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Stability Analysis Of First And Second Order Explicit Finite Difference Scheme of Advection-Diffusion Equation

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Abstract

This article studies two first order schemes, FTBSCS and FTCSCS and propose a second order Lax-Wendroff type scheme for the numerical solutions of advection-diffusion equation (ADE) as an initial and boundary value problem. In previous Lax-Wendroff scheme was introduced only for hyperbolic partial differential equation (PDE) but here a new second order Lax-Wendroff type explicit finite difference scheme is proposed for parabolic ADE. For proposed second order Lax-Wendorff type scheme of ADE, we discretise the first order terms of ADE in second order like Lax-Wendorff scheme for hyperbolic partial differential equation. We perform stability analysis of these numerical schemes and determine the condition of stability in terms of temporal and spatial step sizes, advection co-efficient and diffusion co-efficient. The stability conditions of these schemes lead to determine the efficiency of these schemes in terms of the time step restrictions. Finally, we compare these schemes in terms of stability condition and efficiency as well.

Introduction

ADE is a combination of the advection equation and diffusion equation. It is a parabolic type partial differential equation and is derived on the principle of conservation of mass using Fick's law (Socolofsky and Jirka 2002). Many investigators have been studied analytical and numerical solutions for higher-dimensional and higher order ADE from many years. In numerical analysis, numerical stability is usually a desired property of numerical algorithms. Stability analysis of finite difference schemes for the Navier-Stokes equations is obtained (Rigal 1979). Stability analysis of finite difference schemes for the advection-diffusion equation is studied (Chan 1984). A comparison of some numerical methods for the advection-diffusion equation is presented (Thongmoon and Mckibbin 2006). An analytical solution of the advection diffusion equation for a ground level finite area source is presented (Park and Baik 2008). An analytical solution is obtained of the one dimensional ADE by reducing the original ADE into a diffusion equation by introducing another dependent variable (Al-Niami and Ruston 1977). Analytical solution of 1D ADE with variable coefficients is presented in a finite domain by using Laplace transformation technique. In that process new independent space and time variables have been introduced (Kumar, A., D. K. Jaiswal and N. Kumar. 2010). Two explicit finite difference schemes such as FTBSCS and FTCSCS, for solving the ADE numerically are studied in this article. A numerical technique was proposed in 1960 by P.D. Lax and B. Wendroff for solving, approximately, systems of hyperbolic conservation laws. Here in this article a new explicit second order Lax-Wendroff type scheme is proposed for solving ADE numerically where we discretise the first order terms of ADE in second order similarly as Lax-Wendroff schemes for hyperbolic partial differential equation. Stability conditions for FTBSCS, FTCSCS and proposed second order Lax-Wendorff type explicit finite difference schemes for solving the ADE are determined. The stability analysis for these numerical schemes are verified by numerical experiments in terms of time step restriction. The efficiency of these numerical schemes is verified by elapsed time. Then we compare the stability conditions and efficiency of these numerical schemes. Finally, we conclude in the last section.

Methodology

LeVeque, R. J., & Leveque, R. J. (1992) showed a study on numerical methods for conservation laws. Febi Sanjaya and Sudi Mungkasi. (2017) conducted a study on a simple but accurate explicit finite difference method for the advection-diffusion equation. The way to find the numerical solution of advection-diffusion equation was showed on that article. P.D Lax; B. Wendroff (1960) conducted a study on systems of conservation laws, where Lax-Wendroff scheme for hyperbolic partial differential equation was showed. Azad, T.M.A.K., M. Begum and L.S.Andallah. (2015) conducted a study on an explicit finite difference scheme for advection diffusion equation, where they studied an explicit finite difference scheme for advection-diffusion equation.

Ahmed S.G. (2012) conducted a study on a Numerical Algorithm for Solving Advection-Diffusion Equation with Constant and Variable Coefficients. Murat Sari, Gurhan Gurarslan, and Asuman Zeytinoglu. (2010) showed a study on higher order finite difference approximation for solving advection-diffusion equation. Leon, L. F., & Austria, P. M. (1987) conducted a study on stability Criterion for Explicit Scheme on the solution of Advection Diffusion Equation. Chan, T. F. (1984) conducted a study on stability analysis of finite difference schemes for the advection diffusion equation. Charney, J. G., Fjortoft, R., & Neumann, J. V. (1950) showed a study on numerical integration of the barotropic vorticity equation, a way for stability analysis was shown on this article.

Mathematical Model

The simplest one-dimensional ADE is

$$\frac{\partial c}{\partial x}+u\frac{\partial c}{\partial t}=D\frac{\partial^2 c}{\partial x^2} \tag{1}$$

Where $x\epsilon[a,b], t\epsilon[0,T]$

With initial condition, $c(0,x)=c_0(x);$

Boundary condition $c(t,a)=c_a(t);$

And $c(t,b)=c_b(t);$

Where c is the concentration of the transference elements; *D* is the diffusion co-efficient and *u* is the speed of field.

Numerical Method

We use finite difference method to solve ADE numerically. Here we work with explicit FTBSCS, FTCSCS and proposed second order Lax-Wendroff type scheme of ADE.

