ON THE QUANTUM DIFFERENTIATION OF SMOOTH REAL-VALUED FUNCTIONS

KOLOSOV PETRO

Abstract. Calculating the value of $C^{k\in\{1,\infty\}}$ class of smoothness real-valued function's derivative in point of $\mathbb{R}^+$ in radius of convergence of its Taylor polynomial (or series), applying an analog of Newton's binomial theorem and q-difference operator. (P,q)-power difference introduced in section 5. Additionally, by means of Newton's interpolation formula, the discrete analog of Taylor series, interpolation using q-difference and p,q-power difference is shown.

Keywords. derivative, differential calculus, differentiation, Taylor's theorem, Taylor's formula, Taylor's series, Taylor's polynomial, power function, Binomial theorem, smooth function, real calculus, Newton's interpolation formula, finite difference, q-derivative, Jackson derivative, q-calculus, quantum calculus, (p,q)-derivative, (p,q)-Taylor formula

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e-mail: kolosov_94@mail.ua
ORCID: http://orcid.org/0000-0002-6544-8880

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1. Introduction

Let be Taylor's theorem (see §7 "Taylor's formula", [1])

Theorem 1.1. ***Taylor's theorem.*** *Let be $n \geq 1$ an integer, let function $f(x)$ be $n+1$ times differentiable in neighborhood of $a \in \mathbb{R}$. Let x be an any function's argument from such neighborhood, p - some positive number. Then, there is exist some c between points a and x, such that*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x) \tag{1.2}$$

where $R_{n+1}(x)$ - general form of remainder term

$$R_{n+1}(x) = \left(\frac{x-a}{x-a}\right)^p \frac{(x-c)^{n+1}}{n!p} f^{(n+1)}(c) \tag{1.3}$$

Proof. Denote $\varphi(x,a)$ polynomial related to x of order n, from right part of (1.2), i.e

$$\varphi(x,a) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1.4}$$

Next, denote as $R_{n+1}(x)$ the difference

$$R_{n+1}(x) = f(x) - \varphi(x,a) \tag{1.5}$$

Theorem will be proven, if we will find that $R_{n+1}(x)$ is defined by (1.3). Let fix some x in neighborhood, mentioned in theorem 1.1. By definition, let be $x > a$. Denote by t an variable, such that $t \in [a,x]$, and review auxiliary function $\psi(t)$ of the form

$$\psi(t) = f(x) - \varphi(x,t) - (x-t)^p Q(x) \tag{1.6}$$

where

$$Q(x) = \frac{R_{n+1}(x)}{(x-a)^p} \tag{1.7}$$

More detailed $\psi(t)$ could be written as

$$\psi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - (x-t)^p Q(x) \tag{1.8}$$

Our aim - to express $Q(x)$, going from properties of introduced function $\psi(t)$. Let show that function $\psi(t)$ satisfies to all conditions of Rolle's theorem [2] on $[a,x]$. From (1.8) and conditions given to function $f(x)$, it's obvious, that function $\psi(t)$ continuous on $[a,x]$. Given $t = a$ in (1.6) and keeping attention to equality (1.7), we have

$$\psi(a) = f(x) - \varphi(x,a) - R_{n+1}(x) \tag{1.9}$$

Hence, by means of (1.5) obtain $\psi(a) = 0$. Equivalent $\psi(x) = 0$ immediately follows from (1.8). So, $\psi(t)$ on segment $[a, x]$ satisfies to all necessary conditions of Rolle's theorem [2]. By Rolle's theorem, there is exist some $c \in [a, x]$, such that

$$\psi'(c) = 0 \tag{1.10}$$

Calculating derivative $\psi'(t)$, differentiating equality (1.8), we have

$$\psi'(t) = -f'(t) + \frac{f'(t)}{1!} - \frac{f''(t)}{2!}(x-t) + \frac{f''(t)}{2!}2(x-t) - \cdots \tag{1.11}$$
$$+\frac{f^{(n)}(t)}{n!}n(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n + p(x-t)^{p-1}Q(x)$$

It's seen that all terms in right part of (1.11), except last two items, self-destructs. Hereby,

$$\psi'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + p\cdot(x-t)^{p-1}Q(x) \tag{1.12}$$

Given $t = c$ in (1.12) and applying (1.10), obtain

$$Q(x) = \frac{(x-c)^{n-p+1}}{n!p} f^{(n+1)}(c) \tag{1.13}$$

By means of (1.13) and (1.7), finally, we have

$$R_{n+1}(x) = (x-a)^p Q(x) = \left(\frac{x-a}{x-a}\right)^p \frac{(x-c)^{n+1}}{n!p} f^{(n+1)}(c) \tag{1.14}$$

