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Controllability For Underactuated Compliant Arms

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***Abstract*— Underactuated compliant arms are a promising technology to outperform traditional rigid robots in interaction tasks, thanks to their adaptability and shock-absorbing properties. However, the control problem is yet to be solved. In this paper, we study the controllability for a class of planar underactuated compliant arms under gravity. We prove the Kalman rank condition for a system with one actuated elastic joint, i.e., base joint, and a generic number of unactuated elastic ones. The proof relies on the induction principle on the dimension of the reachability matrix. Finally, we generalize the result for a generic number of actuated elastic joints in the chain.**

I. Introduction

Underactuated compliant robots include systems with elastic elements lumped at the joints, i.e., articulated robots [1], flexible links robots [2], and continuum robots [3]. Despite the great effort into the design of highly performant and compliant structures [4], the control problem is still open.

To tackle this challenge, it is necessary to analyze the controllability [5], and in particular, the small-time local controllability (STLC). Roughly speaking, if the system is STLC from a state, there exists a control action capable to bring the robot from the initial state to the final one in a finite time. Additionally, the reachability set is not empty in a neighborhood of the initial state [6].

In the literature, many articles investigate the controllability for planar rigid structures [7]. However, when the number of robot Degrees of Freedom (DoFs) increases the analysis results are complicated [8]. Thus, strong hypotheses are required to prove the controllability, e.g., in [9], the Authors consider links in the chain without masses. Surprisingly, these systems are accessible as showed in [10]. However, the sufficient conditions developed in [6], [11], and [12] can not prove the STLC also for a 2-DoFs robot, namely Pendubot. For this reason, in [13], the analysis is carried out for a multi-link rigid planar robot with one input deriving rank conditions, which depend on the dynamic parameters of the robot. The same kind of system is studied in [14], the controllability descends from the use of a global stabilizing feedback control law, which guarantees the convergence of the state. Then, in [15], the case of multiple actuators is considered. However, in all these articles, no elasticity is embedded in the robot model. Thus, the controllability of a planar underactuated compliant $n-$link arm with any number of actuators is still an open problem.

Finally, the main contribution of this work is the proof of the STLC property for the class of compliant arms. The proof relies on the Kalman rank condition using the induction principle on the dimension of the reachability matrix.

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II. Problem Definition

We model the underactuated compliant arm such as a combination of active and passive elastic joints [16], i.e.,

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)+D\dot{q}+Kq=S\tau\,, \qquad (1)$$

where $q,\dot{q},\ddot{q}\in\mathbb{R}^n$ are the joint position, velocity, and acceleration vectors, respectively. We indicate with n_{a} the number of actuated joints. $M(q)\in\mathbb{R}^{n\times n}$ such as $M(q)\succ 0\,,\forall q\in\mathbb{R}^n$ and $C(q,\dot{q})\in\mathbb{R}^{n\times n}$ are the inertia and Coriolis matrix of the robot, respectively. $G(q)\in\mathbb{R}^n$ includes the gravity effect, and $D\in\mathbb{R}^{n\times n}$ is diagonal damping matrix such as $D\succ 0$. $K\in\mathbb{R}^{n\times n}$ is the spring matrix computed such as $K\succ 0$. In torque term $S\tau$, τ is the control input, and $S:\mathbb{R}^n\times\mathbb{R}^{n_{\mathrm{a}}}\to\mathbb{R}^n$ is the underactuation matrix.

The system (1) can be written in the classical state-space affine form by defining the state vector $x=\left[q^\top,\dot{q}^\top\right]^\top\in\mathbb{R}^{2n}$

$$\dot{x}(t)=f(x(t))+g(x(t))u(t)\,, \qquad (2)$$

where $t\in[0,t_{\mathrm{f}}]$ is the time variable, while $u\in\mathbb{R}^{n_{\mathrm{a}}}$ is the control action, i.e., τ in (1). $f(\cdot):\mathbb{R}^n\times[0,t_{\mathrm{f}}]\to\mathbb{R}^n$, and $g(\cdot):\mathbb{R}^n\times[0,t_{\mathrm{f}}]\to\mathbb{R}^{n\times n_{\mathrm{a}}}$ are the drift and control vector field, respectively, i.e.,

$$f(x)=\begin{bmatrix}\dot{q}\\ -M^{-1}(q)Q(q,\dot{q})\end{bmatrix},\; g(x)=\begin{bmatrix}0_{n\times n_{\mathrm{a}}}\\ M^{-1}(q)S\end{bmatrix}, \qquad (3)$$

where $Q(q,\dot{q})\triangleq C(q,\dot{q})\dot{q}+G(q)+Kq+D\dot{q}$.

Finally, the goal of this article in to prove that the underactuated compliant arm (1)-(2) is STLC from an equilibrium point, i.e., verify the Kalman rank condition.

III. Main Result

Recalling the systems (1)-(2), any underactuated compliant arm has a rich equilibria set, which is equal to

$$\Theta_{\mathrm{eq}}=\left\{x(0)=\left[q^\top(0)\,,0^\top_{n\times 1}\right]^\top\in\mathbb{R}^{2n}\,,\tau(0)\in\mathbb{R}^{n_{\mathrm{a}}}\;\middle|\;G(q(0))+Kq(0)=S\tau(0)\right\}\,. \qquad (4)$$

It is worth nothing that $G(q(0))+Kq(0)=S\tau(0)$ is a nonlinear system of n equations containing $n+1$ unknowns, thus the number of solutions is infinite.

In the following proposition, we prove the STLC property [5] of the system (1)-(2).