Finite difference formulae

Derivatives in equation (1) are approximated by truncated Taylor Series expansions,

1st order forward difference formula in terms of time,

$$\frac{\partial c(x_i^n)}{\partial t}\approx\frac{c_i^{n+1}-c_i^n}{\Delta t} \tag{2}$$

1st order backward difference formula in terms of space,

$$\frac{\partial c(x_i^n)}{\partial x}\approx\frac{c_i^n-c_{i-1}^n}{\Delta x} \tag{3}$$

1st order central difference formula in terms of space,

$$\frac{\partial c(x_i^n)}{\partial x}\approx\frac{c_{i+1}^n-c_{i-1}^n}{2\Delta x} \tag{4}$$

2nd order central difference formula in terms of space,

$$\frac{\partial^2 c(x_i^n)}{\partial x^2}\approx\frac{c_{i+1}^n-2c_i^n+c_{i-1}^n}{(\Delta x)^2} \tag{5}$$

Explicit Upwind Difference Scheme (FTBSCS)

Substituting equations (2), (3), (5) into equation (1), we get

$$\frac{c_i^{n+1}-c_i^n}{\Delta t}+u\frac{c_i^n-c_{i-1}^n}{\Delta x}=D\frac{c_{i+1}^n-2c_i^n+c_{i-1}^n}{(\Delta x)^2}$$

$$\Rightarrow c_i^{n+1}=c_i^n-\frac{u\Delta t}{\Delta x}(c_i^n-c_{i-1}^n)+\frac{D\Delta t}{(\Delta x)^2}(c_{i+1}^n-2c_i^n+c_{i-1}^n) \tag{6}$$

Taking $\alpha=\frac{u\Delta t}{\Delta x}$ and $\gamma=\frac{D\Delta t}{(\Delta x)^2}$

$$\Rightarrow c_i^{n+1}=c_i^n-\alpha(c_i^n-c_{i-1}^n)+\gamma(c_{i+1}^n-2c_i^n+c_{i-1}^n)$$

$$\Rightarrow c_i^{n+1}=(\alpha+\gamma)c_{i-1}^n+(1-\alpha-2\gamma)c_i^n+\gamma c_{i+1}^n \tag{7}$$

Which is known as the **explicit upwind difference scheme** for ADE and it is also known as **FTBSCS** technique.

Stability condition of FTBSCS

The above scheme (7) satisfies the convex combination,

Therefore, the FTBSCS is stable for

$$0 \le \gamma \le 1 \text{ and } -\gamma \le \alpha \le 1 - 2\gamma$$
$$0 \le \frac{D\Delta t}{(\Delta x)^2} \le 1 \text{ and } -\frac{D\Delta t}{(\Delta x)^2} \le \frac{u\Delta t}{\Delta x} \le 1 - 2\frac{D\Delta t}{(\Delta x)^2} \quad (8)$$

Explicit centered difference scheme (FTCSCS)

Substituting equations (2), (4), (5) into equation (1), we get

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} + u\frac{c_{i+1}^n - c_{i-1}^n}{2\Delta x} = D\frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2}$$
$$\Rightarrow c_i^{n+1} = c_i^n - \frac{u\Delta t}{2\Delta x}(c_{i+1}^n - c_{i-1}^n) + \frac{D\Delta t}{(\Delta x)^2}(c_{i+1}^n - 2c_i^n + c_{i-1}^n) \quad (9)$$

Taking $\alpha = \frac{u\Delta t}{\Delta x}$ and $\gamma = \frac{D\Delta t}{(\Delta x)^2}$

$$\Rightarrow c_i^{n+1} = c_i^n - \frac{\alpha}{2}(c_{i+1}^n - c_{i-1}^n) + \gamma(c_{i+1}^n - 2c_i^n + c_{i-1}^n)$$
$$\Rightarrow c_i^{n+1} = \left(\frac{\alpha}{2} + \gamma\right)c_{i-1}^n + (1 - 2\gamma)c_i^n + \left(\gamma - \frac{\alpha}{2}\right)c_{i+1}^n \quad (10)$$

Which is known as the **explicit centered difference scheme** for ADE and it is also known as **FTCSCS** technique.

Stability condition of FTCSCS

The above scheme (10) satisfies the convex combination,
We can conclude that the FTCSCS is stable for

$$0 \le \alpha \le 1 \text{ and } 0 \le \gamma \le \frac{1}{2}$$
$$0 \le \frac{u\Delta t}{\Delta x} \le 1 \text{ and } 0 \le \frac{D\Delta t}{(\Delta x)^2} \le \frac{1}{2} \quad (11)$$

Explicit second order Lax-Wendroff type Scheme of ADE

For Explicit secondorder Lax-Wendroff type scheme of ADE, we discretize advective part in half time-step Lax-Friedrich scheme, then substituting that value in half-step Leapfrog scheme and combining with centered diffusion part explicit second order Lax-Wendroff type scheme of ADE is found.

Half-time step lax-Friedrich scheme at the point (t^n, x_i):

$$c_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(c_{i+1}^n + c_i^n) - \frac{c\Delta t}{2\Delta x}(c_{i+1}^n - c_i^n) \quad (12)$$
$$c_{i-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(c_i^n + c_{i-1}^n) - \frac{c\Delta t}{2\Delta x}(c_i^n - c_{i-1}^n) \quad (13)$$

Half-step Leapfrog scheme at the point (t^n, x_i):

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} + u\left[\frac{c_{i+\frac{1}{2}}^{n+\frac{1}{2}} - c_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x}\right] = 0 \quad (14)$$

By centered difference discretization of $\frac{\partial^2 c}{\partial x^2}$ at the point (t^n, x_i), we have

$$\frac{\partial^2 c(x_i^n)}{\partial x^2} \approx \frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2} \quad (15)$$

Combining equation $(14), (15)$ in (1) we obtain,

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} + u\left[\frac{c_{i+\frac{1}{2}}^{n+\frac{1}{2}} - c_{i-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta x}\right] = D\frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2}$$
$$\Rightarrow c_i^{n+1} - c_i^n + \frac{u\Delta t}{\Delta x}\left[c_{i+\frac{1}{2}}^{n+\frac{1}{2}} - c_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right] = D\Delta t\left[\frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2}\right]$$
$$\Rightarrow c_i^{n+1} = c_i^n - \frac{u\Delta t}{\Delta x}\left[c_{i+\frac{1}{2}}^{n+\frac{1}{2}} - c_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right] + D\Delta t\left[\frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{(\Delta x)^2}\right] \quad (16)$$