Case $x < a$ is reviewed absolutely similarly. (see for reference [1], pp 246-247) This proves the theorem. □

Let function $f(x) \in C^k$ class of smoothness and satisfies to theorem 1.1, then its derivative by means of its Taylor's polynomial centered at $a \in \mathbb{R}$ in radius of convergence with $f(x)$ and linear nature of derivative, $(gf(x)+um(x))' = gf'(x)+um'(x)$, is

$$\frac{d}{dx}f(x) = \frac{f'(a)}{1!}\frac{d}{dx}(x-a) + \frac{f''(a)}{2!}\frac{d}{dx}(x-a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}\frac{d}{dx}(x-a)^k \tag{1.15}$$
$$+R'_{k+1}(x)$$

Otherwise, if $f \in C^\infty$ we have derivative of Taylor series [5] of f given the same conditions as (1.15)

$$\frac{d}{dx}f(x) = \frac{f'(a)}{1!}\frac{d}{dx}(x-a) + \frac{f''(a)}{2!}\frac{d}{dx}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\frac{d}{dx}(x-a)^n + \tag{1.16}$$
$$+\cdots$$

Hence, derivative of function $f : 1 \leq C(f) \leq \infty$ [1] could be reached by differentiating of its Taylor's polynomial or series in radius of convergence, and consequently summation of power's derivatives being multiplied by coefficient, according theorem 1.1, over k from 1 to $t \leq \infty$, depending on class of smoothness. Hereby, the properties of power function's differentiation holds, in particular, the derivative of power close related to Newton's binomial theorem [4].

Lemma 1.17. *Derivative of power function equals to limit of Binomial expansion of $(x + \Delta x)^n$, iterated from 1 to n, divided by $\Delta x : \Delta x \to 0$.*

[1] For example, let f be a k-smooth function, then $C(f) = k$.

Proof.

$$\frac{d(x^n)}{dx} = \lim_{\Delta x \to 0} \left\{ \sum_{k=1}^{n} \binom{n}{k} x^{n-k} (\Delta x)^{k-1} \right\} = \binom{n}{1} x^{n-1} \tag{1.18}$$

□

According to lemma (1.17), Binomial expansion is used to reach derivative of power, otherwise, let be introduced expansion, based on forward finite differences, discussed in [3]

$$x^n = x^{n-2} + j \sum_{k \in \mathfrak{C}(x)} k \cdot x^{n-2} - k^2 \cdot x^{n-3}, \quad x \in \mathbb{N} \tag{1.19}$$

where $j = 3!$ and $\mathfrak{C}(x) := \{0,\ 1,\ \ldots,\ x\} \subseteq \mathbb{N}$. Particularize[2] (1.19), one has

$$x^n = \sum_{k \in \mathfrak{U}(x)} j \cdot k \cdot x^{n-2} - j \cdot k^2 \cdot x^{n-3} + x^{n-3} \tag{1.20}$$

where $\mathfrak{U}(x) := \{0,\ 1,\ \ldots,\ x-1\} \subseteq \mathbb{N}$.

Property 1.21. *Let $\mathfrak{S}(x)$ be a set $\mathfrak{S}(x) := \{1,\ 2,\ \ldots,\ x\} \subseteq \mathbb{N}$, let be (1.20) written as $T(x,\ \mathfrak{U}(x))$, then we have equality*

$$T(x,\ \mathfrak{U}(x)) \equiv T(x,\ \mathfrak{S}(x)),\ x \in \mathbb{N} \tag{1.22}$$

Let (1.19) be denoted as $U(x,\ \mathfrak{C}(x))$, then

$$U(x,\ \mathfrak{C}(x)) \equiv U(x,\ \mathfrak{S}(x)) \equiv U(x,\ \mathfrak{U}(x)) \tag{1.23}$$

Proof. Let be a plot of $jkx^{n-2} - jk^2x^{n-3} + x^{n-3}$ by k over $\mathbb{R}^+_{\leq 10}$, given $x = 10$

