This Maple worksheet documents the calculations described in Section 2.1.2 of the following paper.

Russell Bradford, James H. Davenport, Matthew England, Hassan Errami, Vladimir Gerdt, Dima Grigoriev, Charles Hoyt, Marek Košta, Ovidiu Rad- ulescu, Thomas Sturm, and Andreas Weber.
A Case Study on the Parametric Occurrence of Multiple Steady States. In Proceedings of ISSAC ’17, Kaiserslautern, Germany, July 25-28, 2017, 8 pages. ACM, 2017.
https://doi.org/http://dx.doi.org/10.1145/3087604.3087622

Note - we are running in Maple 2016 with the latest version of the RegularChains linrary downloaded from www.regularchains.org

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```
> restart;
> libname := "C:/Users/ab9797/Dropbox/Maple_CAD/FromRCorg/Latest",
  libname;
```

libname := "C:/Users/ab9797/Dropbox/Maple_CAD/FromRCorg/Latest",
"C:\Program Files\Maple 2016\lib"

```
> with(RegularChains):
> with(SemiAlgebraicSetTools);
```

[*BoxValues, Complement, CylindricalAlgebraicDecompose, Difference, DisplayParametricBox, DisplayQuantifierFreeFormula, EmptySemiAlgebraicSet, Intersection, IsContained, IsEmpty, IsParametricBox, LinearSolve, PartialCylindricalAlgebraicDecomposition, PositiveInequalities, Projection, QuantifierElimination, RealRootCounting, RealRootIsolate, RefineBox, RefineListBox, RemoveRedundantComponents, RepresentingBox, RepresentingChain, RepresentingQuantifierFreeFormula, RepresentingRootIndex, SignAtBox, VariableOrdering*]

The System

System of interest. Taken from Biomodals database as descibed at the start of Section 2 of the paper

```
> TS := [
  k2*x6 - k1*x1*x4 - k16*x1*x5 + k15*x11 = 0,
  k3*x6 + k5*x7 + k10*x9 + k13*x10 - k11*x2*x5 - k12*x2*x5 - k4*
  x2*x4 = 0,
  k6*x7+k8*x8 - k7*x3*x5 = 0,
  -k1*x1*x4-k4*x2*x4+k2*x6+k3*x6+k5*x7+k6*x7 = 0,
  -k11*x2*x5-k12*x2*x5-k16*x1*x5-k7*x3*x5+k10*x9+k13*x10+k15*x11+
  k8*x8 = 0,
  k1*x1*x4-k2*x6-k3*x6 = 0,
  k4*x2*x4-k5*x7-k6*x7 = 0,
  k7*x3*x5-k8*x8-k9*x8 = 0,
  k11*x2*x5-k10*x9+k9*x8 = 0,
  k12*x2*x5-k13*x10-k14*x10 = 0,
  k16*x1*x5+k14*x10-k15*x11 = 0,
  -k17+x10+x11+x5+x8+x9 = 0,
  -k18+x4+x6+x7 = 0,
  -k19+x1+x10+x11+x2+x3+x6+x7+x8+x9 = 0
  ];
```

$$TS := [-k1\,x1\,x4 - k16\,x1\,x5 + k15\,x11 + k2\,x6 = 0,\ -k11\,x2\,x5 - k12\,x2\,x5 - k4\,x2\,x4 + k10\,x9 + k13\,x10 + k3\,x6 + k5\,x7 = 0,\ -k7\,x3\,x5 + k6\,x7 + k8\,x8 = 0,\ -k1\,x1\,x4 - k4\,x2\,x4 + k2\,x6 + k3\,x6 + k5\,x7 + k6\,x7 = 0,\ -k11\,x2\,x5 - k12\,x2\,x5 - k16\,x1\,x5 - k7\,x3\,x5 + k10\,x9 + k13\,x10 + k15\,x11 + k8\,x8 = 0,\ k1\,x1\,x4 - k2\,x6 - k3\,x6 = 0,\ k4\,x2\,x4 - k5\,x7 - k6\,x7 = 0,\ k7\,x3\,x5 - k8\,x8 - k9\,x8 = 0,\ k11\,x2\,x5 - k10\,x9 + k9\,x8 = 0,\ k12\,x2\,x5 - k13\,x10 - k14\,x10 = 0,\ k16\,x1\,x5 + k14\,x10 - k15\,x11 = 0,\ -k17 + x10 + x11 + x5 + x8 + x9 = 0,\ -k18 + x4 + x6 + x7 = 0,\ -k19 + x1 + x10 + x11 + x2 + x3 + x6 + x7 + x8 + x9 = 0]$$

9 of the parameters have been measured accuretley.

```
> s1 := {k1 = 0.2e-1, k11 = 0.1e-1, k12 = 0.1e-1, k15 = 0.86e-1,
  k16 = 0.11e-2, k3 = 0.1e-1, k4 = 0.32e-1, k7 = 0.45e-1, k9 =
  0.92e-1};
  nops(%);
```

$$s1 := \{k1 = 0.02, k11 = 0.01, k12 = 0.01, k15 = 0.086, k16 = 0.0011, k3 = 0.01, k4 = 0.032, k7 = 0.045, k9 = 0.092\}$$

$$9$$

Another 7 have been estimated with confidence

```
> s2 := {k10 = 1, k13 = 1, k14 = .5, k2 = 1, k5 = 1, k6 = 15, k8
  = 1};
  nops(%);
```

$$s2 := \{k10 = 1, k13 = 1, k14 = 0.5, k2 = 1, k5 = 1, k6 = 15, k8 = 1\}$$

$$7$$

The remaining 3 are estimated as below, but with less confidence.

```
> s3 := {k17 = 100, k18 = 50, k19 = 200};
  nops(%);
```

$$s3 := \{k17 = 100, k18 = 50, k19 = 200\}$$

$$3$$

We will proceed with one free parameter, k19. Hence we substitute the others in and convert to rational numbers

```
> TSX := subs(s1,s2, [k17=100, k18=50], TS):
  convert(TSX, rational):
  TSY := map(X->X*10000, %);
```

$TSY := [-200\, x1\, x4 - 11\, x1\, x5 + 860\, x11 + 10000\, x6 = 0,\ -320\, x2\, x4 - 200\, x2\, x5 + 10000\, x10 + 100\, x6 + 10000\, x7 + 10000\, x9 = 0,\ -450\, x3\, x5 + 150000\, x7 + 10000\, x8 = 0,\ -200\, x1\, x4 - 320\, x2\, x4 + 10100\, x6 + 160000\, x7 = 0,\ -11\, x1\, x5 - 200\, x2\, x5 - 450\, x3\, x5 + 10000\, x10 + 860\, x11 + 10000\, x8 + 10000\, x9 = 0,\ 200\, x1\, x4 - 10100\, x6 = 0,\ 320\, x2\, x4 - 160000\, x7 = 0,\ 450\, x3\, x5 - 10920\, x8 = 0,\ 100\, x2\, x5 + 920\, x8 - 10000\, x9 = 0,\ 100\, x2\, x5 - 15000\, x10 = 0,\ 11\, x1\, x5 + 5000\, x10 - 860\, x11 = 0,\ -1000000 + 10000\, x10 + 10000\, x11 + 10000\, x5 + 10000\, x8 + 10000\, x9 = 0,\ -500000 + 10000\, x4 + 10000\, x6 + 10000\, x7 = 0,\ -10000\, k19 + 10000\, x1 + 10000\, x10 + 10000\, x11 + 10000\, x2 + 10000\, x3 + 10000\, x6 + 10000\, x7 + 10000\, x8 + 10000\, x9 = 0]$

We need to place an ordering on the variables.