Now substituting the value of $c_{i+\frac{1}{2}}^{n+\frac{1}{2}}$ and $c_{i-\frac{1}{2}}^{n+\frac{1}{2}}$ in equation (16) ,we have

$$\Longrightarrow c_i^{n+1} = c_i^n - \frac{u\Delta t}{\Delta x}\left[\frac{1}{2}(c_{i+1}^n + c_i^n) - \frac{u\Delta t}{2\Delta x}(c_{i+1}^n - c_i^n) - \frac{1}{2}(c_i^n + c_{i-1}^n) + \frac{u\Delta t}{2\Delta x}(c_i^n - c_{i-1}^n)\right] + \frac{D\Delta t}{(\Delta x)^2}[c_{i+1}^n - 2c_i^n + c_{i-1}^n]$$

$$\Longrightarrow c_i^{n+1} = c_i^n - \frac{u\Delta t}{\Delta x}\left[\frac{1}{2}(c_{i+1}^n + c_{i-1}^n) - \frac{u\Delta t}{2\Delta x}(c_{i+1}^n - 2c_i^n + c_{i-1}^n)\right] + \frac{D\Delta t}{(\Delta x)^2}[c_{i+1}^n - 2c_i^n + c_{i-1}^n]$$

Taking $\alpha = \frac{u\Delta t}{\Delta x}$ and $\gamma = \frac{D\Delta t}{(\Delta x)^2}$ we have

$$\Longrightarrow c_i^{n+1} = c_i^n - \alpha\left[\frac{1}{2}(c_{i+1}^n + c_{i-1}^n) - \alpha(c_{i+1}^n - 2c_i^n + c_{i-1}^n)\right] + \gamma[c_{i+1}^n - 2c_i^n + c_{i-1}^n]$$

$$\Longrightarrow c_i^{n+1} = \frac{1}{2}(\alpha^2 + \alpha + 2\gamma)c_{i-1}^n + (1 - 2\gamma - \alpha^2)c_i^n + \frac{1}{2}(2\gamma - \alpha + \alpha^2)c_{i+1}^n \quad (17)$$

Which is required **secondorder Lax-Wendroff type scheme** of ADE.

Stability Conditionsecond order Lax-Wendroff scheme with max-principle

The above equation(17) satisfies the convex combination, we obtain,

$$0 \le \frac{1}{2}(\alpha^2 + \alpha + 2\gamma) \le 1 \quad (18)$$

$$0 \le (1 - 2\gamma - \alpha^2) \le 1 \quad (19)$$

$$0 \le \frac{1}{2}(2\gamma - \alpha + \alpha^2) \le 1 \quad (20)$$

Then the new solution is a convex combination of the two previous solutions. That is, the solution at new time-step $(n+1)$ at a spatial node i is an average of the solution at the previous time-step at the spatial-nodes $i-1$, $i\ and\ i+1$. This means that the extreme value of the new solution is the average values of the previous two solutions at the three consecutive nodes. Therefore, the new solution continuously depends on the initial value $c_i^0, i = 1,2,3, \ldots \ldots \ldots \ldots .., M$.

Therefore, from (18), (19), (20) we have

$$0 \le \alpha^2 + 2\gamma \le 1,\ \ 0 \le \gamma < 1 \text{ and } \ 0 \le \alpha < 1 \quad (21)$$

Which is required stability condition for**secondorderLax-Wendroff type scheme** of ADE.

Stability ofsecond order Lax-Wendroff type scheme with Von-Neumann Analysis

The second order Lax-Wendroff type Scheme of ADE can be re-write as

$$c_j^{n+1} = \frac{1}{2}(\alpha^2 + \alpha + 2\gamma)c_{j-1}^n + (1 - 2\gamma - \alpha^2)c_j^n + \frac{1}{2}(2\gamma - \alpha + \alpha^2)c_{j+1}^n \quad (22)$$

where $\alpha = \frac{u\Delta t}{\Delta x}$ and $\gamma = \frac{D\Delta t}{(\Delta x)^2}$

Now we substitute $c_j^n = \xi^n e^{ikj\,\Delta x}$ in equation (22) we get

$$\xi^{n+1}e^{ikj\,\Delta x} = \xi^n e^{ikj\,\Delta x}(1 - \alpha^2 - 2\gamma) + \xi^n e^{ik(j+1)\Delta x}\frac{1}{2}(2\gamma - \alpha + \alpha^2) + \xi^n e^{ik(j-1)\Delta x}\frac{1}{2}(\alpha^2 + \alpha + 2\gamma) \quad (23)$$

Cancelling $\xi^n e^{ikj\,\Delta x}$ from both sides of equation (23), we get

$$\xi = (1 - \alpha^2 - 2\gamma) + e^{ik\Delta x}\frac{1}{2}(2\gamma - \alpha + \alpha^2) + e^{-ik\Delta x}\frac{1}{2}(\alpha^2 + \alpha + 2\gamma)$$

$$\Rightarrow \xi = (1 - \alpha^2 - 2\gamma) + \frac{1}{2}\alpha(e^{ik\Delta x} - e^{-ik\Delta x}) + (e^{ik\Delta x} + e^{-ik\Delta x})\frac{1}{2}(\alpha^2 + 2\gamma) \quad (24)$$

Now using Euler's formula

$$\cos(x) + i * \sin(x) = e^{ix}$$

and

$$\cos(x) - i * \sin(x) = e^{-ix}$$

with $2cosx = e^{ix} + e^{-ix}\ and\ 2 * i * sinx = e^{ix} - e^{-ix}$, equation (24) becomes,

$$\Rightarrow \xi = (1 - \alpha^2 - 2\gamma) + \frac{1}{2}\alpha i * 2\sin(k\Delta x) + \frac{1}{2}(\alpha^2 + 2\gamma) * 2\cos(k\Delta x)$$

$$\Rightarrow \xi = (1 - \alpha^2 - 2\gamma) + \alpha i \sin(k\Delta x) + (\alpha^2 + 2\gamma)\cos(k\Delta x)$$

$$\Rightarrow \xi = 1 + \alpha i\, sin(k\Delta x) - (\alpha^2 + 2\gamma)(1 - cos(k\Delta x))$$

$$\Rightarrow \xi = 1 + \alpha i\, sin(k\Delta x) - (\alpha^2 + 2\gamma) * 2sin^2\left(\frac{k\Delta x}{2}\right)$$

$$\Rightarrow \xi = 1 - (\alpha^2 + 2\gamma) * 2sin^2\left(\frac{k\Delta x}{2}\right) + \alpha i\, sin(k\Delta x) \quad (25)$$