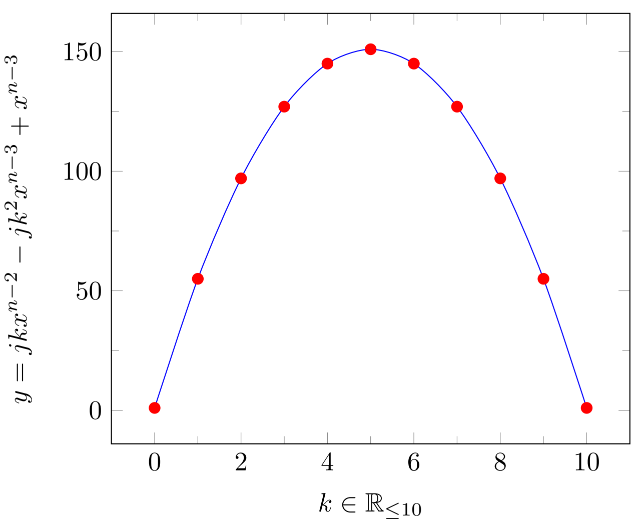


Figure 1. Plot of $jkx^{n-2} - jk^2x^{n-3} + x^{n-3}$ by k over $\mathbb{R}^+_{\leq 10}$, $x = 10$

[2]Transferring x^{n-2} under sigma operator, decreasing the power by 1 and taking summation over $k \in \mathfrak{U}(x)$

Obviously, being a parabolic function, it's symmetrical over $\frac{x}{2}$, hence equivalent $T(x,\ \mathfrak{U}(x)) \equiv T(x,\ \mathfrak{S}(x)),\ x \in \mathbb{N}$ follows. Reviewing (1.19) and denote $u(t) = tx^{n-2} - t^2x^{n-3}$, we can make conclusion, that $u(0) \equiv u(x)$, then equality of $U(x,\ \mathfrak{C}(x)) \equiv U(x,\ \mathfrak{S}(x)) \equiv U(x,\ \mathfrak{U}(x))$ immediately follows.
This completes the proof. □

By definition we will use set $\mathfrak{U}(x) \subseteq \mathbb{N}$ in our next expressions.
Since, for each $x = x_0 \in \mathbb{N}$ we have equivalent

Lemma 1.24. $\forall x = x_0 \in \mathbb{N}$ *holds*

$$\underbrace{\sum_{t=1}^{x}\sum_{k=1}^{n}\binom{n}{k}t^{n-k}}_{x^n} \equiv \sum_{k=0}^{x-1} j \cdot k \cdot x^{n-2} - j \cdot k^2 \cdot x^{n-3} + x^{n-3} \tag{1.25}$$

Proof. Proof can be done by direct calculations. □

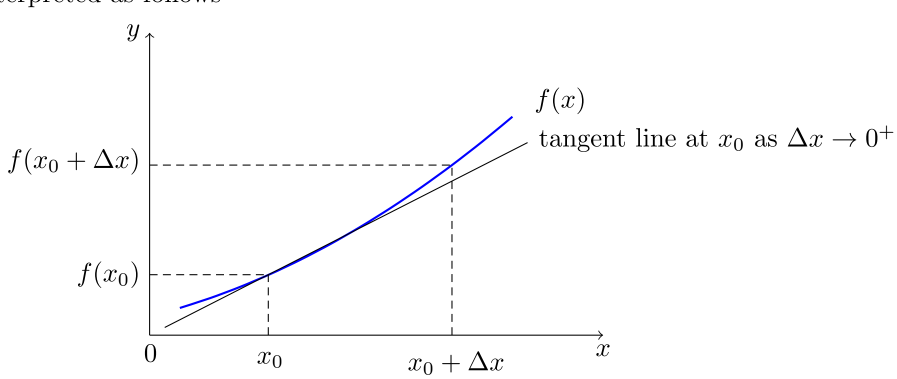
By lemma 1.24 we have right to substitute (1.20) into limit (1.18), replacing Binomial expansion, and represent derivative of power by means of expression (1.20). Note that,

$$\Delta(x^n) = \sum_{k=1}^{n}\binom{n}{k}x^{n-k} \neq j \cdot k \cdot x^{n-2} - j \cdot k^2 \cdot x^{n-3} + x^{n-3} \tag{1.26}$$

As (1.20) is analog of Binomial expansion of power and works only in space of natural numbers, different in sense, that Binomial expansion, for example, could be denoted as $M(x,\ \mathfrak{C}(n))$, where n - exponent. While (1.20) could be denoted $T(x,\ \mathfrak{U}(x) \equiv \mathfrak{S}(x))$, it shows that in case of Binomial expansion the set over which we take summation depends on exponent n of initial function, when for (1.20) it depends on point $x = x_0 \in \mathbb{N}$. To provide expressions' (1.20) usefulness[3] on taking power's derivative over $\mathbb{R}^+$, derivative in terms of quantum calculus should be applied, as next section dedicated to.