```
> indets(TSY);
```

$\{k19, x1, x10, x11, x2, x3, x4, x5, x6, x7, x8, x9\}$

We must have the parameter at the end (so we see how differing this affects the others). We simply place the others in the label ordering

```
> vars := [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2, x1, k19];
  R:=PolynomialRing(vars):
```

$vars := [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2, x1, k19]$

We now add the positivity conditions for variables and remaining parameter:

```
> TSZ := TSY:
  for i from 1 to nops(vars) do:
    var := vars[i]:
    TSZ := [op(TSZ), var>0]:
  od: TSZ;
```

$[-200\, x1\, x4 - 11\, x1\, x5 + 860\, x11 + 10000\, x6 = 0,\ -320\, x2\, x4 - 200\, x2\, x5 + 10000\, x10 + 100\, x6 + 10000\, x7 + 10000\, x9 = 0,\ -450\, x3\, x5 + 150000\, x7 + 10000\, x8 = 0,\ -200\, x1\, x4 - 320\, x2\, x4 + 10100\, x6 + 160000\, x7 = 0,\ -11\, x1\, x5 - 200\, x2\, x5 - 450\, x3\, x5 + 10000\, x10 + 860\, x11 + 10000\, x8 + 10000\, x9 = 0,\ 200\, x1\, x4 - 10100\, x6 = 0,\ 320\, x2\, x4 - 160000\, x7 = 0,\ 450\, x3\, x5 - 10920\, x8 = 0,\ 100\, x2\, x5 + 920\, x8 - 10000\, x9 = 0,\ 100\, x2\, x5 - 15000\, x10 = 0,\ 11\, x1\, x5 + 5000\, x10 - 860\, x11 = 0,\ -1000000 + 10000\, x10 + 10000\, x11 + 10000\, x5 + 10000\, x8 + 10000\, x9 = 0,\ -500000 + 10000\, x4 + 10000\, x6 + 10000\, x7 = 0,\ -10000\, k19 + 10000\, x1 + 10000\, x10 + 10000\, x11 + 10000\, x2 + 10000\, x3 + 10000\, x6 + 10000\, x7 + 10000\, x8 + 10000\, x9 = 0,\ 0 < x11,\ 0 < x10,\ 0 < x9,\ 0 < x8,\ 0 < x7,\ 0 < x6,\ 0 < x5,\ 0 < x4,\ 0 < x3,\ 0 < x2,\ 0 < x1,\ 0 < k19]$

So TSZ is the system to solve.

We start by triangularizing the system. We use the lazy variant, which produces the highest dim component and unevaluated function calls for the others.

```
> Lrt := LazyRealTriangularize( TSZ, R, output=record):
> nops([Lrt]);
```

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One solutions component and 6 unevaluated calls. First we look at the unevaluated calls.

The unevaluated calls

The second solution component may be dismissed easily as the input has both k19=0 and k19>0 in it. These are in compatible so the compoentn evaluates to NULL

```
> Lrt[2];
```

$$\%LazyRealTriangularize\Big(\big[k19 = 0,\ -500000 + 10000\, x4 + 10000\, x6 + 10000\, x7 = 0,\ -1000000$$
$$+ 10000\, x10 + 10000\, x11 + 10000\, x5 + 10000\, x8 + 10000\, x9 = 0,\ x5 + x4 - x3 - x2 - x1$$
$$+ k19 - 150 = 0,\ -10000\, k19 + 10000\, x1 + 10000\, x10 + 10000\, x11 + 10000\, x2 + 10000\, x3$$
$$+ 10000\, x6 + 10000\, x7 + 10000\, x8 + 10000\, x9 = 0,\ -2\, x1\, x4 + 101\, x6 = 0,\ 200\, x1\, x4$$
$$- 10100\, x6 = 0,\ 100\, x2\, x5 - 15000\, x10 = 0,\ 320\, x2\, x4 - 160000\, x7 = 0,\ 450\, x3\, x5 - 10920\, x8$$
$$= 0,\ 11\, x1\, x5 + 5000\, x10 - 860\, x11 = 0,\ 100\, x2\, x5 + 920\, x8 - 10000\, x9 = 0,\ -450\, x3\, x5$$
$$+ 150000\, x7 + 10000\, x8 = 0,\ 101\, x4\, x2 + 1000\, x4\, x1 + 50500\, x4 - 2525000 = 0,\ 2\, x4\, x1$$
$$+ 101\, x7 + 101\, x4 - 5050 = 0,\ -200\, x1\, x4 - 11\, x1\, x5 + 860\, x11 + 10000\, x6 = 0,\ -200\, x1\, x4$$
$$- 320\, x2\, x4 + 10100\, x6 + 160000\, x7 = 0,\ -320\, x2\, x4 - 200\, x2\, x5 + 10000\, x10 + 100\, x6$$
$$+ 10000\, x7 + 10000\, x9 = 0,\ x4\, x2 - x3\, x2 - x2^2 - x2\, x1 + x2\, k19 + 150\, x10 - 150\, x2 = 0,$$
$$15\, x4\, x3 - 15\, x3^2 - 15\, x3\, x2 - 15\, x3\, x1 + 15\, x3\, k19 + 364\, x8 - 2250\, x3 = 0,\ -11\, x1\, x5$$
$$- 200\, x2\, x5 - 450\, x3\, x5 + 10000\, x10 + 860\, x11 + 10000\, x8 + 10000\, x9 = 0,\ 69\, x4\, x3$$
$$+ 182\, x4\, x2 - 69\, x3^2 - 251\, x3\, x2 - 69\, x3\, x1 + 69\, x3\, k19 - 182\, x2^2 - 182\, x2\, x1 + 182\, x2\, k19$$
$$+ 18200\, x9 - 10350\, x3 - 27300\, x2 = 0,\ -1000\, k19\, x1\, x2 - 101\, k19\, x2^2 + 1000\, x1^2\, x2$$
$$+ 1101\, x1\, x2^2 + 1000\, x1\, x2\, x3 + 101\, x2^3 + 101\, x2^2\, x3 - 50500\, k19\, x2 + 200500\, x1\, x2$$
$$+ 65650\, x2^2 + 50500\, x2\, x3 - 150000\, x1 + 5050000\, x2 = 0,$$
$$16838105723097694257603469\, x1^6 - 24078605201553273505077988\, x1^5\, k19$$
$$+ 8176202638735769127032169\, x1^4\, k19^2 + 772396796964497789614868658 0\, x1^5$$
$$- 7723411665463544477701499460\, x1^4\, k19 + 1465408757440589841803452380\, x1^3\, k19^2$$
$$+ 123215435794133887615660681290 0\, x1^4 - 798169557586805582842481309800\, x1^3\, k19$$
$$+ 85462524901276846107251669400\, x1^2\, k19^2 + 8315265524000276772955047764 0000\, x1^3$$
$$- 352664114014276568345720951400 00\, x1^2\, k19$$
$$+ 163168564971970267228250550 0000\, x1\, k19^2$$
$$+ 255680535485331833219748963 6000000\, x1^2$$
$$- 7219895711004618624773423200 00000\, x1\, k19$$
$$+ 288437559383187808232184000000000 00\, x1$$
$$- 70131041394599108765205000000000 0000\, k19 = 0,\ -10\, k19\, x2 - 27\, k19\, x3 + 10\, x1\, x2$$
$$+ 27\, x1\, x3 + 10\, x2^2 + 37\, x2\, x3 - 10\, x2\, x4 + 27\, x3^2 - 27\, x3\, x4 - 600\, k19 + 600\, x1 + 600\, x11$$
$$+ 2100\, x2 + 4650\, x3 - 600\, x4 + 30000 = 0,\ 2327636637521132379740294044 20089\, x2\, x1^5$$
$$- 3328536153010418455776716399 90228\, x2\, x1^4\, k19$$
$$+ 1130247613994501869493906230 74789\, x2\, x1^3\, k19^2$$
$$+ 886463032152050753763081470296 77220\, x2\, x1^4$$

$- 8084390802833149813995452776176274 0\, x2\, x1^{3}\, k19$

$+ 114552323096490343055970487914790 20\, x2\, x1^{2}\, k19^{2}$

$- 30625833064790009548991419920\, x1^{5} + 43795148662369306906962603840\, x1^{4}\, k19$

$- 14871210647782462053693235920\, x1^{3}\, k19^{2}$

$+ 11682465068391769796632986929072776500\, x2\, x1^{3}$

$- 5547251026060433566640620528023877000\, x2\, x1^{2}\, k19$

$+ 290245997063001550130198026458525000\, x2\, x1\, k19^{2}$

$- 3774997922548773180527368650466320 0\, x1^{4}$

$+ 16963336293692750919154910690672400\, x1^{3}\, k19$

$- 15383254482229832299305300492 00\, x1^{2}\, k19^{2}$

$+ 619147207587597001268026254404647600000\, x2\, x1^{2}$

$- 141348286758352762323489548674398500000\, x2\, x1\, k19$

$- 6815925407229297763234036009365120000\, x1^{3}$

$+ 862702164104208291031357996000020000\, x1^{2}\, k19$

$- 2937034169495464810108509900000 0\, x1\, k19^{2}$

$+ 1454728852958138225258707154149460000000 0\, x2\, x1$

$- 1247498501818579946626756931775000000000\, x2\, k19$

$- 279241219028720368578809336249748000000\, x1^{2}$

$+ 12995812279808313524592161760000000\, x1\, k19$

$+ 124749850181857994662675693177500000000000\, x2$

$- 3705960282117523242886769213700000000000\, x1$

$+ 12623587451027839577736900000000000 0\, k19 = 0, 0 < k19, 0 < x1, 0 < x10, 0 < x11, 0 < x2, 0 < x3, 0 < x4, 0 < x5, 0 < x6, 0 < x7, 0 < x8, 0 < x9],$ *polynomial_ring*, *output* = *record*)

