $\mathbb{R}_{>0} := \{x\in\mathbb{R}\ \mid x>0\}$, $\mathbb{R}_{\geq 0} := \{x\in\mathbb{R}\ \mid x\geq 0\}$, $\mathbb{R}^n_{>0} := (\mathbb{R}_{>0})^n$, $\mathbb{R}^n_{\geq 0} := (\mathbb{R}_{\geq 0})^n$. Given $M\in\mathbb{N}$, we define $\langle M\rangle := \{1,\dots,M\}$. The set $\{\mathbf{e}_i\}_{i\in\langle n\rangle}$ is the canonical basis of $\mathbb{R}^n$.

2 Preliminaries

The path-complete Lyapunov functions approach has led to new and useful results in approximating the convergence rate of discrete-time switching systems, see for instance [9] and [7] for a thorough discussion. We briefly recall here the main ideas.

Given $M\in\mathbb{N}$, a *directed and labeled graph* $\mathcal{G}=(S,E)$ on $\langle M\rangle$ is defined by a finite set S (the set of nodes) and $E\subset S\times S\times\langle M\rangle$ (the set of labeled edges).

Definition 1. *A graph $\mathcal{G}=(S,E)$ is* path-complete *on $\langle M\rangle$ if, for any $K\geq 1$ and any word $(j_1\,\dots\,j_K)\in\langle M\rangle^K$, there exists a* path *$\{(s_k,s_{k+1},j_k)\}_{1\leq k\leq K}$ such that $(s_k,s_{k+1},j_k)\in E$, for each $1\leq k\leq K$.*

We consider $F:=\{f_1,\dots,f_M\}\subset\mathcal{C}^0(\mathbb{R}^n,\mathbb{R}^n)$ (the set of dynamics) and we define the discrete-time switching system

$$x(k+1)\ =\ f_{\sigma(k)}(x(k)), \tag{2}$$

where $\sigma:\mathbb{N}\to\langle M\rangle$ is the switching signal. We say that (2) is *stable* if there exists an $\alpha\in\mathcal{K}_\infty$[1] such that $\|x(k)\|\leq\alpha(\|x(0)\|)$, for any initial condition $x(0)\in\mathbb{R}^n$, any switching signal σ and any time $k\in\mathbb{N}$.

Definition 2. *Given $F=\{f_1,\dots,f_M\}\subset\mathcal{C}^0(\mathbb{R}^n,\mathbb{R}^n)$, a* path-complete Lyapunov function *(PCLF) is a pair $(\mathcal{G},V)$ where $\mathcal{G}=(S,E)$ is a path-complete graph, and $V=\{V_s\mid s\in S\}$ is a set of continuous, positive definite and radially unbounded functions such that the following inequalities are satisfied:*

$$\forall(a,b,i)\in E,\forall x\in\mathbb{R}^n: V_b(f_i(x))\ \leq\ V_a(x). \tag{3}$$

If this is the case, we say that V is admissible for $\mathcal{G}$ and F*, and we denote it by $V\in PCLF(\mathcal{G},F)$.*

In [7] it is proved that, given any $M\in\mathbb{N}$, any system $F=\{f_i\,|\,i\in\langle M\rangle\}$ and any path-complete graph $\mathcal{G}=(S,E)$, if $\exists V=\{V_s\,|\,s\in S\}\in PCLF(\mathcal{G},F)$, then system (2) is stable. We now introduce order relations among the set of path-complete graphs, formalizing the idea that one graph "produces less conservative stability conditions" with respect to another.

Definition 3. *Consider two path-complete graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ on $\langle M\rangle$, a set of candidate Lyapunov functions $\mathcal{V}$ (a template) and a family $\mathcal{F}$ of M-tuples of continuous vector fields.*

[1]A function $\alpha:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}_\infty$ ($\alpha\in\mathcal{K}_\infty$) if it is continuous, $\alpha(0)=0$, strictly increasing and unbounded.

(a) We say that

$$\mathcal{G}_1 \ \leq_{\mathcal{V},\mathcal{F}} \ \mathcal{G}_2 \tag{4}$$

if, for any $F \in \mathcal{F}$,

$$\Big[\exists V \in \mathcal{V}^{|S|} \ s.t. \ V \in PCLF(\mathcal{G}_1, F) \ \Big] \ \Rightarrow \ \Big[\exists V \in \mathcal{V}^{|S|} \ s.t. \ V \in PCLF(\mathcal{G}_2, F)\Big].$$

(b) We say that

$$\mathcal{G}_1 \ \leq_{\mathcal{V}} \ \mathcal{G}_2 \tag{5}$$

if the inequality (4) is satisfied for any family $\mathcal{F}$.

