Counterexample of the Riemann Hypothesis

Frank Vega

CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France

Abstract

By proving the existence of a zero-free region for the Riemann zeta-function, de la Vallée-Poussin was able to bound $\theta(x) = x + O(x \times \exp(-c_2 \times \sqrt{\log x}))$, where $\theta(x)$ is the Chebyshev function and c_2 is a positive absolute constant. Under the assumption that the Riemann hypothesis is true, von Koch deduced the improved asymptotic formula $\theta(x) = x + O(\sqrt{x} \times \log^2 x)$. We prove when $\theta(x) = x + \Omega(\sqrt{x} \times \log^2 x)$, then the Riemann hypothesis is false.

Keywords: Riemann hypothesis, Nicolas inequality, Chebyshev function, prime numbers
2000 MSC: 11M26, 11A41, 11A25

1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q \leq p_n} \frac{q}{q-1} > e^{\gamma} \times \log \theta(p_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and p_n is the n^{th} prime number. The importance of this property is:

Theorem 1.1. *[2].* $\mathsf{Nicolas}(p_n)$ *holds for all prime numbers* $p_n > 2$ *if and only if the Riemann hypothesis is true.*

We know the following properties for the Chebyshev function:

Theorem 1.2. *[3]. For a positive absolute constant* c_2*:*

$$\theta(x) = x + O(x \times \exp(-c_2 \times \sqrt{\log x})).$$

Email address: vega.frank@gmail.com (Frank Vega)

Theorem 1.3. *[4]. If the Riemann hypothesis holds, then*

$$\theta(x) = x + O(\sqrt{x} \times \log^2 x).$$

Theorem 1.4. *[5]. For* $2 \leq x \leq 10^8$

$$\theta(x) < x.$$

We also know that

Theorem 1.5. *[6]. If the Riemann hypothesis holds, then*

$$\left(\frac{e^{-\gamma}}{\log x} \times \prod_{q \leq x} \frac{q}{q-1} - 1\right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}$$

for all numbers $x \geq 13.1$.

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [7]. We know from the constant H, the following formula:

Theorem 1.6. *[8].*

$$\sum_{q} \left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows

$$u(x) = \sum_{q > x} \left(\log(\frac{q}{q-1}) - \frac{1}{q}\right).$$

We use the following theorems:

Theorem 1.7. *[9]. For* $x > -1$*:*

$$\frac{x}{x+1} \leq \log(1+x).$$

Theorem 1.8. *[10]. For* $x \geq 1$*:*

$$\log(1 + \frac{1}{x}) < \frac{1}{x + 0.4}.$$

Let's define:

$$\delta(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log x - B\right).$$

Definition 1.9. *We define another function:*

$$\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B\right).$$

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $p \leq x$. In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion and deduce some of its consequences.

2. Results

Theorem 2.1. *The Riemann hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all numbers $x \geq 3$.*

Proof. In the paper [2] is defined the function:

$$f(x) = e^{\gamma} \times (\log \theta(x)) \times \prod_{q \leq x} \frac{q-1}{q}.$$

We know that $f(x)$ is lesser than 1 when $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $2 < p \leq x$. In the same paper, we found that

$$\log f(x) = U(x) + u(x)$$

where $U(x) = -\varpi(x)$ [2]. When $f(x)$ is lesser than 1, then $\log f(x) < 0$. Consequently, we obtain that

$$-\varpi(x) + u(x) < 0$$

which is the same as $\varpi(x) > u(x)$. Therefore, this is a consequence of the theorem 1.1. □

Theorem 2.2. *If the Riemann hypothesis holds, then*

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > 1$$

for all numbers $x \geq 13.1$.

Proof. Under the assumption that the Riemann hypothesis is true, then we would have

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times \log x \times \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right)$$

after of distributing the terms based on the theorem 1.5 for all numbers $x \geq 13.1$. If we apply the logarithm to the both sides of the previous inequality, then we obtain that

$$\sum_{q \leq x} \log(\frac{q}{q-1}) < \gamma + \log \log x + \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right).$$