```
> sol2 := value(Lrt[2]);
```

$sol2 :=$

The same is true for component 3 and 4

```
> Lrt[3]:
  sol3 := value(Lrt[3]);
```

$sol3 :=$

```
> Lrt[4]:
  sol4 := value(Lrt[4]);
```

$sol4 :=$

However, the other 3 are not trivial and the latter two cannot be evaluated in reasonable time. Let us consider their constraints univariate in k19:

```
> select(X->indets(X)={k19}, [op(Lrt[5])][1]);
  evalf(realroot(lhs(%[1])));
```

$$[2319798943341957999492 9\, k19^2 - 894074006154524094530988 00\, k19 - 4822419303419166525491149190000 = 0,\ 0 < k19]$$

$$[[-12619.24219, -12619.23438], [16473.33594, 16473.34375]]$$

So defining two real values of k19, one positive one negative

```
> select(X->indets(X)={k19}, [op(Lrt[6])][1]);
  evalf(realroot(lhs(%[1])));
```

$$[505465566622475867655547880786544637953790406059982726185509\, k19^4 - 1272578045696439189317856051518387368422217896986836692050513412 0\, k19^3 + 117551033091520524183124321323141751700303731556288419365745144 5400\, k19^2 - 2818673598836761598111920829785411936002928043245969118783379725 60000\, k19 - 4243436357021558746566842370156393218505106689274120793187930720 0000000 = 0,\ 0 < k19]$$

$$[[-87.78125000, -87.77343750], [25084.53125, 25084.53906]]$$

Also defining two real values of k19, one positive one negative

```
> select(X->indets(X)={k19}, [op(Lrt[7])][1]);
  evalf(realroot(lhs(%[1])));
```

$$[351590934502740290936895033267017158736060313940693076650155371250411\, k19^{10} - 21369907285215767428399752774639558327303398317042608057480078198909315 6\, k19^9 + 253748516412205547742596056350534694325821098839650158040771191109580340 90\, k19^8 + 1297249301830002270702763926780425925123599161802985288033000450856439159 4\backslash 000\, k19^7 - 8468945963692802414226427249726123493448372439778349029355636316929687020 6\backslash 60000\, k19^6 + 2231098270337406450670301663172664333421440833875848621423683265663846533 0\backslash 79600000\, k19^5 - 3762650089041122582903191731937920520148994855289949259658858955118318734 4\backslash 4245100000\, k19^4 + 3926210154879086940705799498532015650096895836139617890818002684280664376 6\backslash 783104000000\, k19^3 - 2492623990743029234974354081270296106309603462451517057779877596842448287 7\backslash 99337600000000\, k19^2 + 7097885073588747345917699718617597842587326724676002321294061692464317186 8\backslash$$

$478080000000000\, k19$
$- 10628711928389858769480771149238982049904341389014953948347496131846703628\backslash$
$10368000000000000 = 0,\ 0 < k19]$

$$[[-380., -379.9921875], [409.2500000, 409.2578125]]$$

Also defining two real values of k19, one positive one negative

Hence, the evaluated solution component is valid for all k19>0 except for 3 isolated points, which are (to 1dp): 409.3, 16473.3 and 25084.5.

We now proceed to study the evaluated solution component, knowing from above that it is valid for all k19>0 except 3 isolated points.

Investigating the component

(Calculated this way to facilitate printing to pdf)