(c) We say that

$$\mathcal{G}_1 \ \leq \ \mathcal{G}_2 \tag{6}$$

if for any template $\mathcal{V}$, the inequality (5) is satisfied.

In [12, Theorem 3.5], the Authors proposed a complete characterization of the general order in Item (c) of Definition 3 relying on the combinatorial concept of *simulation* between graphs. Here, instead, we want to go further with the comparison analysis, underlining the relations between the analytical properties of the chosen template/class of systems and the conservatism of a path-complete policy with respect to the others.

3 Template-Dependent lifts

In this section, we define several expansions of graphs, called lifts, which enable us to establish ordering relations between graphs, as introduced in Definition 3.

Definition 4. *Given $M \in \mathbb{N}$, we denote with $Graphs_M$ the set of directed and labeled graphs on $\langle M \rangle$. We say that $L : Graphs_M \to Graphs_M$ is a* valid lift with respect to a template $\mathcal{V}$ *if*

1. *$\mathcal{G}$ path-complete implies $L(\mathcal{G})$ is path-complete,*
2. *$\mathcal{G} \ \leq_{\mathcal{V}} \ L(\mathcal{G})$, for all path-complete graph $\mathcal{G}$.*

Similarly, L is a valid lift with respect to a template $\mathcal{V}$ and a family $\mathcal{F} \subset (\mathcal{C}^0(\mathbb{R}^n, \mathbb{R}^n))^M$, *if Item (2) is replaced by*

(2)' $\mathcal{G} \ \leq_{\mathcal{V},\mathcal{F}} \ L(\mathcal{G})$, for all path-complete graph $\mathcal{G}$.

Remark 1. *Note that, given a path-complete graph $\mathcal{G}$ and a valid lift with respect to a template $\mathcal{V}$, the inequality (2) in Definition 4 holds for every path-complete and strongly connected component of the lifted graph $L(\mathcal{G})$. The same remark can be made for a valid lift with respect to a template $\mathcal{V}$ and a family $\mathcal{F}$.*

Similar examples of abstract lifts have already been introduced in [8], such as the M-path-dependent lift and the T-product lift. However, both these lifts (see [8, Definitions 2,3]), even if introduced in the framework of quadratic multinorms, satisfy Definition 4 with respect to *any arbitrary template*. In the following, we introduce lifts that are not valid with respect to arbitrary templates, but *only if* the template satisfies some properties. This approach is motivated by our ultimate goal, i.e. to provide a characterization for order relations (4) and (5), and thus going further the analysis provided in [12], where only the relation (6) is studied.

Assumption 1. *The path-complete graphs considered herein have one strongly connected component and are such that if we remove any edge, the graph is not path-complete.*

This assumption is not restrictive since our aim is to compare stability conditions: we suppose that the inequalities of the form (3) *encoded in the graphs are sufficient conditions for stability (path-completeness) without having redundant/unnecessary inequalities.*

3.1 The T-sum lift

We say that a template $\mathcal{V} \subset \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is *closed under addition* if for any $V_1, V_2 \in \mathcal{V}$ we have $V_1 + V_2 \in \mathcal{V}$. Many templates commonly considered in control theory are closed under addition, for example: quadratic functions, convex functions, Sum-Of-Squares, etc. We now introduce a lift which takes into account this property, exploring the decrease properties of *sums* of candidate Lyapunov functions, which correspond to nodes in the path-complete Lyapunov framework. Given a set S and $T \in \mathbb{N}$, we will denote with $Multi^T(S)$ the set of multi-sets with elements in S of cardinality T, where a multi-set is defined as a set with possible repetitions, see [13] for the formal definition. For example, $\{a, a, b, b, b\}$ is a multi-set with elements in $\{a, b\}$ of cardinality 5.