2. Application of Q-Derivative

Derivative of the function f defined as limit of division of function's grow rate by argument's grow rate, when grow rate tends to zero, and graphically could be interpreted as follows



[3]By classical definition of derivative, we have to use upper summation bound $(x + \Delta x) \in \mathbb{R}^+$ on (1.20), which turns false result as (1.20) works in space of $\mathbb{N}$.

Figure 2. Geometrical sense of derivative

In 1908 Jackson [10] reintroduced [11], [12] the Euler-Jackson q-difference operator [9]

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0 \tag{2.1}$$

The limit as q approaches 1^- is the derivative

$$\frac{df}{dx} = \lim_{q \to 1^-} (D_q f)(x) \tag{2.2}$$

More generalized form of q-derivative

$$\frac{df(x)}{dx} = \lim_{q \to 1^-} \underbrace{\frac{f(x) - f(xq)}{x - xq}}_{(D_q f^-)(x)} \equiv \lim_{q \to 1^+} \underbrace{\frac{f(xq) - f(x)}{xq - x}}_{(D_q f^+)(x)} \tag{2.3}$$

where $(D_q f^+)(x)$ and $(D_q f^-)(x)$ forward and backward q-differences, respectively. The follow figure shows the geometrical sense of above equation as q tends to 1^+

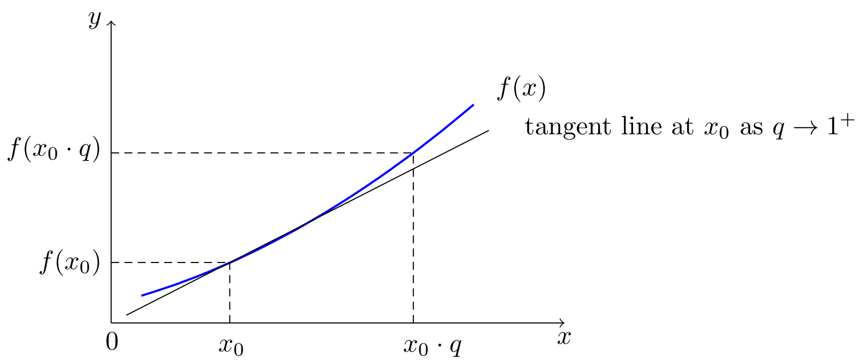


Figure 3. Geometrical sense of right part of (2.3)

Review the monomial x^n, where n-positive integer and applying right part of (2.3), then in terms of q-calculus we have forward q-derivative over $\mathbb{R}$

$$\frac{d(x^n)}{dx} = \lim_{q \to 1^+} (D_q x^{n^+})(x) = \lim_{q \to 1^+} \frac{x^n(q^n - 1)}{x(q-1)} \tag{2.4}$$

$$= \lim_{q \to 1^+} x^{n-1} \sum_{k=0}^{n-1} q^k, \quad q \in \mathbb{R}$$

Otherwise, see reference [9], equation (109).

Generalized view of high-order power's derivative by means of (2.4)

$$\frac{d^k(x^n)}{dx^k} = \lim_{q \to 1^+} (D_q^k x^{n^+})(x) = \lim_{q \to 1^+} x^{n-k} \prod_{j=0}^{k-1} \left(\sum_{m=0}^{n-j} q^m \right) \tag{2.5}$$

Since, the main property of power is

Property 2.6.

$$(x \cdot y)^n = x^n \cdot y^n$$

Let be definition

Definition 2.7. By property (2.6) and (1.20), definition of $c = x \cdot t : t \in \mathbb{R},\ x \in \mathbb{N} \Rightarrow c \in \mathbb{R}$ to power $n \in \mathbb{N}$

$$c^n := \xi(x,\ t)_n := \sum_{k=0}^{x-1} jkx^{n-2} \cdot t^n - jk^2 x^{n-3} \cdot t^n + x^{n-3} \cdot t^n \tag{2.8}$$

Hereby, applying definition (2.7) and (2.4), derivative of monomial $x^n : n \in \mathbb{N}$ by x in point $x_0 \in \mathbb{N}$ is

$$\left.\frac{d(x^n)}{dx}\right|_{x=x_0} = \underbrace{\lim_{q\to 1^+} \frac{\xi(x,\ q)_n - \xi(x,\ 1)_n}{x \cdot q - x}}_{\stackrel{\mathrm{def}}{=} \mathscr{D}_{q>1}[x^n]} \equiv \underbrace{\lim_{q\to 1^-} \frac{\xi(x,\ 1)_n - \xi(x,\ q)_n}{x \cdot q - x}}_{\stackrel{\mathrm{def}}{=} \mathscr{D}_{q<1}[x^n]}, \tag{2.9}$$