That would be equivalent to

$$\sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) < \gamma + \log \log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

where we know that

$$\log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right) < \frac{1}{\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} + 0.4}$$
$$= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)}$$
$$= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

according to theorem 1.8 since $\frac{8\times\pi\times\sqrt{x}}{3\times\log x+5} \geq 1$ for all numbers $x \geq 13.1$. We use the theorem 1.6 to show that

$$\sum_{q\leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) = H - u(x)$$

and $\gamma = H + B$. So,

$$H - u(x) < H + B + \log\log x - \sum_{q\leq x}\frac{1}{q} + \frac{3\times\log x + 5}{8\times\pi\times\sqrt{x} + 1.2\times\log x + 2}$$

which is the same as

$$H - u(x) < H - \delta(x) + \frac{3\times\log x + 5}{8\times\pi\times\sqrt{x} + 1.2\times\log x + 2}.$$

We eliminate the value of H and thus,

$$-u(x) < -\delta(x) + \frac{3\times\log x + 5}{8\times\pi\times\sqrt{x} + 1.2\times\log x + 2}$$

which is equal to

$$u(x) + \frac{3\times\log x + 5}{8\times\pi\times\sqrt{x} + 1.2\times\log x + 2} > \delta(x).$$

Under the assumption that the Riemann hypothesis is true, we know from the theorem 2.1 that $\varpi(x) > u(x)$ for all numbers $x \geq 13.1$ and therefore,

$$\varpi(x) + \frac{3\times\log x + 5}{8\times\pi\times\sqrt{x} + 1.2\times\log x + 2} > \delta(x).$$

Hence,

$$\frac{3\times\log x + 5}{8\times\pi\times\sqrt{x} + 1.2\times\log x + 2} > \log\log\theta(x) - \log\log x.$$

Suppose that $\theta(x) = \epsilon\times x$ for some constant $\epsilon > 1$. Then,

$$\begin{aligned}
\log\log\theta(x) - \log\log x &= \log\log(\epsilon\times x) - \log\log x \\
&= \log(\log x + \log\epsilon) - \log\log x \\
&= \log\left(\log x\times(1 + \frac{\log\epsilon}{\log x})\right) - \log\log x \\
&= \log\log x + \log(1 + \frac{\log\epsilon}{\log x}) - \log\log x \\
&= \log(1 + \frac{\log\epsilon}{\log x}).
\end{aligned}$$

In addition, we know that

$$\log(1 + \frac{\log\epsilon}{\log x}) \geq \frac{\log\epsilon}{\log\theta(x)}$$

using the theorem 1.7 since $\frac{\log\epsilon}{\log x} > -1$ when $\epsilon > 1$. Certainly, we will have that

$$\log(1 + \frac{\log\epsilon}{\log x}) \geq \frac{\frac{\log\epsilon}{\log x}}{\frac{\log\epsilon}{\log x} + 1} = \frac{\log\epsilon}{\log\epsilon + \log x} = \frac{\log\epsilon}{\log\theta(x)}.$$

Thus,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \epsilon}{\log \theta(x)}.$$

If we add the following value of $\frac{\log x}{\log \theta(x)}$ to the both sides of the inequality, then

$$\begin{aligned}\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} &> \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)} \\ &= \frac{\log \epsilon + \log x}{\log \theta(x)} \\ &= \frac{\log \theta(x)}{\log \theta(x)} \\ &= 1.\end{aligned}$$

We know this inequality is satisfied when $0 < \epsilon \leq 1$ since we would obtain that $\frac{\log x}{\log \theta(x)} \geq 1$. Therefore, the proof is done. □

Theorem 2.3. *The Riemann hypothesis is false when*

$$\theta(x) = x + \Omega(\sqrt{x} \times \log^2 x).$$

Proof. If the Riemann hypothesis holds, then

$$\theta(x) = x + O(\sqrt{x} \times \log^2 x)$$

due to the theorem 1.3. Now, suppose there is a real number $x \geq 10^8$ such that $\theta(x) > x + \sqrt{x} \times \log^{1.9} x$. That would be equivalent to

$$\log \theta(x) > \log(x + \sqrt{x} \times \log^{1.9} x)$$

and so,

$$\frac{1}{\log \theta(x)} < \frac{1}{\log(x + \sqrt{x} \times \log^{1.9} x)}$$

for all numbers $x \geq 10^8$. Hence,

$$\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)}.$$

If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)} > 1$$

for those values of x that complies with

$$\theta(x) > x + \sqrt{x} \times \log^{1.9} x$$

due to the theorem 2.2. By contraposition, if there exists some number $y \geq 10^8$ such that for all $x \geq y$ the inequality