Definition 5. *Given $T \in \mathbb{N}$ and a graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$, the T-*sum lift *is a graph $\mathcal{G}^T_{sum} = (S^T_{sum}, E^T_{sum})$ defined as follows :*

1. *For each $\{b_1, \ldots, b_T\} \in Multi^T(S)$ whose nodes $b_1, \ldots, b_T$ can be reached with the same label, i.e. $\exists i_1 \in \langle M \rangle, \exists a_1, \ldots, a_T \in S$ such that for $j = 1, \ldots, T$,*
$$(a_j, b_j, i_1) \ \in \ E,$$
and they can all be left with the same label, i.e. $\exists\ i_2 \in \langle M \rangle, \exists c_1, \ldots, c_T \in S$ such that for $j = 1, \ldots, T$,
$$(b_j, c_j, i_2) \ \in \ E,$$
a new node denoted by $b_1 \oplus \cdots \oplus b_T$ is added in S^T_{sum};

2. *For each multi-set of edges of E of the form $\{(a_1, b_1, i), \ldots, (a_T, b_T, i)\}$ such that $a_1 \oplus \cdots \oplus a_T$ and $b_1 \oplus \cdots \oplus b_T \in S^T_{sum}$, a new edge $(a_1 \oplus \cdots \oplus a_T, b_1 \oplus \cdots \oplus b_T, i)$ is added in E^T_{sum}.*

Note that in Definition 5 we consider only the multi-sets of cardinality T of nodes for which we can reach and leave each node with the same label: this ensures that all the nodes of $\mathcal{G}^T_{sum}$ have at least one incoming and one outgoing edge. On the other hand, observe that $\mathcal{G}^T_{sum}$ might not satisfy Assumption 1, even if $\mathcal{G}$ does, i.e. $\mathcal{G}^T_{sum}$ is possibly composed by more than one strongly connected and path-complete component, as illustrated in the subsequent Example 1. In practice, we will consider each of these components independently, recall Remark 1.

Proposition 1. *The T-sum lift is a valid lift with respect to any template closed under addition.*

Proof. Consider a path-complete graph $\mathcal{G} = (S, E)$. The path-completeness of $\mathcal{G}^T_{sum}$ is direct since $\mathcal{G}$ is a strongly connected component of $\mathcal{G}^T_{sum}$. Indeed, each node $a \in S$ admits an outgoing and an incoming edge thanks to the Assumption 1. This implies that the node $a \oplus \cdots \oplus a \in S^T_{sum}$, and then for every edge $(a, b, i) \in E$, the edge $(a \oplus \cdots \oplus a, b \oplus \cdots \oplus b, i) \in E^T_{sum}$. Now, consider a template $\mathcal{V}$ closed under addition and any family of vector fields $\{f_i\}_{i \in \langle M \rangle}$. Suppose that there exists a PCLF for the initial graph $\mathcal{G}$ with the functions $\{V_s \mid s \in S\} \subset \mathcal{V}$, and, for any $\overline{a} = (a_1 \oplus \cdots \oplus a_T) \in S^T_{sum}$ define

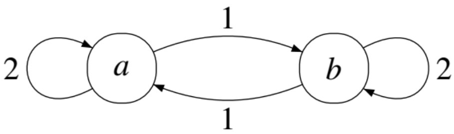
$$V_{\overline{a}} \ := \ V_{a_1} + \cdots + V_{a_T} \in \mathcal{V}. \tag{7}$$

The Lyapunov inequalities (3) of $\mathcal{G}^T_{sum}$ are satisfied because, for every edge $(\overline{a}, \overline{b}, i) \in E^T_{sum}$, we have

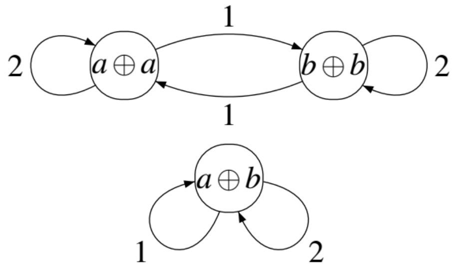
$$\begin{aligned} V_{\overline{b}}(f_i(x)) &= (V_{b_1}(f_i(x)) + \cdots + V_{b_T}(f_i(x)))\,, \\ &\leq (V_{a_1}(x) + \cdots + V_{a_T}(x)) \ = \ V_{\overline{a}}(x), \end{aligned}$$

for all $x \in \mathbb{R}^n$ since (a_1, b_1, i), $\ldots$,$(a_T, b_T, i) \in E$ by Definition 5 (possibly after a re-ordering of $\overline{a}$ and $\overline{b}$). ◻