```
> sol := LazyRealTriangularize( TSZ, R, output=list):
  Display(%,R);
```

$[\{[600\, x11 + (-10\, x2 - 27\, x3 - 600)\, x4 + 27\, x3^2 + (-27\, k19 + 27\, x1 + 37\, x2 + 4650)\, x3$
$+ 10\, x2^2 + (-10\, k19 + 10\, x1 + 2100)\, x2 - 600\, k19 + 600\, x1 + 30000 = 0,],$
$[150\, x10 + x4\, x2 - x3\, x2 - x2^2 + (k19 - x1 - 150)\, x2 = 0,],$
$[18200\, x9 + (69\, x3 + 182\, x2)\, x4 - 69\, x3^2 + (69\, k19 - 69\, x1 - 251\, x2 - 10350)\, x3$
$- 182\, x2^2 + (182\, k19 - 182\, x1 - 27300)\, x2 = 0,],$
$[364\, x8 + 15\, x4\, x3 - 15\, x3^2 + (15\, k19 - 15\, x1 - 15\, x2 - 2250)\, x3 = 0,],$
$[101\, x7 - 5050 + (2\, x1 + 101)\, x4 = 0,],$
$[101\, x6 - 2\, x4\, x1 = 0,],$
$[x5 + x4 - x3 - x2 - x1 + k19 - 150 = 0,],$
$[(101\, x2 + 1000\, x1 + 50500)\, x4 - 2525000 = 0,],$
$[(101\, x2^2 + (1000\, x1 + 50500)\, x2)\, x3 + 101\, x2^3 + (-101\, k19 + 1101\, x1 + 65650)\, x2^2$
$+ (1000\, x1^2 + (-1000\, k19 + 200500)\, x1 - 50500\, k19 + 5050000)\, x2 - 150000\, x1 = 0,],$
$[(232763663752113237974029404420089\, x1^5 + ($
$-332853615301041845577671639990228\, k19 + 88646303215205075376308147029677220)$
$x1^4 + (113024761399450186949390623074789\, k19^2$
$- 80843908028331498139954527761762740\, k19$
$+ 11682465068391769796632986929072776500)\, x1^3$
$+ (11455232309649034305597048791479020\, k19^2$
$- 5547251026060433566640620528023877000\, k19$
$+ 619147207587597001268026254404647600000)\, x1^2$
$+ (290245997063001550130198026458525000\, k19^2$

$- 14134828675835276232348954867439850000\,k19$
$+ 1454728852958138225258707154149460000000)\,x1$
$- 1247498501818579946626756931775000000000\,k19$
$+ 1247498501818579946626756931775000000000000)\,x2$
$- 30625833064790009548991419920\,x1^{5} + (4379514866236930690696260384\,0\,k19$
$- 3774997922548773180527368650466320\,0)\,x1^{4} + ($
$-148712106477824620536932359 20\,k19^{2} + 169633362936927509191549106906724 00\,k19$
$- 681592540722929776323403600936512 0000)\,x1^{3} + ($
$-15383254482229832299305300492 00\,k19^{2}$
$+ 8627021641042082910313579960000200 00\,k19$
$- 279241219028720368578809336249748000000)\,x1^{2} + ($
$-29370341694954648101085099000000\,k19^{2}$
$+ 129958122798083135245921617600000 00\,k19$
$- 370596028211752324288676921370000000000 0)\,x1$
$+ 1262358745102783957773690000000000000\,k19 = 0,]$,
$[16838105723097694257603469\,x1^{6} + (-24078605201553273505077988\,k19$
$+ 77239679696449778961486865 80)\,x1^{5} + (8176202638735769127032169\,k19^{2}$
$- 77234116654635444777014994 60\,k19 + 12321543579413388761566068129 00)\,x1^{4}$
$+ (146540875744058984180345238 0\,k19^{2} - 798169557586805582842481309800\,k19$
$+ 831526552400027677295504776400 00)\,x1^{3} + (854625249012768461072516694 00\,k19^{2}$
$- 3526641140142765683457209514 0000\,k19 + 255680535485331833219748963600000 0)$
$x1^{2} + (1631685649719702672282505500000\,k19^{2}$
$- 7219895711004618624773423200000 00\,k19$
$+ 28843755938318780823218400000000000)\,x1$
$- 7013104139459910876520500000000000\,k19 = 0,]$,
$[x1 > 0,]$,
$[k19 > 0$ **and** $23197989433419579994929\,k19^{2} - 894074006154524094530988 00\,k19$
$- 48224193034191665254911491900 00 \neq 0$ **and**
$505465566622475867655547880786544637953790406059982726185509\,k19^{4}$
$- 1272578045696439189317856051518387368422217896986836692050513412 0\,k19^{3}$
$+ 117551033091520524183124321323141751700303731556288419365745144540 0\,k19^{2}$
$- 28186735988367615981119208297854119360029280432459691187833797256 0000\,k19$
$- 4243436357021558746566842370156393218505106689274120793187930720000000 0$
$\neq 0$ **and**
$3515909345027402909368950332670171587360603139406930766501553712504 11\,k19^{10}$

− 2136990728521576742839975277463955832730339831704260805748007819890931 56
$k19^{9}$
+ 25374851641220554774259605635053469432582109883965015804077119110958034090
$k19^{8}$
+ 12972493018300022707027639267804259251235991618029852880330004508564391594\
000 $k19^{7}$
− 84689459636928024142264272497261234934483724397783490293556363169296870206\
60000 $k19^{6}$
+ 22310982703374064506703016631726643334214408338758486214236832656638465330\
79600000 $k19^{5}$
− 37626500890411225829031917319379205201489948552899492596588589551183187344\
4245100000 $k19^{4}$
+ 39262101548790869407057994985320156500968958361396178908180026842806643766\
783104000000 $k19^{3}$
− 24926239907430292349743540812702961063096034624515170577798775968424482877\
99337600000000 $k19^{2}$
+ 70978850735887473459176997186175978425873267246760023212940616924643171868\
478080000000000 $k19$
− 10628711928389858769480771149238982049904341389014953948347496131846703628\
10368000000000000 $\neq 0$,]]

This is a triangular decomposition, meaning that as we read from the bottom up extra variables are added.
Starting from the bottom we see there are four constraints univariate in k19. That is be greater than zero and not the root of three polynomials. We recognise these as the polynomials from the unevaluated function calls.
I.e. the part univariate in k19 is stating that k19 be positive and away from those 3 isolated points.

Next let us consider the part that is bivariate in x1 and k19.
We have that x1>0 and that the following polynomial is 0.

```
> x1pol := 168381057230976942576 03469*x1^6+
  (-24078605201553273505077988*k19+7723967969644977896148686580)*
  x1^5+(817620263873576912703 2169*k19^2
  -7723411665463544477701499460*
  k19+12321543579413388761566 06812900)*x1^4+
  (14654087574405898418034523 80*k19^2
  -7981695575868055828424813 09800*
  k19+83152655240002767729550477640000)*x1^3+
  (854625249012768461072516 69400*k19^2
  -3526641140142765683457209 5140000*
  k19+25568053548533183321974896 36000000)*x1^2+
  (1631685649719702672282505 500000*k19^2
  -72198957110046186247734232 0000000*
  k19+28843755938318780823218400000000000)*x1
  -70131041394599108765205000000000000*k19
```

$$
\begin{aligned}
x1pol := {} & 168381057230976942576034 69\, x1^6 + (-24078605201553273505077988\, k19 \\
& + 7723967969644977896148686580)\, x1^5 + (8176202638735769127032169\, k19^2 \\
& - 7723411665463544477701499460\, k19 + 1232154357941338876156606812900)\, x1^4 \\
& + (1465408757440589841803452380\, k19^2 - 798169557586805582842481309800\, k19 \\
& + 83152655240002767729550477640000)\, x1^3 + (854625249012768461072516694 00\, k19^2 \\
& - 3526641140142765683457209514 0000\, k19 + 25568053548533183321974896360000 00) \\
& x1^2 + (16316856497197026722825055 00000\, k19^2 \\
& - 7219895711004618624773423200000 00\, k19 \\
& + 28843755938318780823218400000000000)\, x1 \\
& - 70131041394599108765205000000000 00\, k19
\end{aligned}
$$

This polynomial is (7) in the paper. It is degree 6 in x1.

The remaining constraints are all equations linear in their main variable meaning solutions can be easily found for other variables once solutions for x1 and k19 are known.
We print the solution formulae below (given as Table 3 in the paper)

