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The Radiation and Transducer Efficiencies of a Multiport Antenna Array

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ABSTRACT We derive some general properties of generalized Rayleigh ratios, and use these properties to present a detailed theory of two efficiency metrics relevant to a specified excitation of a multiport antenna array (MAA) by a linear time-invariant (LTI) multiport generator. These efficiency metrics are the transducer efficiency and the radiation efficiency. To define the excitation and compute the transducer efficiency and the radiation efficiency, we consider six different variables. We then define and study four new efficiency metrics relevant to unspecified excitations: the minimum transducer efficiency, the transducer efficiency figure, the minimum radiation efficiency, and the radiation efficiency figure. We also look at their connection with the minimum power transfer ratio during emission and the power match figure. We discuss the relevance of the theory presented in this article to the design of an MAA, and to an actual configuration comprising a transmitter, the antenna outputs of which need not behave like an LTI multiport.

INDEX TERMS Antenna array, MIMO, transducer efficiency, radiation efficiency, power transfer ratio.

I. INTRODUCTION

To quantify the efficiency of a multiport antenna array (MAA) as a radiator of electromagnetic power, we will define and discuss some general metrics of this efficiency of the MAA, for specified excitations involving any combinations of the ports, and for unspecified excitations. To this end, we assume that each of the N ports of the MAA is coupled to one and only one of the N ports of a linear time-invariant (LTI) multiport generator (MG). The MG can deliver any excitation. We do not assume that the ports of the MG are uncoupled, so that the internal impedance matrix of the MG need not be diagonal. We neither assume that this impedance matrix is real. This generality is necessary to cover configurations in which the MG comprises a MIMO matching and decoupling network such as the ones described in [1]-[5], or a MIMO antenna tuner (i.e., a tunable or reconfigurable matching network) such as the ones described in [6]-[13].

To define the MAA efficiency metrics without introducing unnecessary assumptions, some general properties of generalized Rayleigh ratios are needed. They are stated and proven in Section II, where Theorem 2, Theorem 3, Corollary 1, Corollary 2 and Corollary 4 seem to be new. This mathematical groundwork is used in the examples of Section III.

The assumptions concerning the MAA, the different variables used to define the excitations, and the computation of the available power from any one of these variables are treated in Section IV and Section V. The computation of the radiated power from any one of these variables is covered in Section VI. The results of Section V and Section VI are new.

Two metrics for the efficiency of the MAA, relevant to a specified excitation, are the transducer efficiency e_T defined in this article, and the radiation efficiency e_R. The minimum transducer efficiency $e_{T\,MIN}$, the transducer efficiency figure F_{TE}, the minimum radiation efficiency $e_{R\,MIN}$, and the radiation efficiency figure F_{RE} are new metrics for the efficiency of the MAA subject to unspecified excitations. Metrics covering matching for unspecified excitations of an MAA have been proposed: the total multiport reflectance [4], [14]; the return figure [10]-[13]; and the power match figure [15]. These earlier metrics ignore antenna efficiency, whereas $e_{T\,MIN}$, F_{TE}, $e_{R\,MIN}$ and F_{RE} depend on the antenna efficiency. Moreover, the total multiport reflectance and the return figure are only relevant to a MG having uncoupled ports presenting the same real impedance, whereas our new metrics are pertinent to the MG defined above, which may present a non-real and/or non-diagonal impedance matrix.

Section VII covers the definition and computation of e_T, $e_{T\,MIN}$ and F_{TE}. Sections VIII and IX covers the definition and computation of e_R, $e_{R\,MIN}$ and F_{RE}. Section X is about the power transfer ratio, which is related to e_T and e_R. Section XI treats single-port excitations and their uses. Efficiency metric computation examples are proposed in Section XII.

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II. GENERALIZED RAYLEIGH RATIO

A. WHAT IS A GENERALIZED RAYLEIGH RATIO?

Let n be a positive integer. The vector space of the complex column vectors of size n is denoted by $\mathbb{C}^n$. For any $E \subset \mathbb{C}^n$, we use $E^\perp$ to denote the orthogonal complement of E, that is the set of all vectors in $\mathbb{C}^n$ that are orthogonal to every vector lying in E.

