Extrait of the Supplement to the Memoir on double refraction

by Augustin Fresnel

(read 13 January 1822?)

with analytical table of contents
by the editors of Fresnel's *Oeuvres complètes*
1866–70

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English translation of A. Fresnel, "Extrait du Supplément au Mémoire sur la double réfraction" (read 13 Jan. 1822?), printed in *Oeuvres complètes d'Augustin Fresnel*, vol. 2 (1868), pp. 335–42, with the corresponding extract from the "Table Analytique" in *Oeuvres complètes...*, vol. 3 (1870), at pp. 645–7.

Translator's abstract

The hypothesis entertained in the First Memoir—that the wave-normal speeds in a biaxial birefringent crystal are given by the maximum and minimum radii of the diametral section of an ellipsoid in a plane parallel to the wavefront—is an approximation valid for weakly birefringent crystals. It cannot be true in general, because it does not give the correct result for calcite—a strongly birefringent (albeit uniaxial) crystal. But the author has accounted for all these cases by supposing the existence of three perpendicular directions (*axes of elasticity*) in which a displacement produces a restoring force parallel thereto. Then, for a general orientation of the wavefront, the permitted polarizations (those which can propagate unchanged) are those for which the restoring force is coplanar with the displacement and the wave-normal. It turns out that the associated directions of displacement are those in which the component of the restoring force parallel to the displacement is a maximum or a minimum (per unit displacement). Hence the wave-normal speed is indeed given by the maximum and minimum radii of the diametral section of a certain surface in a plane parallel to the wavefront. The equation of this surface, which the author calls the *surface of elasticity*, turns out to be

$$v^2 = a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z\,,$$

where v is the wave-normal speed, X, Y, Z are the angles between the wave-normal velocity and the coordinate axes, and the constants a, b, c are the semi-axes of the surface. This surface is indeed well approximated by an ellipsoid when a, b, c are not too different. Like an ellipsoid, it has the property that there are generally two directions in which a plane through the center cuts the surface in a circle, and these directions merge when two of the constants a, b, c are equal—explaining why there are at most two optical axes, and sometimes only one.

If the object point is sufficiently distant from the crystal that the waves can be taken as plane, the refractions can be worked out by knowing the wave-normal speed as a function of direction. If the object point is closer, however, it becomes necessary to know the shape of the *wave surface* (secondary wavefront) within the crystal. This surface is tangential to all the plane wavefronts, with all orientations, that travel from the origin in unit time. For a uniaxial crystal, the wave surface reduces to an ellipsoid of revolution, in agreement with Huygens' theory.

That the wave-normal speed is proportional to the square root of the elasticity in play can be shown by analogy with waves on a stretched string.

The author concludes by drawing attention to the extreme economy of assumptions by which he accounts for the laws of polarization and double refraction of both biaxial and uniaxial crystals.

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Translator's preface

The "Memoir" [3] to which this "*Extrait* of the Supplement" relates was first submitted on 19 November 1821, and revised on 26 November 1821. An *extrait* [4] of the revised version was read to the French Academy of Sciences on the latter date. According to the *Oeuvres complètes d'Augustin Fresnel* [8, vol. 2, p. 335 & vol. 3, p. 645], the "*Extrait* of the Supplement" was read to the Academy on 13 January 1822, whereas the *Procès-verbaux* of the Academy [1] do not record any meeting for that date and do not record any such reading at the meetings of 7 and 14 January. It is clear, however, that the Supplement proper [5] was signed on 13 January and submitted on 22 January, 1822. For further information on this memoir and the related supplements and *extraits*, see my preface to the translation of [6].

Footnotes to the present translation are numbered sequentially. After their sequential numbers, footnotes by the editors of the *Oeuvres complètes* are further identified by their *original letters* in their original parentheses. Unusually, there are no footnotes by Fresnel himself. Footnotes identified by sequential numbers alone, together with all items in *square brackets* (in the analytical table or the main text or the footnotes, and including citations such as "[5, §8]"), are mine.

— *Translator*.