```
> sol := LazyRealTriangularize( TSZ, R, output=list):
  rcPols := map(X->X[4], sol[1][RegularChain][polynomials]):
  ind = 11:
  TriSol := []:
  for var in [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2] do
      tmp := op(select(X->has(X,var), rcPols)):
      rcPols := remove(X->has(X,var), rcPols):
      print(isolate(tmp, var));
      TriSol := [op(TriSol), isolate(tmp, var)]
  od:
```

$$
\begin{aligned}
x11 = {} & \frac{9}{200}\, x4\, x3 + \frac{1}{60}\, x4\, x2 - \frac{9}{200}\, x3^2 - \frac{37}{600}\, x3\, x2 - \frac{9}{200}\, x3\, x1 + \frac{9}{200}\, x3\, k19 - \frac{1}{60}\, x2^2 \\
& - \frac{1}{60}\, x2\, x1 + \frac{1}{60}\, x2\, k19 + x4 - \frac{31}{4}\, x3 - \frac{7}{2}\, x2 - x1 + k19 - 50
\end{aligned}
$$

$$
x10 = -\frac{1}{150}\, x2\, k19 + \frac{1}{150}\, x2\, x1 + \frac{1}{150}\, x2^2 + \frac{1}{150}\, x3\, x2 - \frac{1}{150}\, x4\, x2 + x2
$$

$$
\begin{aligned}
x9 = {} & -\frac{1}{100}\, x2\, k19 - \frac{69}{18200}\, x3\, k19 + \frac{1}{100}\, x2\, x1 + \frac{69}{18200}\, x3\, x1 + \frac{1}{100}\, x2^2 + \frac{251}{18200}\, x3\, x2 \\
& - \frac{1}{100}\, x4\, x2 + \frac{69}{18200}\, x3^2 - \frac{69}{18200}\, x4\, x3 + \frac{3}{2}\, x2 + \frac{207}{364}\, x3
\end{aligned}
$$

$$
x8 = -\frac{15}{364}\, x3\, k19 + \frac{15}{364}\, x3\, x1 + \frac{15}{364}\, x3\, x2 + \frac{15}{364}\, x3^2 - \frac{15}{364}\, x4\, x3 + \frac{1125}{182}\, x3
$$

$$
x7 = -\frac{2}{101}\, x4\, x1 - x4 + 50
$$

$$
x6 = \frac{2}{101}\, x4\, x1
$$

$$
x5 = -x4 + x3 + x2 + x1 - k19 + 150
$$

$$x4 = \frac{2525000}{101\,x2 + 1000\,x1 + 50500}$$

$$\begin{aligned}
x3 = {} & \frac{1}{1000\,x1\,x2 + 101\,x2^2 + 50500\,x2} \big(-101\,x2^3 - 1101\,x2^2\,x1 + 101\,x2^2\,k19 - 1000\,x2\,x1^2 \\
& + 1000\,x2\,x1\,k19 - 65650\,x2^2 - 200500\,x2\,x1 + 50500\,x2\,k19 - 5050000\,x2 + 150000\,x1 \big)
\end{aligned}$$

$$\begin{aligned}
x2 = {} & \big(14871210647782462053693235920\,k19^2\,x1^3 - 43795148662369306906962603840\,k19\,x1^4 \\
& + 30625833064790009548991419920\,x1^5 + 1538325448222983229930530049200\,k19^2\,x1^2 \\
& - 1696333629369275091915491069067240\,k19\,x1^3 \\
& + 3774997922548773180527368650466320\,x1^4 \\
& + 29370341694954648101085099000000\,k19^2\,x1 \\
& - 862702164104208291031357996000020000\,k19\,x1^2 \\
& + 6815925407229297763234036009365120000\,x1^3 \\
& - 1299581227980831352459216176000000\,k19\,x1 \\
& + 27924121902872036857880933624974800000\,x1^2 \\
& - 126235874510278395777369000000000000\,k19 \\
& + 370596028211752324288676921370000000000\,x1 \big) \Big/ \\
& \big(1130247613994501869493906230747 89\,k19^2\,x1^3 \\
& - 332853615301041845577671639990228\,k19\,x1^4 \\
& + 232763663752113237974029404420089\,x1^5 \\
& + 11455232309649034305597048791479020\,k19^2\,x1^2 \\
& - 808439080283314981399545277617627 40\,k19\,x1^3 \\
& + 886463032152050753763081470296772 20\,x1^4 \\
& + 290245997063001550130198026458525000\,k19^2\,x1 \\
& - 5547251026060433566640620528023877000\,k19\,x1^2 \\
& + 11682465068391769796632986929072776500\,x1^3 \\
& - 1413482867583527623234895486743985000000\,k19\,x1 \\
& + 61914720758759700126802625440464760000\,x1^2 \\
& - 1247498501818579946626756931775000000000\,k19 \\
& + 14547288529581382252587071541494600000000\,x1 \\
& + 12474985018185799466267569317750000000000 \big)
\end{aligned}$$

So we have a triangular solution valid for k19>0 excluding 3 points.
For the variables x2...x11 solutions may be calculated easily from solutions for k19 and x1
To demonstrate this we recreate the solutions for k19 =200 and 500 given in Table 1 of the paper

Table 1: Solutions at k19 = 200

At this k19 value only one positive solution to the polynomial in x1,k19

```
> sol := [k19=200]:

  subs(sol, x1pol): realroot(%, 1/100000000): evalf(%);
  S:=select(X->X[1]>0, %): S := op(S):
  x1 = S[1] + (S[2]-S[1])/2;
  sol := [op(sol), %]:

  for entry in ListTools:-Reverse(TriSol) do
    print(subs(sol, entry));
    sol := [op(sol), subs(sol,entry)]:
  od:
```

$[[-110.4712848, -110.4712848], [-95.84066958, -95.84066958], [-46.23133729, -46.23133729], [90.65124025, 90.65124025]]$

$$x1 = 90.65124024$$
$$x2 = 2.673106650$$
$$x3 = 10.49959369$$
$$x4 = 17.85446292$$
$$x5 = 35.9694777$$
$$x6 = 32.05008333$$
$$x7 = 0.09545375$$
$$x8 = 15.56311404$$
$$x9 = 2.393308991$$
$$x10 = 0.641001666$$
$$x11 = 45.4330976$$

Table 1: Solutions at k19 = 500

At this value of k19 there are 3 positive solutions to the polynomial in x1,k19