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)} \leq 1$$

is satisfied, then the Riemann hypothesis should be false. Let's define the function

$$\upsilon(x) = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)} - 1.$$

The Riemann hypothesis would be false when there exists some number $y \geq 10^8$ such that for all $x \geq y$ the inequality $\upsilon(x) \leq 0$ is always satisfied. We ignore when $2 \leq x \leq 10^8$ since $\theta(x) < x$ according to the theorem 1.4. We know that the function $\upsilon(x)$ is monotonically decreasing for every number $x \geq 10^8$. The derivative of $\upsilon(x)$ is negative for all $x \geq 10^8$. The derivative of $\upsilon(x)$ is approximately

$$-\left(\frac{0.1875 \times (0.3 + \pi \times \sqrt{x}) \times (1.66667 + \log(x))}{x \times (0.25 + \pi \times \sqrt{x} + 0.15 \times \log(x))^2}\right) + \frac{3}{2 \times x + 8 \times \pi \times x^{\frac{3}{2}} + 1.2 \times x \times \log(x)}$$

$$-\left(\frac{\sqrt{x} \times \log(x) + 1.9 \times \log^{1.9}(x) + 0.5 \times \log^{2.9}(x)}{x \times (\sqrt{x} + \log^{1.9}(x)) \times \log^2(x + \sqrt{x} \times \log^{1.9}(x))}\right) + \frac{1}{x \times \log(x + \sqrt{x} \times \log^{1.9}(x))}.$$

Indeed, a function $\upsilon(x)$ of a real variable x is monotonically decreasing in some interval if the derivative of $\upsilon(x)$ is lesser than zero and the function $\upsilon(x)$ is continuous over that interval [11]. It is enough to find a value of $y \geq 10^8$ such that $\upsilon(y) \leq 0$ since for all $x \geq y$ we would have that $\upsilon(x) \leq \upsilon(y) \leq 0$, because of $\upsilon(x)$ is monotonically decreasing. We found the value $y = 10^8$ complies with $\upsilon(y) \leq 0$. In this way, we obtain that $\upsilon(x) \leq 0$ for every number $x \geq 10^8$. Consequently, under the assumption that the Riemann hypothesis is true, then

$$\theta(x) < x + \sqrt{x} \times \log^{1.9} x$$

for all $x \geq 10^8$. Hence, this implies that the Riemann hypothesis is false when $\theta(x) = x + \Omega(\sqrt{x} \times \log^2 x)$. □

References

[1] P. B. Borwein, S. Choi, B. Rooney, A. Weirathmueller, The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike, Vol. 27, Springer Science & Business Media, 2008.

[2] J.-L. Nicolas, Petites valeurs de la fonction d'Euler, Journal of number theory 17 (3) (1983) 375–388. doi:10.1016/0022-314X(83)90055-0.

[3] C. J. d. L. V. Poussin, Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inferieurs à une limite donnée, Vol. 51, Hayez, 1899.

[4] H. Von Koch, Sur la distribution des nombres premiers, Acta Mathematica 24 (1) (1901) 159. doi:10.1007/BF02403071.

[5] J. B. Rosser, L. Schoenfeld, Approximate Formulas for Some Functions of Prime Numbers, Illinois Journal of Mathematics 6 (1) (1962) 64–94. doi:10.1215/ijm/1255631807.

[6] J. B. Rosser, L. Schoenfeld, Sharper Bounds for the Chebyshev Functions θ(x) and ψ(x), Mathematics of computation (1975) 243–269doi:10.1090/S0025-5718-1975-0457373-7.

[7] F. Mertens, Ein Beitrag zur analytischen Zahlentheorie., J. reine angew. Math. 1874 (78) (1874) 46–62. doi:10.1515/crll.1874.78.46.

[8] Y. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin's criterion for the Riemann hypothesis, Journal de Théorie des Nombres de Bordeaux 19 (2) (2007) 357–372. doi:10.5802/jtnb.591.

[9] L. Kozma, Useful Inequalities, http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf, accessed on 2022-02-06 (2022).

[10] A. Ghosh, An Asymptotic Formula for the Chebyshev Theta Function, arXiv preprint arXiv:1902.09231 (2019).

[11] G. Anderson, M. Vamanamurthy, M. Vuorinen, Monotonicity Rules in Calculus, The American Mathematical Monthly 113 (9) (2006) 805–816. doi:10.1080/00029890.2006.11920367.