

Counterexample of the Riemann Hypothesis

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Abstract

Under the assumption that the Riemann hypothesis is true, von Koch deduced the improved asymptotic formula $\theta(x) = x + O(\sqrt{x} \times \log^2 x)$, where $\theta(x)$ is the Chebyshev function. We obtain a result which contradicts this asymptotic formula. By contraposition, we deduce that the Riemann hypothesis is false.

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1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. On the other hand, $h(x) + O(f(x))$ denotes the collection of functions having the growth of $h(x)$ plus a part whose growth is limited to that of $f(x)$. Thus,

$$g(x) = h(x) + O(f(x))$$

expresses the same as

$$g(x) - h(x) = O(f(x)).$$

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x . Say $\text{Nicolas}(p_n)$ holds provided

$$\prod_{q \leq p_n} \frac{q}{q-1} > e^\gamma \times \log \theta(p_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, \log is the natural logarithm, and p_n is the n^{th} prime number. The importance of this property is:

Theorem 1.1. [2], [3]. *Nicolas(p_n) holds for all prime numbers $p_n > 2$ if and only if the Riemann hypothesis is true.*

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We know the following properties for the Chebyshev function:

Theorem 1.2. [4]. *If the Riemann hypothesis holds, then*

$$\theta(x) = x + O(\sqrt{x} \times \log^2 x)$$

for all $x \geq 10^8$.

Theorem 1.3. [5]. *For $2 \leq x \leq 10^8$*

$$\theta(x) < x.$$

We also know that

Theorem 1.4. [6]. *If the Riemann hypothesis holds, then*

$$\left(\frac{e^{-\gamma}}{\log x} \times \prod_{q \leq x} \frac{q}{q-1} - 1 \right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}$$

for all numbers $x \geq 13.1$.

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [7]. We know from the constant H , the following formula:

Theorem 1.5. [8].

$$\sum_q \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows

$$u(x) = \sum_{q > x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right).$$

We use the following theorems:

Theorem 1.6. [9]. *For $x > -1$:*

$$\frac{x}{x+1} \leq \log(1+x).$$

Theorem 1.7. [10]. *For $x \geq 1$:*

$$\log\left(1 + \frac{1}{x}\right) < \frac{1}{x+0.4}.$$

Let's define:

$$\delta(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).$$

Definition 1.8. *We define another function:*

$$\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).$$

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\text{Nicolas}(p)$ holds, where p is the greatest prime number such that $p \leq x$. In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion and deduce some of its consequences.

2. Results

Theorem 2.1. *The Riemann hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all numbers $x \geq 3$.*

Proof. In the paper [3] is defined the function:

$$f(x) = e^\gamma \times (\log \theta(x)) \times \prod_{q \leq x} \frac{q-1}{q}.$$

We know that $f(x)$ is lesser than 1 when Nicolas(p) holds, where p is the greatest prime number such that $2 < p \leq x$. In the same paper, we found that

$$\log f(x) = U(x) + u(x)$$

where $U(x) = -\varpi(x)$ [3]. When $f(x)$ is lesser than 1, then $\log f(x) < 0$. Consequently, we obtain that

$$-\varpi(x) + u(x) < 0$$

which is the same as $\varpi(x) > u(x)$. Therefore, this is a consequence of the theorem 1.1. \square

Theorem 2.2. *If the Riemann hypothesis holds, then*

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} > 1$$

for all numbers $x \geq 13.1$.

Proof. Under the assumption that the Riemann hypothesis is true, then we would have

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \times \log x \times \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}} \right)$$

after of distributing the terms based on the theorem 1.4 for all numbers $x \geq 13.1$. If we apply the logarithm to the both sides of the previous inequality, then we obtain that

$$\sum_{q \leq x} \log\left(\frac{q}{q-1}\right) < \gamma + \log \log x + \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right).$$

That would be equivalent to

$$\sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) < \gamma + \log \log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

where we know that

$$\begin{aligned} \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right) &< \frac{1}{\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} + 0.4} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} \end{aligned}$$

according to theorem 1.7 since $\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} \geq 1$ for all numbers $x \geq 13.1$. We use the theorem 1.5 to show that

$$\sum_{q \leq x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) = H - u(x)$$

and $\gamma = H + B$. So,

$$H - u(x) < H + B + \log \log x - \sum_{q \leq x} \frac{1}{q} + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is the same as

$$H - u(x) < H - \delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}.$$

We eliminate the value of H and thus,

$$-u(x) < -\delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is equal to

$$u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Under the assumption that the Riemann hypothesis is true, we know from the theorem 2.1 that $\varpi(x) > u(x)$ for all numbers $x \geq 13.1$ and therefore,

$$\varpi(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Hence,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \log \log \theta(x) - \log \log x.$$