```
> subs(k19=500, x1pol): realroot(%, 1/1000000): evalf(%);
  SS:=select(X->X[1]>0, %);
```

$[[-90.84580302, -90.84580290], [-67.76791108, -67.76791096], [-48.53699827, -48.53699815], [17.63922131, 17.63922143], [122.0339677, 122.0339679], [323.7613771, 323.7613772]]$

$SS := [[17.63922131, 17.63922143], [122.0339677, 122.0339679], [323.7613771, 323.7613772]]$

```
> sol := [k19=500]:

  subs(sol, x1pol): realroot(%, 1/100000000): evalf(%);
  S:=SS[1]:  x1 = S[1] + (S[2]-S[1])/2;
  sol := [op(sol), %]:
  for entry in ListTools:-Reverse(TriSol) do
    print(subs(sol, entry));
    sol := [op(sol), subs(sol,entry)]:
  od:
```

[[-90.84580294, -90.84580294], [-67.76791100, -67.76791100], [-48.53699823, -48.53699823], [17.63922132, 17.63922132], [122.0339678, 122.0339678], [323.7613772, 323.7613772]]

$$x1 = 17.63922137$$
$$x2 = 6.976738402$$
$$x3 = 367.5699821$$
$$x4 = 36.67719330$$
$$x5 = 5.5087486$$
$$x6 = 12.81103231$$
$$x7 = 0.51177439$$
$$x8 = 83.441647$$
$$x9 = 8.0609625$$
$$x10 = 0.256220659$$
$$x11 = 2.7324212$$

```
> sol := [k19=500]:
  subs(sol, x1pol): realroot(%, 1/100000000): evalf(%);
  S:=SS[2]:  x1 = S[1] + (S[2]-S[1])/2;
  sol := [op(sol), %]:

  for entry in ListTools:-Reverse(TriSol) do
    print(subs(sol, entry));
    sol := [op(sol), subs(sol,entry)]:
  od:
```

[[-90.84580294, -90.84580294], [-67.76791100, -67.76791100], [-48.53699823, -48.53699823], [17.63922132, 17.63922132], [122.0339678, 122.0339678], [323.7613772, 323.7613772]]

$$x1 = 122.0339678$$
$$x2 = 14.67212818$$
$$x3 = 234.9735960$$
$$x4 = 14.51017227$$
$$x5 = 7.1695197$$
$$x6 = 35.06403754$$
$$x7 = 0.42579019$$
$$x8 = 69.422300$$
$$x9 = 7.4387728$$
$$x10 = 0.70128074$$
$$x11 = 15.2681256$$

```
> sol := [k19=500]:

  subs(sol, x1pol): realroot(%, 1/100000000): evalf(%);
  S:=SS[3]:
  x1 = S[1] + (S[2]-S[1])/2;
  sol := [op(sol), %]:

  for entry in ListTools:-Reverse(TriSol) do
    print(subs(sol, entry));
    sol := [op(sol), subs(sol,entry)]:
  od:
```

[[-90.84580294, -90.84580294], [-67.76791100, -67.76791100], [-48.53699823, -48.53699823], [17.63922132, 17.63922132], [122.0339678, 122.0339678], [323.7613772, 323.7613772]]

$$x1 = 323.7613772$$

$$x2 = 9.496206124$$

$$x3 = 37.10128319$$

$$x4 = 6.729376568$$

$$x5 = 13.6294899$$

$$x6 = 43.14281634$$

$$x7 = 0.12780709$$

$$x8 = 20.8381138$$

$$x9 = 3.21139097$$

$$x10 = 0.862856308$$

$$x11 = 61.4581492$$

What now remains is to understand when the number of real positive solutions changes from 1 to 3. To understand this we need only study the problem in x1,k19. We use CAD to do this.

CAD of (x1, k19)-space

We decompose against the bivariate polynomial and also x1 (since we require x1>0).

```
> cad := CylindricalAlgebraicDecompose( [x1pol, x1, k19],
  PolynomialRing([x1, k19]), output=list ): nops(%);
```

135

k19 axis divided into 11 (5 points identified)

```
> convert(map(X->X[1][1],cad), set);
```

{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}

```
> select(X->X[1][1]=2, cad):
  convert(map(X->X[-1][-1][1], %), set):
  evalf(%);

  select(X->X[1][1]=4, cad):
  convert(map(X->X[-1][-1][1], %), set):
  evalf(%);

  select(X->X[1][1]=6, cad):
  convert(map(X->X[-1][-1][1], %), set):
  evalf(%);

  select(X->X[1][1]=8, cad):
  convert(map(X->X[-1][-1][1], %), set):
  evalf(%);

  select(X->X[1][1]=10, cad):
  convert(map(X->X[-1][-1][1], %), set):
  evalf(%);
```

{[-379.9930260, -379.9930260]}

{[-87.77612059, -87.77612059]}

{[0., 0.]}

{[409.2533647, 409.2533647]}

{[25084.53589, 25084.53589]}

So there are two positive values of k19 where solution behaviour changes.

Cell index 7 describes the k19 axis above 0 and below 409.25 (to 2dp)

```
> select(X->X[1][1]=7, cad):
  map(X->X[-1][-1][2], %):
  evalf(%);
  nops(%):
```

[[-110.5000000, -110.5000000], [-109.7804117, -109.7804108], [-102.3453851, -102.3453851], [-94.91035938, -94.91035843], [-70.60849333, -70.60849333], [-46.30662823, -46.30662727], [-23.15331364, -23.15331364], [0., 0.], [46.85140133, 46.85140133], [93.70280266, 93.70280361], [94.50000000, 94.50000000]]

We print the sample points. We see the positive x1 quadrant breaks into three pieced - one solutions

Cell index 9 describes the k19 axis above 409.25 and below 25084.53 (to 2dp)

```
> select(X->X[1][1]=9, cad):
  map(X->X[-1][-1][2], %):
  evalf(%);
  nops(%):
```

[[-79.50000000, -79.50000000], [-78.65213013, -78.65212917], [-64.79583025, -64.79583025], [-50.93953133, -50.93953037], [-50.76832724, -50.76832724], [-50.59712410, -50.59712315], [-25.29856157, -25.29856157], [0., 0.], [0.1715278625, 0.1715278625], [0.3430557251, 0.3430566788], [3462.559811, 3462.559811], [6924.776566, 6924.776567], [8974.635913, 8974.635913], [11024.49526, 11024.49526], [11025.50000, 11025.50000]]

We print the sample points. We see the positive x1 quadrant breaks into seven pieced - three solutions

Cell index 11 describes the k19 axis above 25084.53 (to 2dp)

```
> select(X->X[1][1]=11, cad):
  map(X->X[-1][-1][2], %):
  evalf(%);
  nops(%):
```

[[-79.50000000, -79.50000000], [-78.42055225, -78.42055130], [-64.53387451, -64.53387451], [-50.64719772, -50.64719677], [-50.64719057, -50.64719057], [-50.64718437, -50.64718342], [-25.32359171, -25.32359171], [0., 0.], [0.08641815186, 0.08641815186], [0.1728363037, 0.1728372574], [6885.445583, 6885.445583], [13770.71833, 13770.71833], [17796.62871, 17796.62871], [21822.53910, 21822.53910], [21823.50000, 21823.50000]]

We print the sample points. We see the positive x1 quadrant breaks into seven pieced - three solutions

And so we observe the crucial breaking point at k19 = 409.25

The final sections repeat the above analysis for different parameter values and free parameters. These calculations all proceed similarly to the above and so are not annotated. We note that some of them can take up a few minutes to run, so be patient. These results were summarised at the end of Section 2.1.2 in the paper.

Chainging k17 from 100 to 95