Analytical Table of Contents

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1. In the Memoir [3] that I had the honor to submit to the Academy on 26 November last, I assumed that the law of elasticity for crystals endowed with double refraction could be represented by an ellipsoid, at least if the double refraction is weak; for I had noticed that for calcareous spar [calcite], where the difference in speed of the ordinary and extraordinary rays is considerable, this empirical construction no longer agrees with the law of Huygens, which the experiments of Wollaston and Malus seem to have proven exact.[1] One could therefore also assume that, for other crystals whose double refraction has less energy, the ellipsoid was only an approximate representation of the true law of elasticity of the medium. It is this law, which at first seemed to me so difficult to determine *a priori*, that I have succeeded in discovering by a very simple calculation, without making any hypothesis on the nature of the forces which tend to maintain the molecules of the vibrating medium in their relative positions of equilibrium. I assume only three perpendicular axes of elasticity—that is, three perpendicular directions in which each displaced molecule is pushed back in the direction of displacement. For this it suffices that, by reason of a certain symmetry in the arrangement of the particles of the body, each vibrating molecule displaced along one of the three axes is equally repelled to the right and to the left of this axis, in all azimuths, so that the resultant of all these repulsive forces is directed along the axis itself. The hypothesis thus reduced is hardly one any more, strictly speaking; for it is natural to suppose that among crystallized bodies—whose particles are arranged in a regular manner—there must be many which offer in three perpendicular directions the property that I have just stated.

2. When light passes through a diaphanous body, do the molecules belonging to this body participate in the vibrations of light, or are the vibrations propagated only by the ether contained in the body? This is a question which is not yet decided. But even if this ether were the only vehicle of the light waves, one could very well admit that a particular arrangement of the molecules of the body modifies the elasticity of the ether, i.e. the mutual dependence of its consecutive layers, so that it no longer has the same energy in all directions. Thus, without trying to discover whether the whole refractive medium, or only a portion of this medium, participates in the vibrations of light, I consider only the vibrating part, whatever it may be; and the mutual dependence of its molecules is what I call the *elasticity of the medium*. I suppose moreover that, if there is only a portion of the medium which participates in the vibrations of light, this vibrating part always remains the same, in whatever direction the oscillations of the molecules are executed, and that the elasticity alone can vary with this direction.[2]

3. When there are three perpendicular axes of elasticity and the intensities of elasticity along these axes are known, it is easy to deduce its intensity[3] in any direction using the following principle:

As long as there are only small displacements, whatever the law of the forces that the molecules of the medium exert on each other, the displacement of a molecule in any direction produces a repulsive force equal in magnitude and direction to the resultant of the three repulsive forces produced by three perpendicular displacements of this molecule equal to the static components of the first displacement.

4. I demonstrate this principle in the Supplement [5] to my Memoir that I have the honor to submit to the Academy [5, §3], and I then deduce from it the general law of elasticity of media with three axes [5, §§ 4–7].[4] Representing by a^2, b^2, c^2 the intensities of the elasticities parallel to these axes, and by

[1, (a)] See No. XXXIX [reference 4], §§ 15 to 21.

[2] In particular, the effective density of the vibrating part cannot vary with direction.

[3] Stiffness, not compliance.

[4] A similar argument in English may be found in [7], tr. Hobson, pp. 282–5.

v^2 the intensity of the elasticity in a direction which makes the angles X, Y, and Z with these same axes, I find the equation

$$v^2 = a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z\,.$$

Here v^2 represents not the totality of the elastic force which the displacement puts in play, but only the component of this force parallel to the displacement—the only component that we need for calculating the speed of propagation of the waves. Indeed the accelerating force developed by the displacement of a slice of the vibrating medium, sliding on itself, can be resolved into two others, one directed along the same line as the displacement, and the other perpendicular thereto. This second component is not generally perpendicular to the plane of the wave; but in this plane there are always two perpendicular directions for which this condition is fulfilled,[5] and one can conceive of the original motion decomposed into two others parallel to these directions. Now, since the accelerating force developed by each of them resolves into two other forces, one of which is parallel to the displacement and the other perpendicular to the plane of the wave, the latter will have no effect (according to my hypothesis on the constitution of light waves)[6] and the displacement of the next slice will be caused only by the parallel component. We see that in this way the successive displacements of the slices will always be made in the same direction, since the forces which they develop are constantly parallel to them. This would not be so for the other directions, where the component perpendicular to the line of displacement is not also perpendicular to the plane of the wave, because from it there arises in the plane of the wave a component perpendicular to the displacement, by virtue of which the next slice must move obliquely with respect to the first displacement, which thus changes direction from one slice to another, and to the propagation of which we can no longer apply the ordinary laws of wave propagation. This is why I relate the initial motion to the two directions (taken in the plane of the wave) for which this deviation does not take place, because the component perpendicular to the displacement is also perpendicular to the plane of the wave. The calculation shows that the two directions which satisfy this condition are those for which v^2 is a *maximum* or a *minimum* [5, §8].[7]