For stability condition $\|\xi\| \le 1$

i.e. $\left\|1-\ (\alpha^2+2\gamma)*2sin^2\left(\frac{k\Delta x}{2}\right)+\ \alpha i\,sin(k\Delta x)\right\|\leq 1$

For complex number $|A+iB|=\sqrt{A^2+B^2}$

Now

$$\left\|1-\ (\alpha^2+2\gamma)*2sin^2\left(\frac{k\Delta x}{2}\right)+\ \alpha i\,sin(k\Delta x)\right\|=\sqrt{\left(1-\ (\alpha^2+2\gamma)*2sin^2\left(\frac{k\Delta x}{2}\right)\right)^2+(\alpha i\,sin(k\Delta x))^2}$$

Since $\sin(x)$is always positive and varies over [0,1], so we have

$$\sqrt{\left(1-\ (\alpha^2+2\gamma)\right)^2+(\alpha)^2}\leq 1$$

Finding the range of α and γ:

Case 1: when $\alpha=\gamma=0$, then

$$\sqrt{\left(1-\ (\alpha^2+2\gamma)\right)^2+(\alpha)^2}=1 \tag{26}$$

Case 2: when $\alpha=\gamma=1$, then

$$\sqrt{\left(1-\ (\alpha^2+2\gamma)\right)^2+(\alpha)^2}>1 \tag{27}$$

Case 3: when $\alpha,\gamma>0$, then

$(\alpha^2+2\gamma)>0$ and $(\alpha)^2>0$

So,

$$\sqrt{\left(1-\ (\alpha^2+2\gamma)\right)^2+(\alpha)^2}>0 \tag{28}$$

Case 4: when $0\leq\gamma\ll\alpha<1$, then

$$\sqrt{\left(1-\ (\alpha^2+2\gamma)\right)^2+(\alpha)^2}\leq 1 \tag{29}$$

Case 5: when $0\leq\alpha\ll\gamma<1$, then

$$\sqrt{\left(1-\ (\alpha^2+2\gamma)\right)^2+(\alpha)^2}\leq 1 \tag{30}$$

Therefore, the stability condition $\quad 0\leq\gamma\ll\alpha\leq 1$ and $0\leq\alpha\ll\gamma\leq 1 \quad$ (31)

Hence, the scheme is stable conditionally.

Results and Discussion

Stability analysis of ADE by FTBSCS, FTCSCS, and secondorder Lax-Wendroff type schemes

Here for verification of stability we choose $u=0.02m/s$ and $D=0.01m^2/s$ for ADE.

Case 1: Here spatial grid size, $\Delta x=0.25$ for spatial domain (0,20), and temporal grid size, $\Delta t=2.5$ for temporal domain, (0,6) is taken for this analysis .

For this application,

$$\frac{u\Delta t}{\Delta x}=\alpha=\frac{0.02\times 2.5}{0.25}=0.2 \text{ and } \frac{D\Delta t}{\Delta x^2}=\gamma=\frac{0.01\times 2.5}{(0.25)^2}=0.4$$

The stability condition for FTBSCS implies

$$0\leq\frac{D\Delta t}{\Delta x^2}\leq 1 \text{ and } -\frac{D\Delta t}{\Delta x^2}\leq\frac{u\Delta t}{\Delta x}\leq 1-2\frac{D\Delta t}{\Delta x^2}$$
$$\text{or,}0\leq 0.4\leq 1 \text{ and } -0.4\leq 0.2\leq 1-(2\times 0.4)$$
$$\text{or,}0\leq 0.4\leq 1 \text{ and } -0.4\leq 0.2\leq 0.2 \tag{32}$$

The stability condition for FTCSCS implies

$$0\leq\frac{u\Delta t}{\Delta x}\leq 1 \text{ and } 0\leq\frac{D\Delta t}{\Delta x^2}\leq\frac{1}{2}$$
$$\text{or, }0\leq 0.2\leq 1 \text{ and } 0\leq 0.4\leq\frac{1}{2} \tag{33}$$

The stability condition of second order Lax-Wendroff type scheme implies

$$0\leq\left(\frac{u\Delta t}{\Delta x}\right)^2+2\frac{D\Delta t}{\Delta x^2}\leq 1 \text{ and } 0\leq\frac{u\Delta t}{\Delta x}<1\text{and } 0\leq\frac{D\Delta t}{\Delta x^2}<1$$
$$\text{or, }0\leq(0.2)^2+(2\times 0.4)\leq 1 \text{ and } 0\leq 0.2<1\text{and } 0\leq 0.4<1$$
$$\text{or,}0\leq 0.84\leq 1 \text{ and } 0\leq 0.2<1\text{and } 0\leq 0.4<1 \tag{34}$$

Therefore, the stability condition for three schemes are satisfied, and a stable solution is expected. The solutions are to be obtained up to $t=6$ minutes; shown in **figure 1**

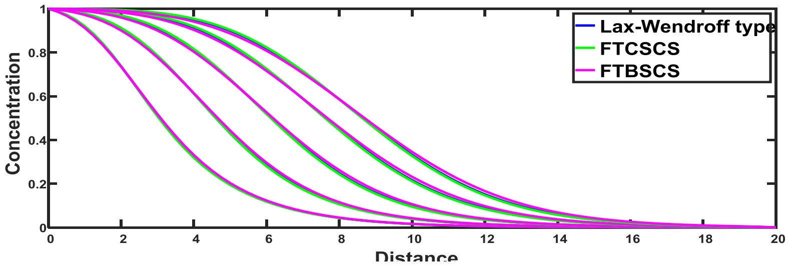

Figure 1: Solutions for the schemes with $\Delta x = 0.25$ and $\Delta t = 2.5(seconds)$.