```
> TSX := subs(s1,s2, [k17=95, k18=50], TS):
  convert(TSX, rational):
  TSY := map(X->X*10000, %):
> vars := [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2, x1, k19]:
  R:=PolynomialRing(vars):
> TSZ := TSY:
  for i from 1 to nops(vars) do:
    var := vars[i]:
    TSZ := [op(TSZ), var>0]:
  od: TSZ:
> sol := LazyRealTriangularize( TSZ, R, output=list):
  rcPols := map(X->X[4], sol[1][RegularChain][polynomials]):
  TriSol := []:
  for var in [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2] do
      tmp := op(select(X->has(X,var), rcPols)):
      rcPols := remove(X->has(X,var), rcPols):
      #print(isolate(tmp, var));
      TriSol := [op(TriSol), isolate(tmp, var)]
  od:
> x1pol := op(select(X->has(X,x1), rcPols)):
> cad := CylindricalAlgebraicDecompose( [x1pol, x1],
  PolynomialRing([x1, k19]), output=list ):
> m := max(convert(map(X->X[1][1],cad), set));
```

$$m := 15$$

```
> for i from 2 by 2 to m-1 do:
    select(X->X[1][1]=i, cad);
    convert(map(X->X[-1][-1][1], %), set):
    print(i, evalf(%));
  od:
```

2, {[-373.3438030, -373.3438030]}

4, {[-250.9856093, -250.9856093]}

6, {[-84.47461554, -84.47461554]}

8, {[0., 0.]}

10, {[213.8741115, 213.8741115]}

12, {[369.9168535, 369.9168535]}

14, {[25093.73877, 25093.73877]}

```
> for ind in [9,11,13,15] do
    select(X->X[1][1]=ind, cad):
    map(X->X[-1][-1][2], %):
    select(X->X[1]>0, %);
    print(ind, "Num sols = ", (nops(%)-1)/2);
  od:
```

9, "Num sols = ", 1

11, "Num sols = ", 1

13, "Num sols = ", 3

15, "Num sols = ", 3

So breaking point is 369.91

Chainging k17 from 100 to 105

```
> TSX := subs(s1,s2, [k17=105, k18=50], TS):
  convert(TSX, rational):
  TSY := map(X->X*10000, %):
> vars := [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2, x1, k19]:
  R:=PolynomialRing(vars):
> TSZ := TSY:
  for i from 1 to nops(vars) do:
    var := vars[i]:
    TSZ := [op(TSZ), var>0]:
  od: TSZ:
> sol := LazyRealTriangularize( TSZ, R, output=list):
  rcPols := map(X->X[4], sol[1][RegularChain][polynomials]):
  TriSol := []:
  for var in [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2] do
      tmp := op(select(X->has(X,var), rcPols)):
      rcPols := remove(X->has(X,var), rcPols):
      #print(isolate(tmp, var));
      TriSol := [op(TriSol), isolate(tmp, var)]
  od:
> x1pol := op(select(X->has(X,x1), rcPols)):
> cad := CylindricalAlgebraicDecompose( [x1pol, x1, k19],
  PolynomialRing([x1, k19]), output=list ):
> m := max(convert(map(X->X[1][1],cad), set));
```

$$m := 11$$

```
> for i from 2 by 2 to m-1 do:
    select(X->X[1][1]=i, cad);
    convert(map(X->X[-1][-1][1], %), set):
    print(i, evalf(%));
  od:
```

$$2, \{[-386.6380072, -386.6380072]\}$$
$$4, \{[-91.09778311, -91.09778311]\}$$
$$6, \{[0., 0.]\}$$
$$8, \{[450.0767880, 450.0767880]\}$$
$$10, \{[25075.33300, 25075.33300]\}$$

```
> for ind in [7,9,11] do
    select(X->X[1][1]=ind, cad):
    map(X->X[-1][-1][2], %):
    select(X->X[1]>0, %);
    print(ind, "Num sols = ", (nops(%)-1)/2);
  od:
```

7, "Num sols = ", 1

9, "Num sols = ", 3

11, "Num sols = ", 3

So breaking point is 450.0

Setting k19 = 200 and allowing k17 free

```
> TSX := subs(s1,s2, [k19=200, k18=50], TS):
  convert(TSX, rational):
  TSY := map(X->X*10000, %):
> vars := [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2, x1, k17]:
  R:=PolynomialRing(vars):
> TSZ := TSY:
  for i from 1 to nops(vars) do:
    var := vars[i]:
    TSZ := [op(TSZ), var>0]:
  od: TSZ:
> sol := LazyRealTriangularize( TSZ, R, output=list):
  rcPols := map(X->X[4], sol[1][RegularChain][polynomials]):
  TriSol := []:
  for var in [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2] do
      tmp := op(select(X->has(X,var), rcPols)):
      rcPols := remove(X->has(X,var), rcPols):
      #print(isolate(tmp, var));
      TriSol := [op(TriSol), isolate(tmp, var)]
  od:
> x1pol := op(select(X->has(X,x1), rcPols)):
> cad := CylindricalAlgebraicDecompose( [x1pol, x1, k17],
  PolynomialRing([x1, k17]), output=list ):
> m := max(convert(map(X->X[1][1],cad), set));
```

$$m := 17$$

```
> for i from 2 by 2 to m-1 do:
    select(X->X[1][1]=i, cad);
    convert(map(X->X[-1][-1][1], %), set):
    print(i, evalf(%));
  od:
```

$$2, \{[-2.611357213\ 10^6, -2.611357213\ 10^6]\}$$
$$4, \{[-91.84759563, -91.84759563]\}$$
$$6, \{[-71.02799821, -71.02799821]\}$$
$$8, \{[-2.657982707, -2.657982707]\}$$
$$10, \{[0., 0.]\}$$
$$12, \{[95.85755288, 95.85755288]\}$$
$$14, \{[8894.131078, 8894.131078]\}$$
$$16, \{[13633.03132, 13633.03132]\}$$

```
> for ind in [11,13,15,17] do
    select(X->X[1][1]=ind, cad):
    map(X->X[-1][-1][2], %):
    select(X->X[1]>0, %);
    print(ind, "Num sols = ", (nops(%)-1)/2);
  od:
```

11, "Num sols = ", 1

13, "Num sols = ", 1

15, "Num sols = ", 1

17, "Num sols = ", 1

So this time there is no bistability!

Setting k19 = 500 and allowing k17 free