We use $\mathbf{1}_n$ to denote the identity matrix of size n by n, and for a positive integer m, the null matrix of size m by n is denoted by $\mathbf{0}_{m,n}$ or by $\mathbf{0}$ when no confusion may arise. We use $\text{diag}_n(a_1, \ldots, a_n)$ to denote the diagonal matrix of diagonal entries $a_{11} = a_1$ to $a_{nn} = a_n$. Let $\mathbf{M}$ be a complex matrix. We use $\ker \mathbf{M}$ to denote the nullspace of $\mathbf{M}$, $\text{rank}\, \mathbf{M}$ to denote the rank of $\mathbf{M}$, and $\mathbf{M}^*$ to denote the hermitian adjoint of $\mathbf{M}$.

Let $\mathbf{A}$ be a positive semidefinite matrix. We know that there exists a unique positive semidefinite matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$ [16, Sec. 7.2.6]. The matrix $\mathbf{B}$ is referred to as the unique positive semidefinite square root of $\mathbf{A}$, and is denoted by $\mathbf{A}^{1/2}$. If $\mathbf{A}$ is positive definite, $\mathbf{A}^{-1}$ and $\mathbf{A}^{1/2}$ are positive definite, and $(\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$, so that we can write $\mathbf{A}^{-1/2} = (\mathbf{A}^{1/2})^{-1} = (\mathbf{A}^{-1})^{1/2}$.

Observation 1. Let $\mathbf{A}$ be a positive semidefinite matrix. For any $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^* \mathbf{A} \mathbf{x} = 0$ if and only if $\mathbf{x} \in \ker \mathbf{A}$.

Proof: If $\mathbf{x} \in \ker \mathbf{A}^{1/2}$, we have $\mathbf{A}\mathbf{x} = \mathbf{A}^{1/2}\mathbf{A}^{1/2}\mathbf{x} = \mathbf{A}^{1/2}\,\mathbf{0} = \mathbf{0}$, so that $\mathbf{x} \in \ker \mathbf{A}$. Conversely, let $\mathbf{x} \in \ker \mathbf{A}$. Since by [16, Sec. 7.2.6] there is a polynomial p with real coefficients such that $\mathbf{A}^{1/2} = p(\mathbf{A})$, we have $\mathbf{A}^{1/2}\mathbf{x} = p(\mathbf{A})\mathbf{x} = \mathbf{0}$, so that $\mathbf{x} \in \ker \mathbf{A}^{1/2}$.

We have proven that $\ker \mathbf{A}^{1/2} = \ker \mathbf{A}$.

For any $\mathbf{x} \in \mathbb{C}^n$, we have $\mathbf{x}^* \mathbf{A} \mathbf{x} = 0$ if and only if $\mathbf{x}^* \mathbf{A}^{1/2}\mathbf{A}^{1/2} \mathbf{x} = 0$ if and only if $(\mathbf{A}^{1/2} \mathbf{x})^*(\mathbf{A}^{1/2} \mathbf{x}) = 0$ if and only if $\mathbf{A}^{1/2} \mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} \in \ker \mathbf{A}^{1/2}$.

Thus, $\mathbf{x}^* \mathbf{A} \mathbf{x} = 0$ if and only if $\mathbf{x} \in \ker \mathbf{A}$. □

Note that there are other proofs of this well-known result [16, Sec. 7.1.6].

Let $\mathbf{A}$ be an hermitian matrix of size n by n. The expression $\mathbf{x}^*\mathbf{A}\mathbf{x}/\mathbf{x}^*\mathbf{x}$, where $x \in \mathbb{C}^n$, is known as a Rayleigh ratio, or Rayleigh-Ritz ratio, or Rayleigh quotient [16, Sec. 4.2] [17, Sec. 4.2]. In this article, this concept is extended as follows. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size n by n, $\mathbf{D}$ being positive semidefinite. The generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$ is a real-valued function $r : \mathbb{C}^n \to \mathbb{R}$ such that, for any $\mathbf{x} \in \mathbb{C}^n$ satisfying $\mathbf{x}^*\mathbf{D}\mathbf{x} \neq 0$, we have

$$r(\mathbf{x}) = \frac{\mathbf{x}^*\mathbf{N}\mathbf{x}}{\mathbf{x}^*\mathbf{D}\mathbf{x}}\,. \tag{1}$$