Case 2: Here spatial grid size, $\Delta x = 0.25$ for spatial domain (0,20), and temporal grid size, $\Delta t = 2.57$ for temporal domain, (0,6) is taken for this analysis.

For this application,

$$\frac{u\Delta t}{\Delta x} = \alpha = \frac{0.02 \times 2.57}{0.25} = 0.2056 \text{ and } \frac{D\Delta t}{\Delta x^2} = \gamma = \frac{0.01 \times 2.57}{(0.25)^2} = 0.4112$$

The stability condition for FTBSCS implies

$$0 \leq \frac{D\Delta t}{\Delta x^2} \leq 1 \text{ and } -\frac{D\Delta t}{\Delta x^2} \leq \frac{u\Delta t}{\Delta x} \leq 1 - 2\frac{D\Delta t}{\Delta x^2}$$

$$\text{or,} 0 \leq 0.4112 \leq 1 \text{ and } -0.4112 \leq 0.2056 \leq 1 - (2 \times 0.4112)$$

$$\text{or,} 0 \leq 0.4112 \leq 1 \text{ and } -0.4112 \leq 0.2056 \leq 0.1776 \quad (35)$$

Which is not possible. It is seen that contradiction of stability condition is growing here.

The stability condition for FTCSCS implies

$$0 \leq \frac{u\Delta t}{\Delta x} \leq 1 \text{ and } 0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

$$\text{or, } 0 \leq 0.2056 \leq 1 \text{ and } 0 \leq 0.4112 \leq \frac{1}{2} \quad (36)$$

The stability condition for second order Lax-Wendroff type scheme implies

$$0 \leq \left(\frac{u\Delta t}{\Delta x}\right)^2 + 2\frac{D\Delta t}{\Delta x^2} \leq 1 \text{ and } 0 \leq \frac{u\Delta t}{\Delta x} < 1 \text{and } 0 \leq \frac{D\Delta t}{\Delta x^2} < 1$$

$$\text{or, } 0 \leq (0.2056)^2 + (2 \times 0.4112) \leq 1 \text{ and } 0 \leq 0.2056 < 1 \text{and } 0 \leq 0.4112 < 1$$

$$\text{or, } 0 \leq 0.86467 \leq 1 \text{ and } 0 \leq 0.2056 < 1 \text{and } 0 \leq 0.4112 < 1 \quad (37)$$

Therefore, the stability condition for three schemes are not satisfied, and so stable solution is not expected. The solutions are to be obtained up to $t = 6$ minutes; shown in **figure 2**

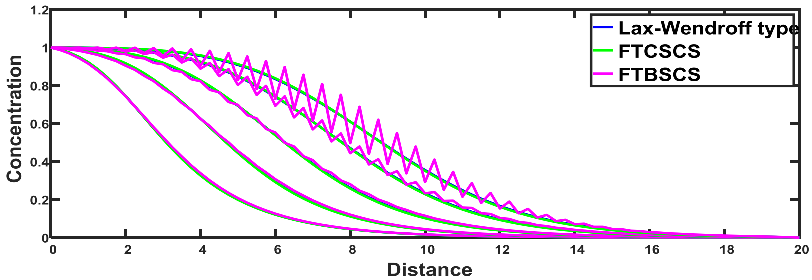

Figure 2: Solutions for the schemes with $\Delta x = 0.25$ and $\Delta t = 2.57(seconds)$

Here it is found that FTBSCS start be unstable at this step but FTCSCS and second order Lax-Wendroff type scheme is still stable.

Case 3: Here spatial grid size, $\Delta x = 0.25$ for spatial domain (0,20), and temporal grid size, $\Delta t = 2.647$ for temporal domain, (0,6) is chosen for this analysis.

For this application,

$$\frac{u\Delta t}{\Delta x} = \alpha = \frac{0.02 \times 2.647}{0.25} = 0.212 \text{ and } \frac{D\Delta t}{\Delta x^2} = \gamma = \frac{0.01 \times 2.647}{(0.25)^2} = 0.424$$

The stability condition for FTBSCS implies

$$0 \leq \frac{D\Delta t}{\Delta x^2} \leq 1 \text{ and } -\frac{D\Delta t}{\Delta x^2} \leq \frac{u\Delta t}{\Delta x} \leq 1 - 2\frac{D\Delta t}{\Delta x^2}$$

$$\text{or,} 0 \leq 0.424 \leq 1 \text{ and } -0.424 \leq 0.212 \leq 1 - 2 \times 0.424$$

$$\text{or,} 0 \leq 0.424 \leq 1 \text{ and } -0.424 \leq 0.21 \leq 0.152 \quad (38)$$

Which is not possible, contradiction of stability condition is seen here.

The stability condition for FTCSCS implies

$$0 \leq \frac{u\Delta t}{\Delta x} \leq 1 \text{ and } 0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

$$\text{or, } 0 \leq 0.212 \leq 1 \text{ and } 0 \leq 0.424 \leq \frac{1}{2} \quad (39)$$

The stability condition of second order Lax-Wendroff type scheme implies

$$0 \leq \left(\frac{u\Delta t}{\Delta x}\right)^2 + 2\frac{D\Delta t}{\Delta x^2} \leq 1 \text{ and } 0 \leq \frac{u\Delta t}{\Delta x} < 1 \text{and } 0 \leq \frac{D\Delta t}{\Delta x^2} < 1$$

$$\text{or, } 0 \leq (0.212)^2 + (2 \times 0.424) \leq 1 \text{ and } 0 \leq 0.212 < 1 \text{and } 0 \leq 0.424 < 1$$
$$\text{or, } 0 \leq 0.8929 \leq 1 \text{ and } 0 \leq 0.212 < 1 \text{and } 0 \leq 0.424 < 1 \quad (40)$$

Therefore, the stability condition for three schemes are not satisfied, and so stable solution is not expected. The solutions are to be obtained up to $t = 6$ minutes; shown in **figure 3**