Let us approach to extend the definition space of expression (2.9) from $x_0 \in \mathbb{N}$ to $x_0 \in \mathbb{R}^+$. Let be $x_0 = \xi(t_0,\ p)_1 \in \mathbb{R}^+ \not\supseteq \mathbb{N}$ as $p \in \mathbb{R}^+ \not\supseteq \mathbb{N}$ and $t_0 \in \mathbb{N}$, then applying (p, q)-difference discussed in [13]

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x},\ x \neq 0 \tag{2.10}$$

by means of definition (2.7) and (2.10), (p, q)-differentiating of monomial $x^n,\ n \in \mathbb{N}$ gives us

$$\left.\frac{d(x^n)}{dx}\right|_{x=t_0} = \lim_{p\to q^+} D_{p,q} x^n = \underbrace{\lim_{p\to q^+} \frac{\xi(x,\ p)_n - \xi(x,\ q)_n}{x \cdot p - x \cdot q}}_{\stackrel{\mathrm{def}}{=} \mathscr{D}_{p\to q}[x^n]} \tag{2.11}$$

$$\equiv \underbrace{\lim_{q\to p^-} \frac{\xi(x,\ p)_n - \xi(x,\ q)_n}{x \cdot p - x \cdot q}}_{\stackrel{\mathrm{def}}{=} \mathscr{D}_{p\leftarrow q}[x^n]}, \qquad t_0 \in \mathbb{N},\ [p,\ q] \in \mathbb{R}^+ \not\supseteq \mathbb{N}$$

Geometrical interpretation is shown below

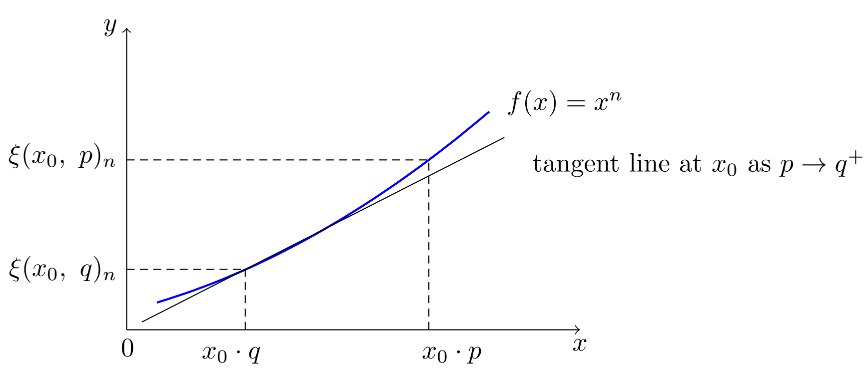


Figure 4. Geometrical interpretation of (2.11)

3. Application on functions of finite class of smoothness

In this section we will get derivative of function $f \in C^n$ in point $x_0 \in \mathbb{R}^+$ by means of its Taylor's polynomial and (2.9), where n - some positive integer. Let

$f(x)$ be an n-smooth function, then derivative of its Taylor's polynomial at radius of convergence with f in $x_0 : (x_0 - a) \in \mathbb{N}$ is

$$\left.\frac{df(x)}{dx}\right|_{x=x_0} = \sum_{k=1}^{n} \left[\frac{f^{(k)}(a)}{k!} \mathscr{D}_{q>1}[(x-a)^k] \right] + \mathscr{D}_{q>1}[R_{n+1}(x)] \equiv \sum_{k=1}^{n} \left[\frac{f^{(k)}(a)}{k!} \mathscr{D}_{q<1}[(x-a)^k] \right] + \mathscr{D}_{q<1}[R_{n+1}(x)] \tag{3.1}$$

Otherwise, let $(x_0 - a)$ satisfies to conditions of (2.11), i.e $(x_0 - a) \in \mathbb{R}^+$, then applying operator $\mathscr{D}$, defined in (2.9) we can reach derivative of $f : f \in C^n$ in point $x_0 : (x_0 - a) \in \mathbb{R}^+$, by differentiation of its Taylor's polynomial in radius of convergence with f, that is

$$\left.\frac{df(x)}{dx}\right|_{x=t_0} = \sum_{k=1}^{n} \left[\frac{f^{(k)}(a)}{k!} \mathscr{D}_{p\to q}[(x-a)^k] \right] + \mathscr{D}_{p\to q}[R_{n+1}(x)] \equiv \sum_{k=1}^{n} \left[\frac{f^{(k)}(a)}{k!} \mathscr{D}_{p\leftarrow q}[(x-a)^k] \right] + \mathscr{D}_{p\leftarrow q}[R_{n+1}(x)] \tag{3.2}$$