The generalized Rayleigh ratio r may be viewed as a ratio of two hermitian quadratic forms [18, ch. 3]–[19, Sec. 10.1] (also called "hermitian forms" [20, ch. X]) in the variable $\mathbf{x}$: the hermitian quadratic form $f_\mathbf{N} : \mathbb{C}^n \to \mathbb{R}$ such that $f_\mathbf{N}(\mathbf{x}) = \mathbf{x}^*\mathbf{N}\mathbf{x}$ and the positive definite hermitian quadratic form $f_\mathbf{D} : \mathbb{C}^n \to \mathbb{R}$ such that $f_\mathbf{D}(\mathbf{x}) = \mathbf{x}^*\mathbf{D}\mathbf{x}$.

By Observation 1, the domain of definition of r, denoted by D_r, is

$$D_r = \{\mathbf{x} \in \mathbb{C}^n : \mathbf{x} \notin \ker \mathbf{D}\}\,, \tag{2}$$

where the colon means "such that". Let $d = \dim \ker \mathbf{D}$ be the nullity of $\mathbf{D}$. By Observation 1, $\mathbf{D}$ is positive definite if and only if $d = 0$, that is to say if and only if $\ker \mathbf{D} = \{\mathbf{0}\}$.

Observation 2. Let $||\mathbf{x}||_2 = \sqrt{\mathbf{x}^*\mathbf{x}}$ be the euclidian vector norm of an arbitrary complex column vector $\mathbf{x}$. We use $\mathbb{S}_n$ to denote the hypersphere of the unit vectors of $\mathbb{C}^n$. It follows from (1) that, for a fixed $\mathbf{x}/||\mathbf{x}||_2$, if $r(\mathbf{x})$ is defined, it does not depend on $||\mathbf{x}||_2$. Thus, the set of the values of $r(\mathbf{x})$ such that $\mathbf{x} \in D_r$ is equal to the set of the values of $r(\mathbf{x})$ such that $\mathbf{x} \in D_r \cap \mathbb{S}_n$.

Observation 3. If $\mathbf{N}$ is positive semidefinite, for any $\mathbf{x} \in D_r$ we have $r(\mathbf{x}) \geqslant 0$.

B. BOUNDS OF GENERALIZED RAYLEIGH RATIOS

To investigate the bounds of generalized Rayleigh ratios, we will first cover the special case where $\mathbf{D}$ is positive definite. Afterwards, we will address the general case, which is more involved.

Theorem 1. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size n by n. We assume that $\mathbf{D}$ is positive definite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$. Since $\mathbf{D}$ is positive definite, $D_r = \{\mathbf{x} \in \mathbb{C}^n : \mathbf{x} \neq \mathbf{0}\}$ and we can define

$$\mathbf{M} = \mathbf{D}^{-1/2}\mathbf{N}\mathbf{D}^{-1/2}\,. \tag{3}$$

$\mathbf{M}$ is of size n by n, and hermitian. Thus, its eigenvalues are real. Let $\lambda_{\max}$ be the largest eigenvalue of $\mathbf{M}$ and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{M}$. For any $\mathbf{x} \in \mathbb{C}^n$ satisfying $\mathbf{x} \neq \mathbf{0}$, we have

$$\lambda_{\min} = \min_{\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}^*\mathbf{M}\mathbf{y}}{\mathbf{y}^*\mathbf{y}} \leqslant r(\mathbf{x}) \leqslant \lambda_{\max} = \max_{\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}^*\mathbf{M}\mathbf{y}}{\mathbf{y}^*\mathbf{y}}\,. \tag{4}$$

Moreover,

- the equality $r(\mathbf{x}) = \lambda_{\max}$ is satisfied if and only if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\max}$;
- the equality $r(\mathbf{x}) = \lambda_{\min}$ is satisfied if and only if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\min}$; and
- the eigenvalues of $\mathbf{N}\mathbf{D}^{-1}$ are real, $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$.