5. Taking v for the radius vector, I give the name *surface of elasticity* to the surface represented by the equation of elasticity,

$$v^2 = a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z\,,$$

in which X, Y, and Z represent the angles that the radius vector makes with the three axes; then a, b, and c are the semi-axes of this surface, whose radius vector is generally equal to the square root of the parallel component of the accelerating force produced by a displacement directed along this same radius vector. Hence, if we make in this surface a diametral section by the plane of the wave, the largest and smallest of the radius vectors included in this section will give the two directions in which we must decompose the oscillatory movement, in order that each component movement propagates without deviation. These movements will generally produce two wavetrains whose propagation speeds will be respectively proportional to the largest and the smallest radius vector; so these two radius vectors will measure the speeds of the ordinary and extraordinary rays (counted perpendicular to the plane of the wave), and, giving the directions of their vibrations, will determine the directions of their planes of polarization, which must be perpendicular. Such was also the construction that I had indicated in my first Memoir, except that I employed an ellipsoid instead of the true surface of elasticity; but these two surfaces nearly coincide when the three semi-axes a, b, and c differ little, which is the case for nearly all crystals, except calcareous spar. So the conclusions that I had drawn from the ellipsoid also apply to the true surface of elasticity, when the double refraction is not stronger than that of the various crystals

[5] That is, within the plane of the wavefront, there are two perpendicular directions of displacement for which the restoring force component normal to the displacement is also normal to the wavefront—in other words, two perpendicular directions of displacement for which the restoring force is in the plane of the displacement and the wave-normal.

[6, (a)] See below.

[7] The same "calculation" is given in English in [7], tr. Hobson, pp. 286–8.

of two axes studied so far. The new elastic surface, determined *a priori*, is therefore as well supported as the ellipsoid by the facts observed so far in the double refraction of crystals with two axes.

6. However different its three axes may be, this surface always has, like the ellipsoid, the property of being cut along a circle by two of its diametrical planes,[8] and only by two—whence it follows that a medium having three perpendicular axes of elasticity must always present two optical axes, and present only two, whatever the energy of its double refraction. When two of the axes of the surface of elasticity are equal to each other, it becomes [a surface] of revolution: the two optical axes merge into one, perpendicular to the plane of the equator, and the equation of the surface leads to the law of Huygens.[9]

7. As long as one supposes that the object point observed through the crystal is infinitely distant from it, the waves, being taken as plane at their arrival on the first surface of the prism, are still so in its interior and at their exit; and hence, to know the deviation of the rays, it suffices to determine the mutual inclination of the incident wave and the emerging wave, because it is perpendicular to the plane of each that the object point is seen without the prism and through the prism:[10] now the mutual inclination of the incident and emergent waves can, in a pinch, be calculated from the knowledge of only the speed of propagation of the plane wave[11] introduced into the crystal, and without having first determined the nature of the curved surface that would be assumed by the light waves produced in the interior of the crystal. Thus, in the case of an infinitely distant object point, the verification of the surface of elasticity by the law of Huygens was easy.[12]

8. But when the object point is close enough for the curvature of the wave to become noticeable, as in the experiments of Malus (where the proximity of this point was even an essential element, since he observed it through plates of calcareous spar *with parallel faces*), then it becomes necessary to know the shape of the wavefronts in the interior of the crystal in order to calculate, by the principle of the shortest path, the direction of the visual ray.[13]

9. Using the principle of the composition[14] of small movements, I easily succeed in proving the following theorem:

"To obtain the wave surface [*surface de l'onde*] produced by a center of disturbance in any medium—that is, to obtain the set of all points of the medium simultaneously disturbed at the end of a unit of time—it suffices to know the propagation speeds of plane waves (speeds measured perpendicular to the plane of the wave), and, starting these plane waves from the center of disturbance, to determine, for all the initial directions of their planes, the distance to which they will be transported at the end of the unit of time; the surface simultaneously tangential to all these planes will be the wavefront produced by the center of disturbance."