Suppose that $\theta(x) = \epsilon \times x$ for some constant $\epsilon > 1$. Then,

$$\begin{aligned} \log \log \theta(x) - \log \log x &= \log \log(\epsilon \times x) - \log \log x \\ &= \log(\log x + \log \epsilon) - \log \log x \\ &= \log\left(\log x \times \left(1 + \frac{\log \epsilon}{\log x}\right)\right) - \log \log x \\ &= \log \log x + \log\left(1 + \frac{\log \epsilon}{\log x}\right) - \log \log x \\ &= \log\left(1 + \frac{\log \epsilon}{\log x}\right). \end{aligned}$$

In addition, we know that

$$\log\left(1 + \frac{\log \epsilon}{\log x}\right) \geq \frac{\log \epsilon}{\log \theta(x)}$$

using the theorem 1.6 since $\frac{\log \epsilon}{\log x} > -1$ when $\epsilon > 1$. Certainly, we will have that

$$\log\left(1 + \frac{\log \epsilon}{\log x}\right) \geq \frac{\frac{\log \epsilon}{\log x}}{\frac{\log \epsilon}{\log x} + 1} = \frac{\log \epsilon}{\log \epsilon + \log x} = \frac{\log \epsilon}{\log \theta(x)}.$$

Thus,

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \frac{\log \epsilon}{\log \theta(x)}.$$

If we add the following value of $\frac{\log x}{\log \theta(x)}$ to the both sides of the inequality, then

$$\begin{aligned} \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} &> \frac{\log \epsilon}{\log \theta(x)} + \frac{\log x}{\log \theta(x)} \\ &= \frac{\log \epsilon + \log x}{\log \theta(x)} \\ &= \frac{\log \theta(x)}{\log \theta(x)} \\ &= 1. \end{aligned}$$

We know this inequality is satisfied when $0 < \epsilon \leq 1$ since we would obtain that $\frac{\log x}{\log \theta(x)} \geq 1$. Therefore, the proof is done. \square

Theorem 2.3. *The Riemann hypothesis is false.*

Proof. If the Riemann hypothesis holds, then

$$\theta(x) = x + O(\sqrt{x} \times \log^2 x)$$

for all $x \geq 10^8$ due to the theorem 1.2. Now, suppose there is a real number $x \geq 10^8$ such that $\theta(x) > x + \sqrt{x} \times \log^{1.9} x$. That would be equivalent to

$$\log \theta(x) > \log(x + \sqrt{x} \times \log^{1.9} x)$$

and so,

$$\frac{1}{\log \theta(x)} < \frac{1}{\log(x + \sqrt{x} \times \log^{1.9} x)}$$

for all numbers $x \geq 10^8$. Hence,

$$\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)}.$$

If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)} > 1$$

for those values of x that complies with

$$\theta(x) > x + \sqrt{x} \times \log^{1.9} x$$

due to the theorem 2.2. By contraposition, if there exists some number $y \geq 10^8$ such that for all $x \geq y$ the inequality

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)} \leq 1$$

is satisfied, then the Riemann hypothesis should be false. Let's define the function

$$v(x) = \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + \sqrt{x} \times \log^{1.9} x)} - 1.$$

The Riemann hypothesis would be false when there exists some number $y \geq 10^8$ such that for all $x \geq y$ the inequality $v(x) \leq 0$ is always satisfied. We ignore when $2 \leq x \leq 10^8$ since $\theta(x) < x$ according to the theorem 1.3. We know that the function $v(x)$ is monotonically decreasing for every number $x \geq 10^8$. The derivative of $v(x)$ is negative for all $x \geq 10^8$. The derivative of $v(x)$ is

$$v'(x) \approx - \left(\frac{0.1875 \times (0.3 + \pi \sqrt{x}) \times (1.66667 + \log(x))}{x \times (0.25 + \pi \times \sqrt{x} + 0.15 \times \log(x))^2} \right) + \frac{3}{2 \times x + 8 \times \pi \times x^{\frac{3}{2}} + 1.2 \times x \times \log(x)}$$

$$- \left(\frac{\sqrt{x} \times \log(x) + 1.9 \times \log^{1.9}(x) + 0.5 \times \log^{2.9}(x)}{x \times (\sqrt{x} + \log^{1.9}(x)) \times \log^2(x + \sqrt{x} \times \log^{1.9}(x))} \right) + \frac{1}{x \times \log(x + \sqrt{x} \times \log^{1.9}(x))}.$$

Indeed, a function $v(x)$ of a real variable x is monotonically decreasing in some interval if the derivative of $v(x)$ is lesser than zero and the function $v(x)$ is continuous over that interval [11]. It is enough to find a value of $y \geq 10^8$ such that $v(y) \leq 0$ since for all $x \geq y$ we would have that $v(x) \leq v(y) \leq 0$, because of $v(x)$ is monotonically decreasing. We found the value $y = 10^8$ complies with $v(y) \leq 0$. In this way, we obtain that $v(x) \leq 0$ for every number $x \geq 10^8$. Consequently, under the assumption that the Riemann hypothesis is true, then

$$\theta(x) < x + \sqrt{x} \times \log^{1.9} x$$

for all $x \geq 10^8$. Hence, this implies that the Riemann hypothesis is false using the theorem 1.2. \square

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