Proof: For any $\mathbf{x} \in \mathbb{C}^n$, let $\mathbf{y} = \mathbf{D}^{1/2}\mathbf{x}$. We have $\mathbf{x} \neq \mathbf{0}$ if and only if $\mathbf{y} \neq \mathbf{0}$. Since $\mathbf{D}$ is positive definite, we have $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$ and, for $\mathbf{y} \neq \mathbf{0}$,

$$r(\mathbf{x}) = \frac{(\mathbf{D}^{1/2}\mathbf{x})^*\mathbf{M}(\mathbf{D}^{1/2}\mathbf{x})}{(\mathbf{D}^{1/2}\mathbf{x})^*(\mathbf{D}^{1/2}\mathbf{x})} = \frac{\mathbf{y}^*\mathbf{M}\mathbf{y}}{\mathbf{y}^*\mathbf{y}}\,. \tag{5}$$

Using Rayleigh's theorem [16, Sec. 4.2.2], we obtain (4). The other assertions of Theorem 1 relating to the equalities $r(\mathbf{x}) = \lambda_{\max}$ and $r(\mathbf{x}) = \lambda_{\min}$ result from Rayleigh's theorem and the definition of $\mathbf{y}$. Moreover, we observe that

$$\mathbf{N}\mathbf{D}^{-1} = \mathbf{D}^{1/2}\mathbf{M}\mathbf{D}^{-1/2}\,, \tag{6}$$

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so that $\mathbf{M}$ is similar to $\mathbf{N}\mathbf{D}^{-1}$. It follows that $\mathbf{M}$ and $\mathbf{N}\mathbf{D}^{-1}$ have the same eigenvalues, counting mutiplicities, by [16, Sec. 1.3.4]. □

Observation 4. If $\mathbf{D}$ is positive definite and $\mathbf{N}$ is positive semidefinite, then $\mathbf{M}$ defined in Theorem 1 is positive semidefinite, so that $\lambda_{\min} \geqslant 0$.

Theorem 2. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size n by n, $\mathbf{D}$ being positive semidefinite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$, and let D_r be the domain of definition of r. If $D_r \neq \varnothing$ and if there exists $\mathbf{x} \in \ker \mathbf{D}$ such that $\mathbf{x}^*\mathbf{N}\mathbf{x} \neq 0$, then the image of D_r under r, denoted by $r(D_r)$, is not bounded.

Proof: We assume that $D_r \neq \varnothing$. It follows that there exists $\mathbf{y} \in D_r$. We have $\mathbf{y}^*\,\mathbf{D}\,\mathbf{y} \neq 0$. If there exists $\mathbf{x} \in \ker \mathbf{D}$ such that $\mathbf{x}^*\mathbf{N}\mathbf{x} \neq 0$, we observe that for any $\lambda \in \mathbb{R}$,

$$(\mathbf{x} + \lambda\mathbf{y})^*\,\mathbf{D}\,(\mathbf{x} + \lambda\mathbf{y}) = \lambda^2\,\mathbf{y}^*\,\mathbf{D}\,\mathbf{y}\,, \tag{7}$$

so that $\mathbf{x} + \lambda\mathbf{y} \in \ker \mathbf{D}$ if and only if $\lambda = 0$. It follows that we can define $g : \mathbb{R} \to \mathbb{R}$ such that for any $\lambda \neq 0$, $g(\lambda) = |r(\mathbf{x} + \lambda\mathbf{y})|$. For any nonzero $\lambda \in \mathbb{R}$, we have

$$g(\lambda) = \left| \frac{\mathbf{x}^*\mathbf{N}\mathbf{x} + \lambda(\mathbf{y}^*\mathbf{N}\mathbf{x} + \mathbf{x}^*\mathbf{N}\mathbf{y}) + \lambda^2\mathbf{y}^*\mathbf{N}\mathbf{y}}{\lambda^2\,\mathbf{y}^*\,\mathbf{D}\,\mathbf{y}} \right| , \tag{8}$$

which becomes arbitrarily large as λ approaches 0, because $\mathbf{x}^*\mathbf{N}\mathbf{x} \neq 0$. Thus, $r(D_r)$ is not bounded. □

Corollary 1. Let $\mathbf{N}$ and $\mathbf{D}$ be positive semidefinite matrices of size n by n. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$, and let D_r be the domain of definition of r. If $D_r \neq \varnothing$ and if $r(D_r)$ is bounded, then $\ker \mathbf{D} \subset \ker \mathbf{N}$.