10. Applying this theorem to the law of propagation speeds deduced from the equation of elasticity, I find that in crystals of one axis the extraordinary wavefronts must indeed be ellipsoids of revolution,[15]

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[8] Indeed it is easily demonstrated that the surface given by Fresnel is obtainable from the ellipsoid $a^2x^2+b^2y^2+c^2z^2=1$ by inversion in the unit sphere centered on the origin; and under this inversion a circle centered on the origin is transformed to a circle in the same plane, centered on the origin.

[9] In other words, the relation between the wave-normal speed and the wave-normal direction, as given by the surface of elasticity, is consistent with spheroidal secondary wavefronts.

[10] That is, "seen" by an observer in the isotropic air, in which the rays are perpendicular to the wavefronts.

[11] As a function of the wave-normal direction.

[12] See [5], §11—essentially repeated in English in [7], tr. Hobson, pp. 291–4, except that some features of the diagram are missing. Here the "law of Huygens" apparently refers to his use of secondary wavefronts, of which the only ones used in the diagram are spherical.

[13] That the "shortest path" means the path of least *time* is not made explicit in this "Extrait of the Supplement", but is clear enough in the Supplement proper [5, §§ 11, 13–15].

[14] Superposition.

[15] The corresponding entry in the Analytical Table [8, vol. 3, 1870, p. 646] incorrectly states that the *surface of elasticity* becomes an ellipsoid of revolution. In the original French, this is an obvious slip of the pen, substituting *surface d'élasticité* for *surface de l'onde*. The error has been corrected [in square brackets] in the present translation of the Analytical Table.

as Huygens had supposed; and thus I complete the demonstration of the agreement between the law resulting from his ingenious construction, and the equation of elasticity.

11. I have been able to demonstrate the theorem which I have just cited only for the case where the wave is already removed from the center of disturbance by a very large distance relative to the wavelength, as I have been able to account for the general laws of reflection and refraction, and calculate those of the various phenomena of diffraction, only when the wave is distant from the refractive or diffractive surface by a very large measure relative to the wavelength. But if we note that a millimeter already contains nearly two thousand times the mean wavelength of light, we will sense that the formulae thus deduced from the theory of waves apply with sufficient accuracy to the ordinary circumstances of observations.

12. By supposing that the light waves have the constitution that I have indicated, all the known laws of light can be deduced from the principle of the composition of small movements. It seems to me that as soon as one accepts this principle as general and without exception, one cannot reject the consequences that I have drawn from it: to me they appear to be mathematical. A learned geometer [Poisson], who was kind enough to pay some attention to them, has indeed judged them very susceptible to controversy; and while admitting the principle of the composition of small movements in all the generality of his statement, he has made several objections to the consequences that I have deduced from them;[16] but I think it is easy to respond. That is what I have tried to do in this Supplement by briefly setting out the demonstration of the principle of the shortest path,[17] which is the basis of the laws of refraction in the wave theory. I intend to publish a more detailed version of this demonstration.[18] But in submitting it now to the judgment of the Academy, I have the honor to offer to give Mssrs. the Commissioners[19] all the clarifications and elaborations that they will deem necessary on this subject.

13. I have supposed that when we had reduced the oscillatory movements, directed in any way, to two other perpendicular movements directed along the largest and smallest radius vector included in the plane of the wave, we could consider the propagation speeds of these two movements as proportional to the square roots of the elasticities that they put in play, because the accelerating forces developed are then parallel to the displacement and propagate it without altering its direction; but, as the application of a principle demonstrated for a medium of a uniform elasticity, and waves of a different constitution, could appear hazardous when the question concerns elastic media such as those which I consider, it was necessary to show that the speed of propagation measured perpendicular to the plane of the wave was still proportional to the square root of the elasticity put in play. This I did without calculation by bringing the question back, by a small artifice of reasoning, to the ordinary cases of vibrating strings [5, §20].