4. Application on analytic functions

If $f \in C^\infty$ (i.e analytic), then approximation by means of Taylor series holds in neighborhood of its center at $a \in \mathbb{R}$. Suppose that f is real-valued and satisfies to conditions of Taylor's theorem 1.1, then derivative of f at $x_0 : x < x_0 < a$ is

$$\frac{df(x)}{dx} = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = \left[\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} \frac{d}{dx} (x-a)^k \right]_{x=x_0} \tag{4.1}$$

Let x_0 satisfies to conditions of (3.1), then, applying definition (2.7), we have derivative of f in point $x_0 \in \mathbb{R}^+$

$$\frac{df(x)}{dx} = \left[\sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} \mathscr{D}_{q>1}[(x-a)^k] \equiv \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} \mathscr{D}_{q<1}[(x-a)^k] \right]_{x=x_0} \tag{4.2}$$

Otherwise, if x_0 satisfies to conditions of (3.2) and x_0 in radius of convergence with f, then derivative of $f \in C^\infty$, by means of its Taylor's series and (2.7), is

$$\frac{df(x)}{dx} = \left[\sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} \mathscr{D}_{p\to q}[(x-a)^k] \equiv \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} \mathscr{D}_{p\leftarrow q}[(x-a)^k] \right]_{x=t_0} \tag{4.3}$$

5. Introduction of $(P,\ q)$-power difference

Lemma 5.1. *Let be $m \in \mathbb{R}/\mathbb{I}$ and m could be represented as $m = at$, then exists some $c \in \mathbb{R}/\mathbb{I}$, such that*

$$m = a^c \tag{5.2}$$

Reviewing (2.3), we can see, that argument's differential Δx is given by $x \cdot q - x$, according to lemma 5.1 $\exists c \in \mathbb{R}/\mathbb{I},\ x \cdot t - x = x^c - x$, then, from (2.4) immediately follows q-power difference, (see [14], page 2, equation 3)

$$\mathcal{D}_{q>1} f(x) := \frac{f(x^q) - f(x^1)}{x^q - x^1}, \quad x \neq 0 \tag{5.3}$$

As q tends to 1^+ we have reached derivative

$$\frac{df(x)}{dx} = \lim_{q\to 1^+} \mathcal{D}_{q>1} f(x) = \underbrace{\lim_{q\to 1^+} \frac{f(x^q) - f(x^1)}{x^q - x^1}}_{\stackrel{\text{def}}{=} \mathbf{D}_{q>1}[f(x)]} \tag{5.4}$$

$$\equiv \underbrace{\lim_{q\to 1^-} \frac{f(x^1) - f(x^q)}{x^1 - x^q}}_{\stackrel{\text{def}}{=} \mathbf{D}_{q<1}[f(x)]} =: \lim_{q\to 1^-} \mathcal{D}_{q<1} f(x)$$

where $\lim\limits_{q\to 1^-} \mathcal{D}_{q<1} f(x)$ denotes the derivative through backward q-power difference By lemma 5.1 from (2.10) immediately follows (p,q)-power difference

$$\mathcal{D}_{p\to q} f(x) := \frac{f(x^p) - f(x^q)}{x^p - x^q}, \quad x \neq 0 \tag{5.5}$$

Hence, for $v = x^p,\ p \in \mathbb{R}$

$$\frac{df(x)}{dx}(v) = \lim_{p\to q^+} \mathcal{D}_{p\to q} f(x) = \underbrace{\lim_{p\to q^+} \frac{f(x^p) - f(x^q)}{x^p - x^q}}_{\stackrel{\text{def}}{=} \mathbf{D}_{p\to q}[f(x)]} \tag{5.6}$$

$$\equiv \underbrace{\lim_{q\to p^-} \frac{f(x^p) - f(x^q)}{x^q - x^p}}_{\stackrel{\text{def}}{=} \mathbf{D}_{p\leftarrow q}[f(x)]} =: \lim_{q\to p^-} \mathcal{D}_{p\leftarrow q} f(x)$$

where $\mathbf{D}_{p\to q}[f(x)],\ \mathbf{D}_{p\leftarrow q}[f(x)]$ denote derivative through forward and backward (p,q)-power differences. Let us to show geometrical interpretation of (5.4) and (5.6)