Proof: We assume that $D_r \neq \varnothing$ and $r(D_r)$ is bounded. By Theorem 2, there is no $\mathbf{x} \in \ker \mathbf{D}$ such that $\mathbf{x}^*\mathbf{N}\mathbf{x} \neq 0$. Since $\mathbf{N}$ is positive semidefinite, we can use Observation 1 to conclude that there is no $\mathbf{x} \in \ker \mathbf{D}$ such that $\mathbf{x} \notin \ker \mathbf{N}$. □

Theorem 3. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size n by n, $\mathbf{D}$ being positive semidefinite. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$, let D_r be the domain of definition of r, and let d be the nullity of $\mathbf{D}$. We assume that $D_r \neq \varnothing$ and $\ker \mathbf{D} \subset \ker \mathbf{N}$.

$\mathbf{D}$ being positive semidefinite, it has n eigenvalues, counting multiplicities, and these values are real and nonnegative by [16, Sec. 7.2.1]. Let us label these eigenvalues according to a non-decreasing order $\mu_1, \ldots, \mu_n$. Since $D_r \neq \varnothing$, we have $d \leqslant n - 1$, so that $0 < \mu_{d+1} \leqslant \ldots \leqslant \mu_n$. For any positive integer i such that $i \leqslant d$, we have $\mu_i = 0$. $\mathbf{D}$ being hermitian, by [16, Sec. 2.5.6] there exists a unitary matrix $\mathbf{L}$ of size n by n such that

$$\mathbf{D} = \mathbf{L}\,\mathrm{diag}_n(\mu_1, \ldots, \mu_n)\,\mathbf{L}^* \tag{9}$$

For any $i \in \{1, \ldots, n\}$, let the i-th column vector of $\mathbf{L}$ be denoted by $\mathbf{L}^{<i>}$. Let $\mathcal{L}$ be the submatrix of $\mathbf{L}$, of size n by $n - d$, whose column vectors are $\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<n>}$, in this order. Let

$$\mathbf{P} = \mathcal{L}\,\mathrm{diag}_{n-d}\left(\frac{1}{\sqrt{\mu_{d+1}}}, \ldots, \frac{1}{\sqrt{\mu_n}}\right) \tag{10}$$

and

$$\mathbf{Q} = \mathbf{P}^*\,\mathbf{N}\,\mathbf{P}\,. \tag{11}$$

The matrix $\mathbf{P}$ is of size n by $n - d$. The matrix $\mathbf{Q}$ is clearly hermitian, of size $n - d$ by $n - d$. Thus, its eigenvalues are real. Let $\kappa_{\max}$ be the largest eigenvalue of $\mathbf{Q}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{Q}$. For any $\mathbf{x} \in D_r$, we have

$$\kappa_{\min} = \min_{\mathbf{u} \neq \mathbf{0}} \frac{\mathbf{u}^*\mathbf{Q}\mathbf{u}}{\mathbf{u}^*\mathbf{u}} \leqslant r(\mathbf{x}) \leqslant \kappa_{\max} = \max_{\mathbf{u} \neq \mathbf{0}} \frac{\mathbf{u}^*\mathbf{Q}\mathbf{u}}{\mathbf{u}^*\mathbf{u}}\,. \tag{12}$$

Moreover,

- we have $r(\mathbf{x}) = \kappa_{\max}$ if $\mathbf{x} = \mathbf{P}\mathbf{w}$, where $\mathbf{w}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\max}$;
- we have $r(\mathbf{x}) = \kappa_{\min}$ if $\mathbf{x} = \mathbf{P}\mathbf{w}$, where $\mathbf{w}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\min}$; and
- the eigenvalues of

$$\mathbf{R} = \mathcal{L}^*\mathbf{N}\mathcal{L}\mathrm{diag}_{n-d}\left(\frac{1}{\mu_{d+1}}, \ldots, \frac{1}{\mu_n}\right) \tag{13}$$

are real, $\kappa_{\max}$ is the largest eigenvalue of $\mathbf{R}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{R}$.

Proof: Since $\mathbf{D}\mathbf{L} = \mathbf{L}\mathrm{diag}_n(\mu_1, \ldots, \mu_n)$, we know that, for any $i \in \{1, \ldots, n\}$, $\mathbf{L}^{<i>}$ is an eigenvector of $\mathbf{D}$ associated with the eigenvalue μ_i. It follows that $\mathbf{L}^{<1>}$ to $\mathbf{L}^{<d>}$ are vectors of $\ker \mathbf{D}$. $\mathbf{L}$ being unitary, $(\mathbf{L}^{<1>}, \ldots, \mathbf{L}^{<n>})$ is an orthonormal basis of $\mathbb{C}^n$. Thus, $(\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<n>})$ is an orthonormal basis of $(\ker \mathbf{D})^{\perp}$.