14. Thus the theoretical results presented in this Supplement are mathematical consequences of the very simple definition that I have given, of crystals with one and two axes. I supposed that in these the vibrating medium had three perpendicular axes of elasticity—that is, three directions in which the displacement of a molecule produced a repulsive force directed in the very line of displacement. When the intensity of these forces is the same for two of the axes, the medium exhibits the properties of crystals with one axis, such as calcareous spar. It is quite remarkable that, without making any further hypothesis on the nature and the law of the forces which the molecules of the medium exert on each other, and supposing only a certain symmetry of elasticity which, moreover, the regular arrangement of the molecules of the crystal makes quite probable, we arrive at the elliptical waves of Huygens, together with all the known laws of polarization and double refraction of crystals with two axes.

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[16, (a)] See No. XXXIV [8, vol. 2, pp. 183–238].

[17] See footnote 13.

[18] *Cf.* [7], tr. Hobson, pp. 292–7, 305–6, 309–11.

[19] Arago, Ampère, Fourier, and Poisson [1, p. 248].

References

[1] Académie des Sciences, *Procès-verbaux des séances de l'Académie tenues depuis la fondation de l'Institut jusqu'au mois d'août 1835*, vol. 7 (for 1820–23), Hendaye, Basses Pyrénées: Imprimerie de l'Observatoire d'Abbadia, 1916.

[2] A. Fresnel, "Note sur le calcul des teintes que la polarisation développe dans les lames cristallisées" et seq., *Annales de Chimie et de Physique*, Ser. 2, vol. 17, pp. 102–11 (May 1821), 167–96 (June 1821), 312–15 ("Postscript", July 1821); reprinted (with added section nos.) in [8], vol. 1, pp. 609–48; translated by G.R. Putland as "On the calculation of the tints that polarization develops in crystalline plates, & postscript", doi.org/10.5281/zenodo.4058004 (2021).

[3] A. Fresnel, "Premier Mémoire sur la double réfraction", submitted 19 Nov. 1821, revised 26 Nov. 1821; printed in [8], vol. 2, pp. 261–308.

[4] A. Fresnel, "Extrait d'un Mémoire sur la double réfraction", signed 25 Nov. 1821, read 26 Nov. 1821; printed in [8], vol. 2, pp. 309–329.

[5] A. Fresnel, "Supplément au Mémoire sur la double réfraction", signed 13 Jan. 1822, submitted 22 Jan. 1822; printed in [8], vol. 2, pp. 343–67.

[6] A. Fresnel, "Extrait d'un Mémoire sur la double réfraction", *Annales de Chimie et de Physique*, Ser. 2, vol. 28, pp. 263–79 (March 1825); reprinted as "Extrait du second Mémoire sur la double réfraction" in [8], vol. 2, pp. 465–78; translated by G.R. Putland as "*Extrait* of a [second] memoir on double refraction", doi.org/10.5281/zenodo.5442206 (2022).

[7] A. Fresnel, "Mémoire sur la double réfraction", *Mémoires de l'Académie Royale des Sciences de l'Institut de France*, vol. VII (nominally for 1824, printed 1827), pp. 45–176; reprinted as "Second mémoire..." in [8], vol. 2, pp. 479–596; translated by A.W. Hobson as "Memoir on double refraction" in R. Taylor (ed.), *Scientific Memoirs*, vol. V (London: Taylor & Francis, 1852), pp. 238–333, archive.org/details/scientificmemoir05memo/page/238. For notable errata in the original printing, and consequently in the translation, see [8], vol. 2, p. 596*n*.

[8] A. Fresnel (ed. H. de Sénarmont, E. Verdet, & L. Fresnel), *Oeuvres complètes d'Augustin Fresnel* (3 vols.), Paris: Imprimerie Impériale, 1866, 1868, 1870.