Proposition 1. *The system* (1)-(2) *is* STLC *from the equilibrium point* $x(0) = \left[q^\top(0),\ 0_{n\times 1}^\top\right]^\top \in \mathbb{R}^{2n}$*,* $u(0) \in \mathbb{R}^{n_\mathrm{a}}$ *with* $(x(0), u(0)) \in \Theta_{\mathrm{eq}}$ *, if* $\mathrm{rank}\{\partial G(q)/\partial q\big|_{q(0)} + K\} = n$.

Proof. Linearizing the system (2) around the equilibrium point $(x(0), u(0)) \in \Theta_{\mathrm{eq}}$, leads to $\dot{\xi} = A\xi + B\upsilon$, where $\xi \in \mathbb{R}^{2n}$, $\upsilon \in \mathbb{R}^{n_\mathrm{a}}$, $A \in \mathbb{R}^{2n\times 2n}$, and $B \in \mathbb{R}^{2n\times n_\mathrm{a}}$ are computed as Taylor series. Without loss of generality, let us consider $q(0) = 0 \in \mathbb{R}^n$, $n_\mathrm{a} = 1$, and $u(0) = 0 \in \mathbb{R}$. If

$$\mathrm{rank}\left\{\left[B,\ AB,\ A^2B,\ \cdots,\ A^{2n-1}B\right]\right\} = \mathrm{rank}\{\mathscr{R}\} = 2n\ , \quad (5)$$

then the system (1)-(2) is STLC from $x(0) \in \mathbb{R}^{2n}$.

The expressions of A and B are

$$A \triangleq \begin{bmatrix} 0_n & I_n \\ A_{21} & A_{22} \end{bmatrix}, \ B \triangleq \begin{bmatrix} 0_{n\times n_\mathrm{a}} \\ B_2 \end{bmatrix}, \quad (6)$$

where

$$\begin{aligned} A_{21} &= -M^{-1}\left(\tfrac{\partial G}{\partial q} + K\right)\Big|_{q(0),u(0)}\ , \\ A_{22} &= -M^{-1}D\big|_{q(0),u(0)}\ , \\ B_2 &= M^{-1}S\big|_{q(0),u(0)}\ . \end{aligned} \quad (7)$$

Let us define the block-diagonal matrix $\mathscr{B} = \texttt{blkdiag}(B_2, \cdots, B_2) \in \mathbb{R}^{n^2\times n n_\mathrm{a}}$. Then, we can rewrite (5) such as

$$\begin{aligned} &\mathscr{R} \triangleq \begin{bmatrix} R_0 & R_1 & R_2 & R_3 & \cdots \end{bmatrix}\mathscr{B} = \\ &\begin{bmatrix} 0_n & I_n & A_{22} & A_{21}+A_{22} & \cdots \\ I_n & A_{22} & A_{21}+A_{22} & A_{22}\left(A_{21}+A_{22}\right)+A_{21}A_{22} & \cdots \end{bmatrix}\mathscr{B}\ , \end{aligned} \quad (8)$$

where

$$R_i \triangleq \begin{bmatrix} A^i_{[1]} \\ A^i_{[2]} \end{bmatrix} = \begin{bmatrix} A^{i-1}_{[2]} \\ A_{21}A^{i-1}_{[1]} + A_{22}A^{i-1}_{[2]} \end{bmatrix}, i = 1, \cdots, 2n-1\ . \quad (9)$$

From direct computation, we have that

i) $\mathrm{rank}\{A_{21}\} = \mathrm{rank}\{A_{22}\} = n$.
ii) $\mathrm{rank}\{A_{21} + A_{22}\} = n$.
iii) $\mathrm{rank}\{A_{21}A_{22}\} = \mathrm{rank}\{A_{22}A_{21}\} = n$.
iv) $\mathrm{rank}\{A_{21}B_2\} = \mathrm{rank}\{A_{22}B_2\} = \mathrm{rank}\{B_2\} = n_\mathrm{a} = 1$.

Now, we proceed by induction on the dimension of $\mathscr{R}$ in (8). Let $h = 0, \cdots, 2n-1$ be the inductive index.

The base step $h = 0$ leads to $\mathrm{rank}\{R_0B_2\} = n_\mathrm{a} = 1$. Analogously, from direct computation we have $\mathrm{rank}\left\{\left[R_0B_2 \quad R_1B_2\right]\right\} = 2n_\mathrm{a} = 2$ when $h = 1$.

Now, we apply the induction hypotheses on $h = 2, \cdots, 2n-2$, which yields to $\mathrm{rank}\left\{\left[R_0B_2 \quad \cdots \quad R_{2n-2}B_2\right]\right\} = 2n-1$.

The induction step $h = 2n-1$ leads to $\mathscr{R}$ in (8). Due to the inductive hypotheses, $R_{2n-2}B_2$ is linear independent form $[R_0B_2, \ldots, R_{2n-3}B_2]$. Hence, recalling (9) and (i)-(iv) we have also that $R_{2n-1}B_2$ is full rank and linearly independent from $[R_0B_2, \ldots, R_{2n-2}B_2]$ leading to the thesis, i.e., (5) holds. □

Remark 1. *The extension to* $n_\mathrm{a} \in \mathbb{N}$ *such as* $n_\mathrm{a} \le n$ *actuators can be trivially obtained considering* $A^{\lceil 2n/n_\mathrm{a}\rceil - 1}$ *products instead of* A^{2n-1} *in* (5)*, and following analogous steps.*

IV. Conclusion

This article proposes proof of the controllability for compliant underactuated arms. The proof relies on the Kalman rank condition using the induction principle on the dimension of the reachability matrix. Future work will investigate the observability.

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