For any $\mathbf{x} \in \mathbb{C}^n$, there is a unique $p_1(\mathbf{x}) \in \ker \mathbf{D}$, and a unique $p_2(\mathbf{x}) \in (\ker \mathbf{D})^{\perp}$ such that $\mathbf{x} = p_1(\mathbf{x}) + p_2(\mathbf{x})$. We have $\mathbf{x}^*\,\mathbf{D}\,\mathbf{x} = p_2(\mathbf{x})^*\,\mathbf{D}\,p_2(\mathbf{x})$. Thus, if $\mathbf{x} \in D_r$, then $p_2(\mathbf{x}) \neq \mathbf{0}$. Since we assume that $\ker \mathbf{D} \subset \ker \mathbf{N}$, we also have $\mathbf{x}^*\,\mathbf{N}\,\mathbf{x} = p_2(\mathbf{x})^*\,\mathbf{N}\,p_2(\mathbf{x})$. Thus, we can assert that, if $\mathbf{x} \in D_r$, then

$$r(\mathbf{x}) = \frac{p_2(\mathbf{x})^*\,\mathbf{N}\,p_2(\mathbf{x})}{p_2(\mathbf{x})^*\,\mathbf{D}\,p_2(\mathbf{x})} = r(\mathbf{p_2}(\mathbf{x}))\,. \tag{14}$$

It follows that

$$r(D_r) = r((\ker \mathbf{D})^{\perp})\,. \tag{15}$$

Let $\mathbf{x} \in D_r$ and $\mathbf{z} = p_2(\mathbf{x})$. Let $\zeta_{d+1}, \ldots, \zeta_n$ be the coordinates of $\mathbf{z}$ in the basis $(\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<n>})$ of $(\ker \mathbf{D})^{\perp}$. We introduce a column vector of size $n - d$, given by

$$\mathfrak{z} = \begin{pmatrix} \zeta_{d+1} \\ \vdots \\ \zeta_n \end{pmatrix}. \tag{16}$$

The product $\mathcal{L}\,\mathfrak{z}$ is a column vector of size n. Using the rule for the multiplication of block matrices, we find

$$\mathcal{L}\,\mathfrak{z} = \sum_{i=d+1}^{n} \mathbf{L}^{<i>}\zeta_i = \mathbf{z}\,. \tag{17}$$

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Using (14) and (17), we get

$$r(\mathbf{x}) = \frac{\mathfrak{z}^* \mathcal{L}^* \mathbf{N} \mathcal{L} \mathfrak{z}}{\mathfrak{z}^* \mathcal{L}^* \mathbf{D} \mathcal{L} \mathfrak{z}}, \tag{18}$$

and (9) leads us

$$r(\mathbf{x}) = \frac{\mathfrak{z}^* \mathcal{L}^* \mathbf{N} \mathcal{L} \mathfrak{z}}{\mathfrak{z}^* \mathcal{L}^* \mathbf{L} \operatorname{diag}_n(\mu_1, \ldots, \mu_n) \mathbf{L}^* \mathcal{L} \mathfrak{z}}. \tag{19}$$

$\mathbf{L}^* \mathcal{L}$ is of size n by $n-d$. Since $\mathbf{L}$ is unitary, we find that $\mathbf{L}^* \mathcal{L}$ is given by

$$\mathbf{L}^* \mathcal{L} = \begin{pmatrix} \mathbf{0}_{d,\, n-d} \\ \mathbf{1}_{n-d} \end{pmatrix}. \tag{20}$$

Using (19) and (20), we obtain

$$r(\mathbf{z}) = \frac{\mathfrak{z}^* \mathcal{L}^* \mathbf{N} \mathcal{L} \mathfrak{z}}{\mathfrak{z}^* \operatorname{diag}_{n-d}(\mu_{d+1}, \ldots, \mu_n) \mathfrak{z}}. \tag{21}$$