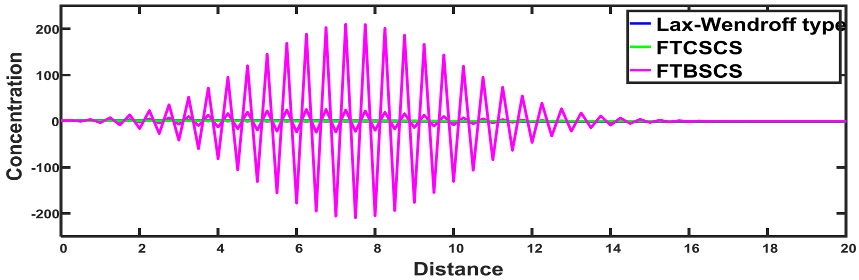

Figure 3: Solutions for the schemes with $\Delta x = 0.25$ and $\Delta t = 2.647(seconds)$

Now up to this step, it is found that the FTBSCS scheme is become totally unstable but the FTCSCS and the second order Lax-Wendroff type schemes is still stable, so we will move to next step of analysis with FTCSCS and second order Lax-Wendroff type scheme.

Case 4: Here spatial grid size, $\Delta x = 0.25$ for spatial domain (0,20), and temporal grid size, $\Delta t = 2.9$ for temporal domain, (0,6) is chosen for this analysis.

For this application,

$$\frac{u\Delta t}{\Delta x} = \alpha = \frac{0.02 \times 2.9}{0.25} = 0.232 \text{ and } \frac{D\Delta t}{\Delta x^2} = \gamma = \frac{0.01 \times 2.9}{(0.25)^2} = 0.464$$

The stability condition for FTCSCS implies

$$0 \leq \frac{u\Delta t}{\Delta x} \leq 1 \text{ and } 0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$$
$$\text{or, } 0 \leq 0.232 \leq \frac{1}{2} \text{ and } 0 \leq 0.464 \leq 1 \quad (41)$$

The stability condition for second order Lax-Wendroff type scheme implies

$$0 \leq \left(\frac{u\Delta t}{\Delta x}\right)^2 + 2\frac{D\Delta t}{\Delta x^2} \leq 1 \text{ and } 0 \leq \frac{u\Delta t}{\Delta x} < 1 \text{and } 0 \leq \frac{D\Delta t}{\Delta x^2} < 1$$
$$\text{or, } 0 \leq (0.232)^2 + (2 \times 0.464) \leq 1 \text{ and } 0 \leq 0.232 < 1 \text{and } 0 \leq 0.464 < 1$$
$$\text{or,} 0 \leq 0.9809 \leq 1 \text{ and } 0 \leq 0.232 < 1 \text{and } 0 \leq 0.464 < 1 \quad (42)$$

Therefore, the stability condition for both schemes are satisfied, and a stable solution is expected. The solutions are to be obtained up to $t = 6$ minutes; shown in **figure 4**

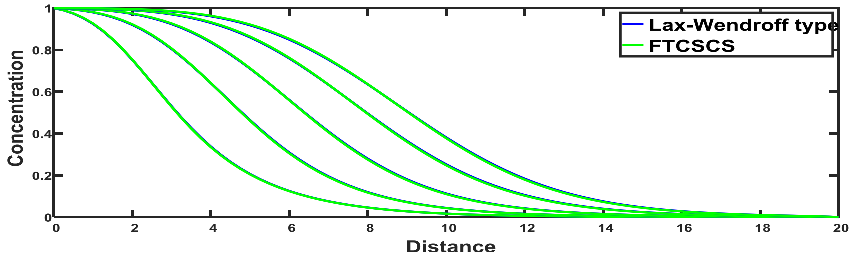

Figure 4: Solutions for the schemes with $\Delta x = 0.25$ and $\Delta t = 2.9(seconds)$

Case 5: Here spatial grid size, $\Delta x = 0.25$ for spatial domain (0,20), and temporal grid size, $\Delta t = 3$ for temporal domain, (0,6) is chosen for this analysis.

For this application,

$$\frac{u\Delta t}{\Delta x} = \alpha = \frac{0.02 \times 3}{0.25} = 0.24 \text{ and } \frac{D\Delta t}{\Delta x^2} = \gamma = \frac{0.01 \times 3}{(0.25)^2} = 0.48$$

The stability condition for FTCSCS implies

$$0 \leq \frac{u\Delta t}{\Delta x} \leq 1 \text{ and } 0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$$
$$\text{or, } 0 \leq 0.24 \leq 1 \text{ and } 0 \leq 0.48 \leq \frac{1}{2} \quad (43)$$

The stability condition for second order Lax-Wendroff type scheme implies

$$0 \leq \left(\frac{u\Delta t}{\Delta x}\right)^2 + 2\frac{D\Delta t}{\Delta x^2} \leq 1 \text{ and } 0 \leq \frac{u\Delta t}{\Delta x} < 1 \text{and } 0 \leq \frac{D\Delta t}{\Delta x^2} < 1$$
$$\text{or, } 0 \leq (0.24)^2 + (2 \times 0.48) \leq 1 \text{ and } 0 \leq 0.24 < 1 \text{and } 0 \leq 0.48 < 1$$
$$\text{or, } 0 \leq 1.0176 \leq 1 \text{ and } 0 \leq 0.24 < 1 \text{and } 0 \leq 0.48 < 1 \quad (44)$$

Which is impossible, so the stability condition is not justified here.

Therefore, the stability condition for both schemes are not satisfied, and so stable solution is not expected. The solutions are to be obtained up to $t = 6$ minutes; shown in **figure 5**

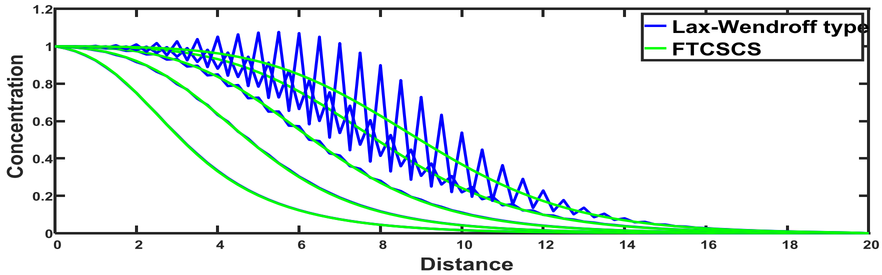

Figure 5: Solutions for the schemes with $\Delta x = 0.25$ and $\Delta t = 3$(seconds)

Here it is found that the second order Lax-Wendroff type scheme start to be unstable with contradicting its stability condition but still the FTCSCS is stable.