Example 1. *Consider the path-complete graph $\mathcal{G}_1 = (S_1, E_1)$ on $\langle M \rangle = \{1, 2\}$ in Figure 1a. We apply Definition 5 to $\mathcal{G}_1$ and construct the corresponding 2-sum lift as presented in Figure 1b. The graph $(\mathcal{G}_1)^2_{sum}$ has two strongly connected and path-complete components: one isomorphic to $\mathcal{G}_1$ itself, and the other isomorphic to the common Lyapunov function graph $\mathcal{G}_0 := (\{z\}, \{(z, z, i)_{i \in \langle M \rangle}\})$. Considering any template $\mathcal{V}$ closed under addition, by definition we have that $\mathcal{G}_1 \leq_{\mathcal{V}} (\mathcal{G}_1)^2_{sum}$. Moreover, recalling*



(a) The graph $\mathcal{G}_1 = (S_1, E_1)$ in Example 1.



(b) The 2-sum-lift $(\mathcal{G}_1)^2_{sum}$

Figure 1: Example of a 2-sum-lifted graph

Remark 1, we obtain in particular the relation $\mathcal{G}_1 \leq_{\mathcal{V}} \mathcal{G}_0$. Furthermore it can be easily shown that $\mathcal{G}_0 \leq \mathcal{G}$ for any path-complete graph $\mathcal{G}$ (indeed, $\mathcal{G}_0$ is the most conservative graph, since it represents the common Lyapunov function case). In other words, we have proven that, for a template closed under addition, the inequalities encoded in $\mathcal{G}_1$ are as conservative as *the ones encoded in $\mathcal{G}_0$. We can conclude that for this type of templates, the path-complete policy of $\mathcal{G}_1$ is somehow a "poor choice", since we are increasing the number of inequalities (i.e. edges), without reducing the conservatism with respect to $\mathcal{G}_0$.*

3.2 The min lift

We say that a template $\mathcal{V} \subset \mathcal{C}^0(\mathbb{R}^n, \mathbb{R})$ is *closed under minimum* if for any $V_1, V_2 \in \mathcal{V}$ we have that $V(x) := \min\{V_1(x), V_2(x)\}$ (pointwise-minimum function) satisfies $V \in \mathcal{V}$. We introduce in what follows a lift which exploits this closure property.

Definition 6. *Given a graph $\mathcal{G} = (S, E)$ on the alphabet $\langle M \rangle$, the* min-lift *is a graph $\mathcal{G}_{\min} = (S_{\min}, E_{\min})$ defined as follows:*

(1) The set of nodes $S_{\min}$ is defined by

$$S_{\min} \;:=\; \{S' \subset S \mid S' \neq \emptyset\}.$$

(2) An edge $(A, B, i) \in E_{min}$ with $A, B \in S_{\min}$ and $i \in \langle M \rangle$ if and only if for all $a \in A$, there exists at least one $b \in B$ such that $(a, b, i) \in E$.

A combinatorial construction similar to Definition 6, the so-called *co-observer graph*, was introduced in [14, Definition 5.29] in order to prove the existence of an explicit common Lyapunov function from a path-complete Lyapunov function. Here, instead, we leverage the same construction in order to exhibit a novel lift for which the lifted nodes (i.e. the lifted Lyapunov functions) are associated to pointwise minima of initial functions. Note that the min-lift, as well as the T-sum lift, can have more than one strongly connected and path-complete component, as the subsequent Example 2 will underline.

Proposition 2. *The min lift is a valid lift with respect to any template closed under minimum.*

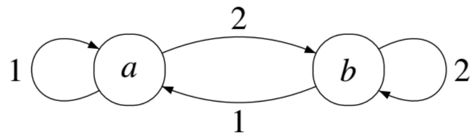
Proof. The proof is similar to the proof for Proposition 1. Path-completeness is trivial. Now, consider a template $\mathcal{V}$ closed under minimum and any family of vector fields $\{f_i\}_{i \in \langle M \rangle}$. Suppose that there exists a PCLF for the initial graph $\mathcal{G}$ of the form $\{V_s \mid s \in S\} \subset \mathcal{V}$. Given any $A \in S_{min}$ the corresponding Lyapunov function $V_A \in \mathcal{V}$ is defined by

