Special Relativity from Lagrangian Formalism Part II?

Francesco R. Ruggeri Hanwell, N.B. Jan. 10, 2021

In part I of this note we argued that setting p=E(v,mo)v where p is momentum and E energy of a free particle and using d/dx (partial) A(x,t) = p and d/dt (partial) A(x,t) = -E one may obtain special relativistic results for a free particle with a rest mass of mo. In particular A(x,t) is set equal to L(v)t with v=x/t yielding p=dL/dv and E= pv-L from which one may create a differential equation using p=Ev. From L, one has p=dL/dv and finally E=p/v and finds that E=mo/sqrt(1-vv) (c=1) and p=Ev. This leads to the well-known EE=pp+momo. If one sets mo=0, one has E=pc, but for mo, E=mo/sqrt(1-vv)=0 so we wish to examine the photon result in more detail. The photon solution is linked to a special solution of A=-Et+px which does not hold for a particle with rest mass, namely dA=0= -Edt + pdx. This yields the classical wave equation for a photon, which also follows from Maxwell's equations.

In addition one may consider rest mass to be created by bouncing photon. In such a case, the bouncing particle need not have a rest mass, yet its localized energy acts as rest mass and this system may be given "momentum" i.e. Ev. Given that the particle, however, has no rest mass, it is all momentum and this must be linked to energy in a manner consistent to EE=pp+momo and also to E=pc. Thus one finds a Doppler shift for the photon.

We examine some of these ideas in this note.

Special Relativistic Results

The two flow equations for a relativistic (or nonrelativistic) particle are:

d/dx (partial) A(x,t) = p ((1a)) and d/dt (partial) A(x,t) = -E ((1b))

These do not explicitly contain rest mass, and yet they hold for both particles with rest mass (relativitically and nonrelativistically) and for a photon which has no rest mass. In particular, if A=-Et+px, then ((1a)) and ((1b)) are satisfied. If one imagines a particle moving so that A remains constant, then E/p=dx/dt. The wave equation for a photon is c= f/lambda ((2)) or c=E/p so the two equations are the same. Thus even though p=Ev does not hold for a photon and this is explicitly used in the derivation of special relativistic results from p=dL/dv and E=pv-L, the resulting equation: EE=ppcc + momocccc ((3)) may still hold if the photon's momentum may display itself solely as energy.

In order to have a photon with momentum p appear as having no momentum i.e. as being energy only one may place it in a box so that it bounces back and forth and lower the resolution of time and space. I.e. one thinks of the center of mass of a box which now contains rest mass equal to the photon energy (even though the photon) itself has no rest mass. The photon has momentum, but bouncing back and forth the average momentum is zero. In such a case, one may apply E=pv to the box system. This scenario also shows that momentum p is linked to force because the box must "do something" to have the photon momentum change sign i.e. bounce back and forth.

One might argue, however, that at any given instant in time (i.e. apply high time resolution), the photon is moving in one direction and only has momentum and no rest mass. This view must be consistent with EE = pp+momo.

Let Eo be the photon energy such that Eo=mo. The photon is in a box bouncing back and forth, but the box is moving with velocity v. Let P be the photon momentum at an instant in this scenario. The E=pc from dA=-Edt+pdx=0. One knows that the photon at a given instant has no rest mass, but has energy which must be equivalent to E. E=Eo/sqrt(1-vv) so the photon energy is Doppler shifted.

One may examine these ideas in a different manner. A photon in a vacuum has a fixed velocity of c. In order to associate it with a velocity v, it must be localized or trapped say in a box, bouncing back and forth. By symmetry, one might expect this box to be exaclty half a wavelength long (or longer). If the box moves with velocity v, from special relativity linked to an object with rest mass, the box should shrink in size to Lo(1-vv) as viewed from the lab. If p=b/L, then p seen in the lab should be p/[sqrt(1-vv)Lo], but then p behaves exactly as E so using E=p seems to be reasonable.

An alternative approach is to use the classical wave equation c=f lambda, but then one must link 6.28f with energy and 6.28/lambda with momentum. These results, however, follow from the derivation of the photon from Maxwell's equation. Thus, a separate set of equations (i.e. Maxwell's classical EM equations) give an independent link between p and E. The EE=pp+momo approach only yields the behaviour of E as v changes (i.e. from different frames). (The wavelength approach, however, seems to link E and p as being proportional.) Furthermore, Maxwell's EM equations are related to special relativity so one could find Lorentz boost properties of a photon's (p,E) four vector by using this classical theory only.

Using ((1a)) and ((1b)), however, seems to be an approach which may apply both to photons and particles with rest mass regardless of whether it has charge. Furthermore ((1a)) and ((1b)) may be converted into eigenfunction equations and yield both free particle relativistic and nonrelativistic quantum mechanics i.e.

-id/dx (partial) exp(iA) = p exp(iA) and id/dt (partial) exp(iA) = E exp(iA) ((4))

Using these together with EE=pp+momo yields the Klein-Gordon equation which in such a case is not usually applied to a bouncing photon in a box system. Interestingly, however, one may write EE=pp+momo as a 2x2 matrix equation with solutions consisting of basis vectors (1,0) and (0,1) which in (1) were interpreted as representing forward and backward motion i.e. a bouncing photon. See (1) for details of this model.

Conclusion

In conclusion we note that one may use two flux equations d/dx (partial) A(x,t) = p and d/dt (partial) A(x,t) = -E to obtain special relativity results for a free particle with rest mass mo namely E=mo/sqrt(1-vv) (c=1) and p=Ev. These are consistent with EE=pp+momo. We argue that A(x,t)=-Et+px also contains another solution, namely E=pc not consistent with a particle with rest mass mo. This solution follows from insisting that A remain constant as one changes x by dx and t by dt. E=pc is the classical wave equation which also applies to a photon (from Maxwell's em equations). One may use EE=pp+momo for a photon if one considers it to be trapped e.g. bouncing back and forth in a box. In such a case it represents rest mass mo and the equation holds. E represents the energy of the photon which moves in the box as seen from the lab. Thus the photon undergoes a Doppler shift. This together with the solution E=Pc yields a result for P. One may see that at any instant the photon bouncing in the box is really moving in one direction with momentum P. Thus pp+momo is simply a mathematical way of treating the trapped photon, but E=Pc is the photon equation and it is consistent with Lorentz transformations just as the particle with mass mo. These results for a photon could be directly obtained from Maxwell's classical em equations. The flux approach, however, links particles with rest mass with those with none and also leads to quantum mechanical equations for free particles with rest mass.

We also note that EE=pp+momo may be written as 2x2 matrix equation (linked to Pauli matrices) which may be interpreted as representing a photon moving back and forth as described in (1).

References

(1) Ruggeri, Francesco R. A 2x2 Matrix Model is Contained in Relativstic Energy Momentum Equation (preprint, zenodo, 2018)