Case 6: Here spatial grid size, $\Delta x = 0.25$ for spatial domain (0,20), and temporal grid size, $\Delta t = 3.075$ for temporal domain, (0,6) is chosen for this analysis.

For this application,

$$\frac{u\Delta t}{\Delta x} = \alpha = \frac{0.02\times 3.075}{0.25} = 0.246 \text{ and } \frac{D\Delta t}{\Delta x^2} = \gamma = \frac{0.01\times 3.075}{(0.25)^2} = 0.492$$

The stability condition for FTCSCS implies

$$0 \le \frac{u\Delta t}{\Delta x} \le 1 \text{ and } 0 \le \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2}$$

$$\text{or, } 0 \le 0.246 \le 1 \text{ and } 0 \le 0.492 \le \frac{1}{2} \quad (45)$$

The stability condition for second order Lax-Wendroff type scheme implies

$$0 \le \left(\frac{u\Delta t}{\Delta x}\right)^2 + 2\frac{D\Delta t}{\Delta x^2} \le 1 \text{ and } 0 \le \frac{u\Delta t}{\Delta x} < 1 \text{and } 0 \le \frac{D\Delta t}{\Delta x^2} < 1$$

$$\text{or, } 0 \le (0.246)^2 + (2\times 0.492) \le 1 \text{ and } 0 \le 0.246 < 1 \text{and } 0 \le 0.492 < 1$$

$$\text{or, } 0 \le 1.044516 \le 1 \text{ and } 0 \le 0.246 < 1 \text{and } 0 \le 0.492 < 1 \quad (46)$$

Which is impossible, so the stability condition is not justified here.

Therefore, the stability condition for both schemes are not satisfied, and so stable solution is not expected. The solutions are to be obtained up to $t = 6$ minutes; shown in **figure 6**

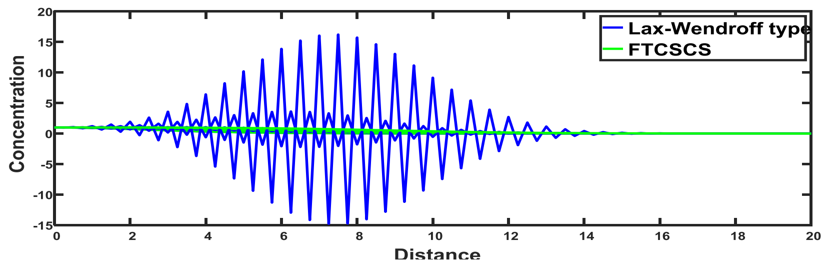

Figure 6: Solutions for the schemes with $\Delta x = 0.25$ and $\Delta t = 3.075(seconds)$.

Form the above figure we see that second order Lax-Wendroff type schemes become totally unstable but the FTCSCS scheme is still stable. So, we will move to next step with FTCSCS scheme to find the stability limit of it with respect to time step.

Case 7: Here spatial grid size, $\Delta x = 0.25$ for spatial domain (0,20), and temporal grid size, $\Delta t = 3.1034$ for temporal domain, (0,6) is taken for this analysis.

For this application,

$$\frac{u\Delta t}{\Delta x} = \alpha = \frac{0.02\times 3.1034}{0.25} = 0.248 \text{ and } \frac{D\Delta t}{\Delta x^2} = \gamma = \frac{0.01\times 3.1034}{(0.25)^2} = 0.497$$

The stability condition FTCSCS implies

$$0 \le \frac{u\Delta t}{\Delta x} \le 1 \text{ and } 0 \le \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2}$$

$$\text{or, } 0 \le 0.248 \le 1 \text{ and } 0 \le 0.497 \le \frac{1}{2} \quad (47)$$

Therefore, the stability condition for this scheme is satisfied, and a stable solution is expected. The solutions are to be obtained up to $t = 6$ minutes; shown in **figure 7.**

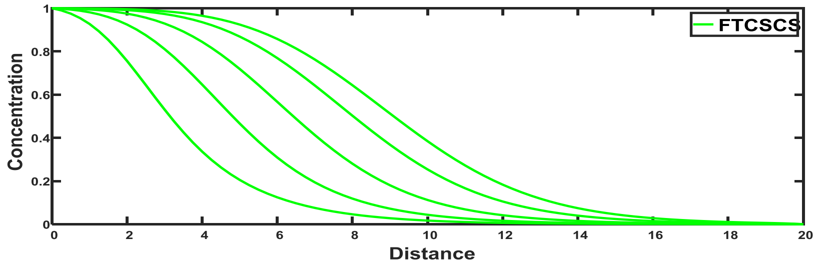

Figure 7: Solutions for the schemes with $\Delta x = 0.25$ and $\Delta t = 3.1034(seconds)$

Case 8: Here spatial grid size, $\Delta x = 0.25$ for spatial domain (0,20), and temporal grid size, $\Delta t = 3.214$ for temporal domain, (0,6) is chosen for this analysis.

For this application,

$$\frac{u\Delta t}{\Delta x} = \alpha = \frac{0.02\times3.214}{0.25} = 0.257 \text{ and } \frac{D\Delta t}{\Delta x^2} = \gamma = \frac{0.01\times3.214}{(0.25)^2} = 0.514$$

The stability condition for FTCSCS implies

$$0 \leq \frac{u\Delta t}{\Delta x} \leq 1 \text{and } 0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

$$\text{or, } 0 \leq 0.255 \leq 1 \text{and } 0 \leq 0.514 \leq \frac{1}{2} \quad (48)$$

Which is not possible, so the stability condition is not satisfied here.