$$V_A(x) \;:=\; \min_{a \in A} V_a(x). \tag{8}$$

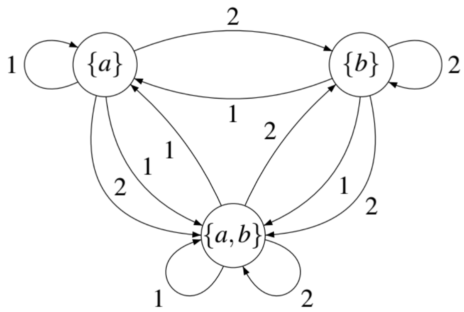
Given $(A, B, i) \in E_{min}$, we have

$$V_B(f_i(x)) \;=\; \min_{b \in B} V_b(x) \;\leq\; \min_{a \in A} V_a(x) \;=\; V_A(x),$$

for any $x \in \mathbb{R}^n$, since, by Definition 6, for all $a \in A$ there exists at least a $b \in B$ such that $V_b(f_i(x)) \leq V_a(x)$, concluding the proof. $\square$



(a) The graph $\mathcal{G}_2 = (S_2, E_2)$ in Example 2.



(b) The min-lift $(\mathcal{G}_2)_{\min}$

Figure 2: Example of a min-lifted graph

Example 2. *Consider the graph $\mathcal{G}_2 = (S_2, E_2)$ in Figure 2a, path-complete on $\langle M \rangle := \{1, 2\}$. If we apply the min-lift procedure as introduced in Definition 8, we obtain $(\mathcal{G}_2)_{min}$ represented in Figure 2b. We see that $(\mathcal{G}_2)_{min}$ has two strongly-connected and path-complete components: one isomorphic to $\mathcal{G}_2$ itself (the subgraph induced by $\{a\}$ and $\{b\}$); and one induced by $\{a, b\}$, which is isomorphic to $\mathcal{G}_0$, as defined in Example 1 (the common Lyapunov function case). We can conclude that*

$$\mathcal{G}_2 \;\leq_{\mathcal{V}}\; \mathcal{G}_0,$$

for any template $\mathcal{V}$ closed by minimum, or, in other words, the graph $\mathcal{G}_2$ is as conservative as $\mathcal{G}_0$ (the "worst" graph, see Example 1) for this kind of templates.

Remark 2. *It can be seen that, simply exchanging the quantifiers in Item (2) of Definition 8, one can obtain a definition of* max-lift *(denoted with $\mathcal{G}_{max}$) which would be valid for templates closed under maximum. The definitions and characterization of this and other possible template-dependent lifts can be studied with tools similar to the ones used in this section, and it is indeed an open route for future research.*

4 Application to Positive Systems

In this section we apply the ideas previously developed in studying stability of *positive* switching systems of the form

$$x(k+1) \;=\; A_{\sigma(k)} x(k) \tag{9}$$

where $\sigma : \mathbb{N} \to \langle M \rangle$, and $\mathcal{A} := \{A_1, \ldots, A_M\} \subset \mathbb{R}^{n \times n}_{\geq 0}$ is a set of positive matrices.

4.1 Copositive Norms: Definition and Properties

We introduce two particular templates of Lyapunov functions, which are used in studying stability of system (9).

Definition 7. *Given $v \in \mathbb{R}^n_{>0}$, we define the* primal *and* dual copositive norms induced by v *on $\mathbb{R}^n_{\geq 0}$ by*

$$\|x\|_v := v^\top x, \qquad \textit{and} \tag{10}$$

$$\|x\|^\star_v := \max_i \left\{ \frac{x_i}{v_i} \right\}, \tag{11}$$

for all $x \in \mathbb{R}^n_{\geq 0}$. We denote with $\mathcal{P}$ and $\mathcal{D}$ the set of all primal and dual copositive norms, respectively.

In the context of positive switching systems, primal norms as in (10) were considered in [11, 10, 15]. The definition of copositive dual norms in (11) is obtained by convex-functions duality (see [16, A.1.6]), since an equivalent definition is given by

$$\forall\, x \in \mathbb{R}^n_{\geq 0},\;\; \|x\|^\star_v = \sup_{y \in \mathbb{R}^n_{\geq 0}, \|y\|_v = 1} \{y^\top x\}. \tag{12}$$

It follows from Definition 7 that, given any $v \in \mathbb{R}^n_{>0}$, the functions $\|\cdot\|_v, \|\cdot\|^\star_v : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ are positive definite and radially unbounded, and thus $\mathcal{P}$ and $\mathcal{D}$ represent legitimate templates when studying stability of (9). In order to apply the lifts introduced in Section 3, we need to demonstrate the closure properties mentioned above for the template of copositive norms. For this, we introduce the following auxiliary notation.