```
> TSX := subs(s1,s2, [k19=500, k18=50], TS):
  convert(TSX, rational):
  TSY := map(X->X*10000, %):
> vars := [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2, x1, k17]:
  R:=PolynomialRing(vars):
> TSZ := TSY:
  for i from 1 to nops(vars) do:
    var := vars[i]:
    TSZ := [op(TSZ), var>0]:
  od: TSZ:
> sol := LazyRealTriangularize( TSZ, R, output=list):
  rcPols := map(X->X[4], sol[1][RegularChain][polynomials]):
  TriSol := []:
  for var in [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2] do
      tmp := op(select(X->has(X,var), rcPols)):
      rcPols := remove(X->has(X,var), rcPols):
      #print(isolate(tmp, var));
      TriSol := [op(TriSol), isolate(tmp, var)]
  od:
> x1pol := op(select(X->has(X,x1), rcPols)):
> cad := CylindricalAlgebraicDecompose( [x1pol, x1, k17],
  PolynomialRing([x1, k17]), output=list ):
> m := max(convert(map(X->X[1][1],cad), set));
```

$$m := 21$$

```
> for i from 2 by 2 to m-1 do:
    select(X->X[1][1]=i, cad);
    convert(map(X->X[-1][-1][1], %), set):
    print(i, evalf(%));
  od:
```

$$2, \{[-2.611056006\ 10^6, -2.611056006\ 10^6]\}$$
$$4, \{[-137.6954999, -137.6954999]\}$$
$$6, \{[-73.61489489, -73.61489489]\}$$
$$8, \{[0., 0.]\}$$
$$10, \{[0.6164194692, 0.6164194692]\}$$
$$12, \{[55.09668286, 55.09668286]\}$$
$$14, \{[85.98772184, 85.98772184]\}$$
$$16, \{[110.8692721, 110.8692721]\}$$
$$18, \{[8893.842566, 8893.842566]\}$$
$$20, \{[13469.76024, 13469.76024]\}$$

```
> for ind in [9,11,13,15,17,19,21] do
    select(X->X[1][1]=ind, cad):
    map(X->X[-1][-1][2], %):
    select(X->X[1]>0, %);
    print(ind, "Num sols = ", (nops(%)-1)/2);
  od:
```

9, "Num sols = ", 1

11, "Num sols = ", 1

13, "Num sols = ", 1

15, "Num sols = ", 3

17, "Num sols = ", 1

19, "Num sols = ", 1

21, "Num sols = ", 1

So this time there is bistability for k17 greater than 85.99 but less than 110.87

Setting k19 = 200 and allowing k18 free

```
> TSX := subs(s1,s2, [k19=200, k17=100], TS):
  convert(TSX, rational):
  TSY := map(X->X*10000, %):
> vars := [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2, x1, k18]:
  R:=PolynomialRing(vars):
> TSZ := TSY:
  for i from 1 to nops(vars) do:
    var := vars[i]:
    TSZ := [op(TSZ), var>0]:
  od: TSZ:
> sol := LazyRealTriangularize( TSZ, R, output=list):
  rcPols := map(X->X[4], sol[1][RegularChain][polynomials]):
  TriSol := []:
  for var in [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2] do
      tmp := op(select(X->has(X,var), rcPols)):
      rcPols := remove(X->has(X,var), rcPols):
      #print(isolate(tmp, var));
      TriSol := [op(TriSol), isolate(tmp, var)]
  od:
> x1pol := op(select(X->has(X,x1), rcPols)):
> cad := CylindricalAlgebraicDecompose( [x1pol, x1, k18],
  PolynomialRing([x1, k18]), output=list ):
> m := max(convert(map(X->X[1][1],cad), set));
```

$$m := 17$$

```
> for i from 2 by 2 to m-1 do:
    select(X->X[1][1]=i, cad);
    convert(map(X->X[-1][-1][1], %), set):
    print(i, evalf(%));
  od:
```

$$2, \{[-123.3420386, -123.3420386]\}$$
$$4, \{[-71.93371102, -71.93371102]\}$$
$$6, \{[-69.13335740, -69.13335740]\}$$
$$8, \{[0., 0.]\}$$
$$10, \{[0.5365879183, 0.5365879183]\}$$
$$12, \{[0.9162006659, 0.9162006659]\}$$
$$14, \{[51.83277400, 51.83277400]\}$$
$$16, \{[6436.018741, 6436.018741]\}$$

```
> for ind in [7,9,11,13,15,17] do
    select(X->X[1][1]=ind, cad):
    map(X->X[-1][-1][2], %):
    select(X->X[1]>0, %);
    print(ind, "Num sols = ", (nops(%)-1)/2);
  od:
```

7, "Num sols = ", 1

9, "Num sols = ", 1

11, "Num sols = ", 1

13, "Num sols = ", 1

15, "Num sols = ", 1

17, "Num sols = ", 1

So this time there is no bistability!

Setting k19 = 500 and allowing k18 free

```
> TSX := subs(s1,s2, [k19=500, k17=100], TS):
  convert(TSX, rational):
  TSY := map(X->X*10000, %):
> vars := [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2, x1, k18]:
  R:=PolynomialRing(vars):
> TSZ := TSY:
  for i from 1 to nops(vars) do:
    var := vars[i]:
    TSZ := [op(TSZ), var>0]:
  od: TSZ:
> sol := LazyRealTriangularize( TSZ, R, output=list):
  rcPols := map(X->X[4], sol[1][RegularChain][polynomials]):
  TriSol := []:
  for var in [x11, x10, x9, x8, x7, x6, x5, x4, x3, x2] do
      tmp := op(select(X->has(X,var), rcPols)):
      rcPols := remove(X->has(X,var), rcPols):
      #print(isolate(tmp, var));
      TriSol := [op(TriSol), isolate(tmp, var)]
  od:
> x1pol := op(select(X->has(X,x1), rcPols)):
> cad := CylindricalAlgebraicDecompose( [x1pol, x1, k18],
  PolynomialRing([x1, k18]), output=list ):
> m := max(convert(map(X->X[1][1],cad), set));
```

$$m := 21$$

```
> for i from 2 by 2 to m-1 do:
    select(X->X[1][1]=i, cad);
    convert(map(X->X[-1][-1][1], %), set):
    print(i, evalf(%));
  od:
```

$$2, \{[-205.6521320, -205.6521320]\}$$
$$4, \{[-68.09991250, -68.09991250]\}$$
$$6, \{[-43.27661084, -43.27661084]\}$$
$$8, \{[0., 0.]\}$$
$$10, \{[0.5521414522, 0.5521414522]\}$$
$$12, \{[1.479376353, 1.479376353]\}$$
$$14, \{[44.43445109, 44.43445109]\}$$
$$16, \{[58.32899443, 58.32899443]\}$$
$$18, \{[64.20774832, 64.20774832]\}$$
$$20, \{[6744.969650, 6744.969650]\}$$

```
> for ind in [9,11,13,15,17,19,21] do
    select(X->X[1][1]=ind, cad):
    map(X->X[-1][-1][2], %):
    select(X->X[1]>0, %);
    print(ind, "Num sols = ", (nops(%)-1)/2);
  od:
```

9, "Num sols = ", 1

11, "Num sols = ", 1

13, "Num sols = ", 1

15, "Num sols = ", 3

17, "Num sols = ", 1

19, "Num sols = ", 1

21, "Num sols = ", 1

So bistability between 44.43 and 58.32