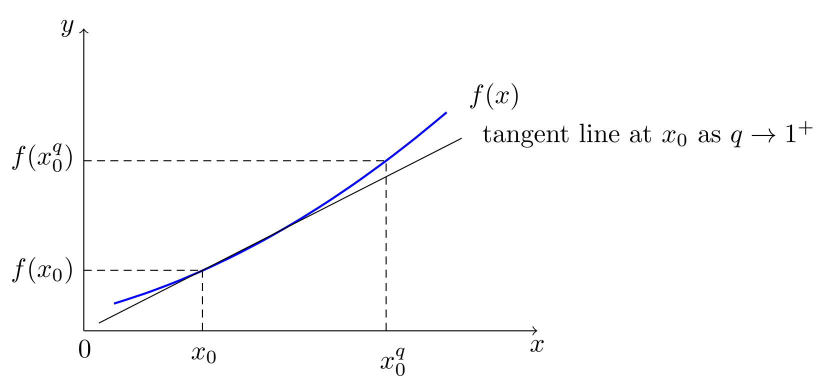


Figure 5. Geometrical sense of (5.4)

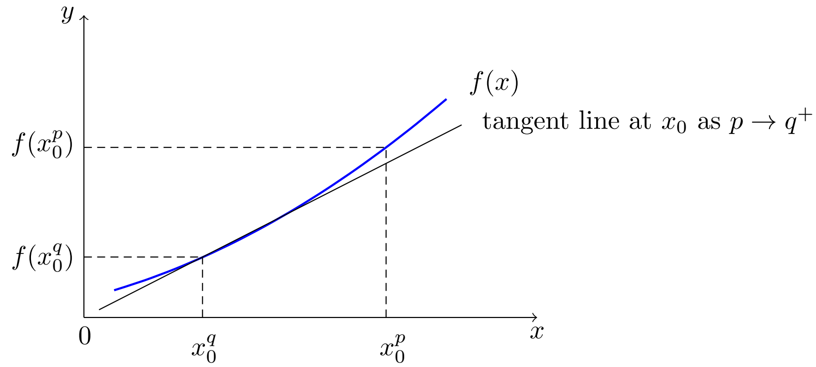


Figure 6. Geometrical sense of (5.6)

Applying (5.4) with monomial $x^m : m \in \mathbb{N}$, we get

$$\frac{d(x^m)}{dx} = \mathbf{D}_{q>1}[x^m] = \lim_{q \to 1^+} \left[\sum_{k=1}^{m} (x^q)^{m-k} \cdot x^{k-1} \right] = mx^{m-1} \tag{5.7}$$

$$\equiv \lim_{q \to 1^-} \mathbf{D}_{q<1}[x^m] = \lim_{q \to 1^-} \left[\sum_{k=1}^{m} x^{k-1} \cdot (x^q)^{m-k} \right] = mx^{m-1}$$

Note that $\mathbf{D}_{q<1}[x^m]$, $\mathbf{D}_{q>1}[x^m]$ defined by (5.4). The high order $N \le m$ derivative, derived from (5.7)

$$\frac{d^N(x^m)}{dx^N} = \mathbf{D}^N_{q>1}[x^m] = \lim_{q \to 1^+} \prod_{j=0}^{N-1} \left(\sum_{k=1}^{m-j} (x^q)^{m-k} \cdot x^{k-j-1} \right) \tag{5.8}$$

$$\equiv \mathbf{D}^N_{q<1}[x^m] = \lim_{q \to 1^-} \prod_{j=0}^{N-1} \left(\sum_{k=1}^{m-j} x^{k-j-1} \cdot (x^q)^{m-k} \right)$$

Let be analytic function f and let f satisfies to Taylor's theorem 1.1 on segment of (a, x), $a \in \mathbb{R}$, then, applying (5.4), in radius of convergence of its Taylor's series, we obtain derivative

$$\frac{df(x)}{dx} = \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} \mathbf{D}_{q<1}[(x-a)^k] \equiv \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} \mathbf{D}_{q>1}[(x-a)^k] \tag{5.9}$$

Using $\mathbf{D}_{p \to q}[f(x)]$, $\mathbf{D}_{p \leftarrow q}[f(x)]$ defined by (5.8), for each $v = x^p$, we receive

$$\frac{df(x)}{dx} = \left[\sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} \mathbf{D}_{p \to q}[(x-a)^k] \equiv \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!} \mathbf{D}_{p \leftarrow q}[(x-a)^k] \right]_{x=v} \tag{5.10}$$