Definition 8. *Given $v, w \in \mathbb{R}^n_{>0}$, define $v \vee w \in \mathbb{R}^n_{>0}$ as*

$$v \vee w \ := \ \sum_i \min\{v_i, w_i\}\mathbf{e}_i, \tag{13}$$

i.e. the componentwise minimum between v and w.

Proposition 3. *Given any $v, w \in \mathbb{R}^n_{>0}$, any $A \in \mathbb{R}^{n\times n}_{\geq 0}$, any $x \in \mathbb{R}^n_{\geq 0}$, any $\lambda > 0$ we have*

1. $\|x\|_{v+w} = \|x\|_v + \|x\|_w$;
2. $\|x\|_{\lambda v} = \lambda \|x\|_v$;
3. $\|x\|_{v\vee w} = \inf_{\substack{x_1+x_2=x,\\ x_1,x_2\in\mathbb{R}^n_{\geq 0}}} \|x_1\|_v + \|x_2\|_w$.
4. $\forall x \in \mathbb{R}^n_{\geq 0},\ \|Ax\|_v \leq \|x\|_w \ \Leftrightarrow\ A^\top v \leq_c w$.

Proof. Items *(1)*, *(2)* and *(4)* are trivial. For Item *(3)*, given $v \in \mathbb{R}^n_{>0}$ we denote with $B_v := \{x \in \mathbb{R}^n_{\geq 0} \mid \|x\|_v \leq 1\}$ the unit ball of the corresponding primal copositive norm. Given $v, w \in \mathbb{R}^n_{>0}$, the function $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$f(x) := \inf_{\substack{x_1+x_2=x,\\ x_1,x_2\in\mathbb{R}^n_{\geq 0}}} \|x_1\|_v + \|x_2\|_w,$$

is called the *infimal convolution* of $\|\cdot\|_v$ and $\|\cdot\|_w$, see [17, Theorem 5.4], where it is also proved that its unit ball is defined by $B = \mathrm{conv}\{B_v \cup B_w\}$. We thus need to prove that $B_{v\vee w} = \mathrm{conv}\{B_v \cup B_w\}$, then the result follows from the correspondence between norms and their unit balls, see [17, Corollary 16.4.1].
($\subset$): Without loss of generality, consider $x \in B_{v\vee w}$ such that $\|x\|_{v\vee w} = 1$, i.e. $\sum_i \min\{v_i, w_i\}x_i = 1$. Define the index sets

$$\begin{aligned} I_v &:= \{i \in \{1,\dots,n\} \mid v_i < w_i\},\\ I_w &:= \{i \in \{1,\dots,n\} \mid w_i \leq v_i\}. \end{aligned}$$

Now define $x_1 := \frac{\sum_{i\in I_v} x_i\mathbf{e}_i}{\sum_{i\in I_v} x_i v_i}$ and $x_2 := \frac{\sum_{i\in I_w} x_i\mathbf{e}_i}{\sum_{i\in I_w} x_i w_i}$. Note that, by definition, $\|x_1\|_v = 1$ and $\|x_2\|_w = 1$. Consider the positive scalar $\lambda = \sum_{i\in I_v} x_i v_i$, we have that $1-\lambda = \sum_{i\in I_w} x_i w_i$, since $\sum_{i\in I_v} x_i v_i + \sum_{i\in I_w} x_i w_i = \sum_i x_i \min\{v_i, w_i\} = 1$. Now, computing, we have $\lambda x_1 + (1-\lambda)x_2 = x$, proving that $x \in \mathrm{conv}\{B_v \cup B_w\}$.
($\supset$) The inclusions $B_v \subset B_{v\vee w}$ and $B_w \subset B_{v\vee w}$ are trivial. Consider thus $y \in B_v$, $z \in B_w$ (i.e. $\sum_i v_i y_i \leq 1$ and $\sum_i w_i z_i \leq 1$) and any $\lambda \in [0,1]$. Computing

$$\begin{aligned} &\sum_i \min\{v_i, w_i\}(\lambda y_i + (1-\lambda)z_i) \leq \sum_i v_i\lambda y_i + w_i(1-\lambda)z_i)\\ &= \lambda \sum_i v_i y_i + (1-\lambda)\sum_i w_i z_i \leq 1, \end{aligned}$$

concluding the proof. $\square$

We can now study which lifts defined in Section 3 are valid, when considering primal norms. We need the following preliminary result.