Therefore, the stability condition for this scheme is not satisfied, and so stable solution is not expected. The solutions are to be obtained up to $t = 6$ minutes; shown in **figure 8.**

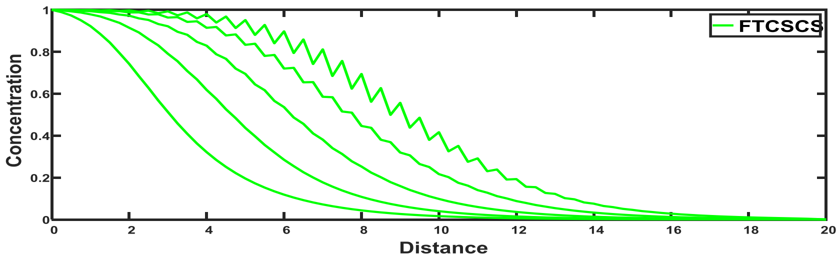

Figure 8: Solutions for the schemes with $\Delta x = 0.25$ and $\Delta t = 3.214(seconds)$

From the above analysis it is seen that the FTCSCS scheme is start to be unstable by contradicting its stability condition.

Case 9: Here spatial grid size, $\Delta x = 0.25$ for spatial domain (0,20), and temporal grid size, $\Delta t = 3.3$ for temporal domain, (0,6) is chosen for this analysis.

For this application,

$$\frac{u\Delta t}{\Delta x} = \alpha = \frac{0.02\times3.3}{0.25} = 0.267 \text{ and } \frac{D\Delta t}{\Delta x^2} = \gamma = \frac{0.01\times3.3}{(0.25)^2} = 0.528$$

The stability condition for FTCSCS implies

$$0 \leq \frac{u\Delta t}{\Delta x} \leq 1 \text{and } 0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

$$\text{or, } 0 \leq 0.258 \leq 1 \text{and } 0 \leq 0.516 \leq \frac{1}{2} \quad (49)$$

Which is not possible, so the stability condition is not satisfied here.

Therefore, the stability condition for the FTCSCS scheme is not satisfied, and so stable solution is not expected. The solutions are to be obtained up to $t = 6$ minutes; shown in **figure 9**

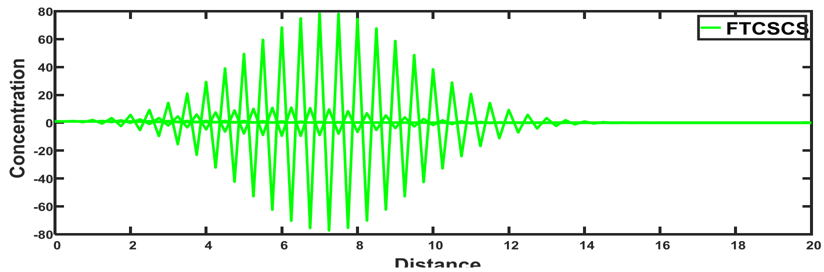

Figure 9: Solutions for the schemes with $\Delta x = 0.25$ and $\Delta t = 3.3(seconds)$

From the above figure it is seen that the FTCSCS scheme become totally unstable with contradicting its stability.

Efficiency of the Numerical schemes

To the check the efficiency of these numerical schemes, various methods are used here the efficiency of the numerical schemes are checked by Elapsed time.

Elapsed Time

Elapsed time is the time which is needed for computation of any system or phenomena. So, in comparison of numerical schemes which has less elapsed time, is more efficient and it will take less computational cost to work with that scheme.

So, by calculating elapsed time we can find which scheme is more efficient than other.

We can check the elapsed time of the scheme to find out efficiency for last stable time step (critical time step) of the finite difference FTBSCS, FTCSCS and proposed secondorder Lax-Wendroff type schemes from the previous stability analysis of this article.

Here, for calculating elapsed time we use velocity $u = 0.02$ and diffusion co-efficient.

$D = 0.01$, spatial domain (0,20) and temporal domain (0,6).

Table 1: Elapsed time with last stable time step (critical time step) of FTBSCS, FTCSCS and secondorder Lax-Wendroff type scheme.

Scheme	Spatial step Δx	Critical time step Δt	Elapsed time of scheme with last stable temporal step
FTBSCS	0.25	2.5	0.064039
Lax-Wendroff type	0.25	2.9	0.067143
FTCSCS	0.25	3.1034	0.065534

From the above analysis it is seen that the finite different FTBSCS scheme take less elapsed time than FTCSCS and proposed secondorder Lax-Wendroff type scheme. The FTBSCS scheme will cost less than FTCSCS and proposed secondorder Lax-Wendroff type scheme.

Conclusion

In this paper, we have discussed about the different numerical schemes of ADE such as FTBSCS, FTCSCS. We have proposed a secondorder Lax-Wendroff type scheme for ADE like as Lax-Wendroff scheme of hyperbolic partial differential equation. Here for proposed new Lax-Wendroff type scheme of ADE the discretisation of first order terms are in second order same as Lax-Wendroff scheme of hyperbolic partial differential equation. We have determined the stability condition of FTBSCS, FTCSCS and secondorder Lax-Wendroff type scheme by maximum principle and Von-Neumann stability analysis. We have compared the stability of these schemes with respect to different temporal step and verify the efficiency of these schemes with elapsed time. We have observed that FTBSCS scheme is more efficient with less accuracy than FTCSCS and secondorder Lax-Wendroff type scheme, on the other hand secondorder Lax-Wendroff type scheme is less efficient but more accurate than FTBSCS and FTCSCS.

Recommendation

According to numerical analysis results of this research, second order Lax-Wendroff type scheme of ADE is more accurate and less efficient according to elapse time for computational work compare to FTBSCS and FTCSCS. In future work, it would be useful to further investigate with higher order and higher dimensional explicit and implicit scheme of ADEwhich will provide good results.

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