Or, by means of definition (2.7) and (5.9), when $(x_0 - a) \in \mathbb{N}$ derivative could be taken as follows

(5.11)

$$\frac{df(x)}{dx} = \sum_{k=1}^{\infty} \left\{ \frac{f^{(k)}(a)}{k!} \cdot \lim_{n \to 1^+} \sum_{k=1}^{m} \xi(x-a,\ 1)_{nm-nk} x' \cdot \xi(x-a,\ 1)_{k-1} x' \right\} \Bigg|_{x=x_0}$$

Given x_0, such that $(x_0 - a) \in \mathbb{R}^+$, then conditions of (3.2) is reached, and, applying definition (2.7), derivative f' follows

(5.12)
$$\frac{df(x)}{dx} = \sum_{k=1}^{\infty} \left\{ \frac{f^{(k)}(a)}{k!} \cdot \lim_{n \to 1^+} \sum_{k=1}^{m} \xi(x-a,\ 1)_{nm-nk} x' \cdot \xi(x-a,\ 1)_{k-1} x' \right\} \Bigg|_{x=t_0}$$

Otherwise, let be $f : f \in C^n$, where n - positive integer, then under similar conditions as (5.11) and (5.13), derivative could be reached by differentiating of n-order Taylor's polynomial of f in terms of q-power difference (5.3) under limit notation over n

$$\frac{df(x)}{dx} = \sum_{k=1}^{n} \left\{ \frac{f^{(k)}(a)}{k!} \cdot \lim_{n \to 1^+} \sum_{k=1}^{m} (x-a)^{nm-nk} x' \cdot (x-a)^{k-1} x' \right\} + R'_{n+1}(x) \tag{5.13}$$

Similarly, as (5.13), derivative of $f \in C^n$ in point $x = x_0$, such that $(x_0 - a) \in \mathbb{N}$

(5.14)
$$\frac{df(x)}{dx} = \sum_{k=1}^{n} \left\{ \frac{f^{(k)}(a)}{k!} \cdot \lim_{n \to 1^+} \sum_{k=1}^{m} \xi(x-a,\ 1)_{nm-nk} x' \cdot \xi(x-a,\ 1)_{k-1} x' \right\} \Bigg|_{x=x_0} + R'_{n+1}(x)$$

Otherwise, going from (5.14), $\forall (x_0 - a) \in \mathbb{R}^+$

(5.15)
$$\frac{df(x)}{dx} = \sum_{k=1}^{n} \left\{ \frac{f^{(k)}(a)}{k!} \cdot \lim_{n \to 1^+} \sum_{k=1}^{m} \xi(x-a,\ 1)_{nm-nk} x' \cdot \xi(x-a,\ 1)_{k-1} x' \right\} \Bigg|_{x=t_0} + R'_{n+1}(x)$$

6. Newton's interpolation formula

Being a discrete analog of Taylor's series, the Newton's interpolation formula [6], first published in his Principia Mathematica in 1687, hereby, by author's opinion, supposed to be discussed

$$f(x) = \sum_{k=0}^{\infty} \binom{x-a}{k} \Delta^k f(a) \tag{6.1}$$

Given $q = const$ in (2.3) divided q-difference $f[xq;\ x]$ is reached. Let be $\Delta f = f[xq;\ x](xq - x)$, then, by means of generalized high order forward finite difference $\Delta^k f,\ k \geq 2$, ([7], [8]), revised according to (2.3), Newton's formula (6.1) takes the form

$$f(x) = \sum_{k=0}^{\infty} \left[\binom{x-a}{k} \sum_{m=0}^{k} (-1)^m \binom{m}{k} f(x \cdot t^m) \right] \tag{6.2}$$

Review (5.4) and given $q = const$ divided q-power difference follows, by similar way as (6.2) reached, (6.1) could be written as

$$f(x) = \sum_{k=0}^{\infty} \left[\binom{x-a}{k} \sum_{m=0}^{k} (-1)^m \binom{m}{k} f(x^{n^{k-m}}) \right] \tag{6.3}$$

7. Conclusion

In this paper was discussed a way of obtaining real-valued smooth function's derivative in radius of convergence of it's Taylor's series or polynomial by means of analog of Newton's binomial theorem (1.20) in terms of q-difference (3.1) and (p, q)-power difference operators (5.12). In the last section reviewed a discrete analog of Taylor's series - Newton's interpolation formula (6.1), and applying operators of q-difference, (p, q)-power difference interpolation of initial function is shown (6.2), (6.3).

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