Lemma 1. *If $v_1, w_1, v_2, w_2 \in \mathbb{R}^n_{>0}$ and $A \in \mathbb{R}^{n\times n}_{\geq 0}$. Then*

$$\begin{gathered} (\forall\, x \in \mathbb{R}^n_{\geq 0},\ \ \min\{\|Ax\|_{v_2}, \|Ax\|_{w_2}\} \leq \min\{\|x\|_{v_1}, \|x\|_{w_1}\})\\ \Downarrow\\ (\,\forall\, x \in \mathbb{R}^n_{\geq 0},\ \ \|Ax\|_{v_2\vee w_2} \leq \|x\|_{v_1\vee w_1}\,). \end{gathered}$$

Proof. We recall that levels sets of a pointwise-minimum function $f := \min\{f_a, f_b\}$ are union of levels sets of f_a and f_b. We thus need to prove that $x \in B_{v_1} \cup B_{w_1} \Rightarrow Ax \in B_{v_2} \cup B_{w_2}$ implies $x \in B_{v_1 \vee w_1} \Rightarrow Ax \in B_{v_2 \vee w_2}$. Consider $x \in B_{v_1 \vee w_1}$, recalling proof of Item (3) in Proposition 3, we know that $x \in \text{conv}\{B_{v_1} \cup B_{w_1}\}$ i.e. there exist $x_1, x_2 \in B_{v_1} \cup B_{w_1}$, and $\lambda \in [0,1]$ such that $x = \lambda x_1 + (1-\lambda)x_2$. Computing

$$Ax \;=\; A\lambda x_1 + A(1-\lambda)x_2 \;=\; \lambda Ax_1 + (1-\lambda)Ax_2.$$

Since $Ax_1, Ax_2 \in B_{v_2} \cup B_{w_2}$, we have proved that $Ax \in \text{conv}(B_{v_2} \cup B_{w_2}) = B_{v_2 \vee w_2}$, concluding the proof. □

Theorem 1. *Consider $\mathcal{G} = (S, E)$ a path-complete graph on $\langle M \rangle$, any $n \in \mathbb{N}$ and any $T \in \mathbb{N}$. Denote by $\mathcal{P}$ the template of primal norms on $\mathbb{R}^n_{\geq 0}$ and by $\mathcal{L}$ the set of all the M-tuples of positive matrices in $\mathbb{R}^{n \times n}_{\geq 0}$. We have*

- $\mathcal{G} \;\leq_{\mathcal{P}}\; \mathcal{G}^T_{sum}$,
- $\mathcal{G} \;\leq_{\mathcal{P},\mathcal{L}} \mathcal{G}_{min}$,

Sketch of the proof: The first relation follows from Proposition 1 and recalling Item (1) of Proposition 3, which establishes that the template of primal norms is closed under addition. The second relation follows from Proposition 2 and Lemma 1. Note that the template of primal copositive norms is not closed under minimum, but the construction proposed in Proposition 2 is still possible by Lemma 1, which requires the linearity of the subvector fields. We can thus conclude $\mathcal{G} \;\leq_{\mathcal{P},\mathcal{L}} \mathcal{G}_{\min}$, but not, in general, $\mathcal{G} \;\leq_{\mathcal{P}} \mathcal{G}_{\min}$. □

Remark 3. *By duality, it is possible to develop similar arguments (as in Proposition 3 and Theorem 1) for the template of dual copositive norms $\mathcal{D}$, as defined in* (11) *in Definition 7. These arguments are omitted due to space constraint, and the analysis of the duality relation between graphs is a possible direction for future work.*

4.2 Numerical Example

In this section, we consider a planar positive switching system, already introduced in [10, Remark 2].

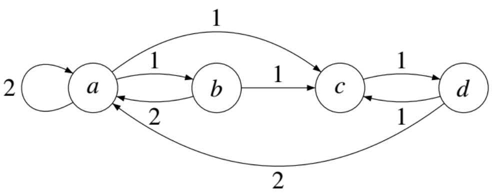
Example 3. *We consider the positive switching system* (9) *with*

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{2}{3} & \frac{1}{30} \end{bmatrix}, \;\; A_2 = \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{3} \end{bmatrix}. \tag{14}$$

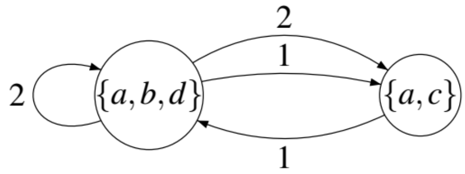
We want to estimate the infimum $\gamma \geq 0$ (denoted with $\gamma^\star$) for which the switching system defined by

$$\mathcal{A}_\gamma \;:=\; \{A_1/\gamma,\; A_2/\gamma\} \tag{15}$$

is stable. The scalar $\gamma^\star$ represents the decay rate of the system (9) (also called joint spectral radius (JSR) *of $\mathcal{A} = \{A_1, A_2\}$) (see [1] for further information). As already proven in [10], a common copositive Lyapunov function for $\mathcal{A}_\gamma$ exists if and only if $\gamma \geq 1$. Considering a common* quadratic *Lyapunov function, i.e. choosing $V(x) = x^\top Px$ with $P > 0$, and verifying $A_i^\top PA_i - \gamma^2 P \leq 0$ for $i \in \{1,2\}$ provides an upper bound for the JSR of* (9) *of $\gamma = 0.913$. In what follows we show how the estimation of the JSR can be improved considering a PCLF in the template of primal copositive norms, and how the template-dependent lifts can provide a useful tool in improving the stability analysis. Let us consider the path-complete graph $\mathcal{G}_3 = (S_3, E_3)$ in Figure 3a. The smallest value $\gamma \geq 0$ such that the Lyapunov inequalities (3) encoded by $\mathcal{G}_3$ are satisfied for the system (9) with the set of matrices (15) is $\gamma = 1$. In other words, this graph does not allow us to improve our estimation of the JSR with respect to the common copositive Lyapunov norm case. To refine our approximation, we can consider the min-lift of $\mathcal{G}_3$, as introduced in Definition 6. More specifically, we consider a particular strongly connected and path-complete component of $(\mathcal{G}_3)_{\min}$ given by $\mathcal{G}_4$ in Figure 3b, which satisfies, by Theorem 1, $\mathcal{G}_3 \leq_{\mathcal{P},\mathcal{L}} \mathcal{G}_4$. As it turns out, this graph allows to improve the approximation of the joint spectral radius, since the inequalities are feasible for $\gamma = 0.903$, and the obtained solution corresponding to this bound is given by $v_{\{a,b,d\}} = [1.000, 2.4219]^\top$ and $v_{\{a,c\}} = [1.7881, 1.1968]^\top$.*



(a) The graph $\mathcal{G}_3 = (S_3, E_3)$ in Example 3



(b) $\mathcal{G}_4 = (S_4, E_4)$, the strongly connected and path-complete component of $(\mathcal{G}_3)_{\min}$ considered in Example 3

Figure 3: The path-complete graphs $\mathcal{G}_3$ and $\mathcal{G}_4$ in Example 3

This example highlights that, given a particular path-complete structure, the lifting approach can provide better estimation of the joint spectral radius while decreasing the number of Lyapunov inequalities (half in the example). It also shows that we can provide a better approximation with a multiple Lyapunov function in the template of copositive norms than a common quadratic/copositive Lyapunov function. More specifically, the example illustrates a possible generalization of the technique proposed in [10]: considering path-complete Lyapunov functions in the template of copositive norms we reduce the conservatism of the stability analysis, without resorting to LMIs but simply relying on a set of linear inequalities encoded in a path-complete graph.

5 Conclusion

The path-complete approach is an appealing tool for stability analysis of switching systems because it provides a way of building ad-hoc, nonstandard, Lyapunov stability criteria while alleviating the combinatorial explosion of classical optimization techniques. In this paper, we studied the problem of establishing relations among different path-complete structures, with the goal of optimizing this structure, while at the same time controlling the computational cost. We have demonstrated the strong connections between templates of candidate Lyapunov functions and the ordering relations between graph-based conditions.

We proposed novel graph-lifts, whose validity depends on the properties of the chosen template. We illustrated our approach in the context of positive systems and copositive functions, for which we showed that our techniques indeed outperform the state of the art. This work has opened the way for a much larger research avenue, such as the generalization of this approach to a wider class of templates/systems, (notably the quadratic functions cases).

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