Since $\mathfrak{z}$ is the column vector of the coordinates of $\mathbf{z}$ in the basis $(\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<n>})$ of $(\ker \mathbf{D})^{\perp}$, it follows from (15) that $r(D_r)$ is the set of all $r(\mathbf{z})$ given by (21) when $\mathfrak{z}$ takes on any nonzero value in $\mathbb{C}^{n-d}$. Consequently, using Theorem 1, we obtain (12), and

- we have $r(\mathbf{x}) = \kappa_{\max}$ if we have $\mathbf{x} = \mathcal{L}\mathbf{x}'$ in which $\mathbf{x}' = \operatorname{diag}_{n-d}(\mu_{d+1}, \ldots, \mu_n)^{-1/2}\mathbf{w}$, where $\mathbf{w}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\max}$;
- we have $r(\mathbf{x}) = \kappa_{\min}$ if we have $\mathbf{x} = \mathcal{L}\mathbf{x}'$ in which $\mathbf{x}' = \operatorname{diag}_{n-d}(\mu_{d+1}, \ldots, \mu_n)^{-1/2}\mathbf{w}$, where $\mathbf{w}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\min}$; and
- the eigenvalues of $\mathbf{R}$ given by (13) are real, $\kappa_{\max}$ is the largest eigenvalue of $\mathbf{R}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{R}$.

This leads to the final results of Theorem 3. □

In the case $d = 0$, we can use Theorem 1 and Theorem 3, the latter giving the same results as the former.

Corollary 2. Let $\mathbf{N}$ and $\mathbf{D}$ be positive semidefinite matrices of size n by n. Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$, and let D_r be the domain of definition of r. We assume that $D_r \neq \varnothing$. Then $r(D_r)$ is bounded if and only if $\ker \mathbf{D} \subset \ker \mathbf{N}$.

Proof: This is a direct consequence of Corollary 1 and Theorem 3. □

C. RELATED RESULTS THAT DO NOT USE A RATIO

Corollary 3. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size n by n. We assume that $\mathbf{D}$ is positive definite, so that we can define $\mathbf{M} = \mathbf{D}^{-1/2}\mathbf{N}\mathbf{D}^{-1/2}$. The matrix $\mathbf{M}$ is of size n by n, and hermitian. Thus, its eigenvalues are real. Let $\lambda_{\max}$ be the largest eigenvalue of $\mathbf{M}$ and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{M}$. For any $\mathbf{x} \in \mathbb{C}^n$, we have

$$\lambda_{\min}\, \mathbf{x}^*\mathbf{D}\mathbf{x} \leqslant \mathbf{x}^*\mathbf{N}\mathbf{x} \leqslant \lambda_{\max}\, \mathbf{x}^*\mathbf{D}\mathbf{x}. \tag{22}$$

Moreover,

- we have $\mathbf{x}^*\mathbf{N}\mathbf{x} = \lambda_{\max}\, \mathbf{x}^*\mathbf{D}\mathbf{x}$ if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\max}$;
- we have $\mathbf{x}^*\mathbf{N}\mathbf{x} = \lambda_{\min}\, \mathbf{x}^*\mathbf{D}\mathbf{x}$ if $\mathbf{x} = \mathbf{D}^{-1/2}\mathbf{y}$, where $\mathbf{y}$ is an eigenvector of $\mathbf{M}$ associated with $\lambda_{\min}$; and
- the eigenvalues of $\mathbf{N}\mathbf{D}^{-1}$ are real, $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$ and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{N}\mathbf{D}^{-1}$.

Proof: This is a direct consequence of Theorem 1. □

Corollary 4. Let $\mathbf{N}$ and $\mathbf{D}$ be hermitian matrices of size n by n, $\mathbf{D}$ being positive semidefinite. Let d be the nullity of $\mathbf{D}$. We assume that $\mathbf{D} \neq \mathbf{0}$ and $\ker \mathbf{D} \subset \ker \mathbf{N}$.

$\mathbf{D}$ being positive semidefinite, it has n eigenvalues, counting multiplicities, and these values are real and nonnegative. Let us label these eigenvalues according to a non-decreasing order $\mu_1, \ldots, \mu_n$. Since $\mathbf{D} \neq \mathbf{0}$, we have $d \leqslant n-1$, so that $0 < \mu_{d+1} \leqslant \ldots \leqslant \mu_n$. For any positive integer i such that $i \leqslant d$, we have $\mu_i = 0$. $\mathbf{D}$ being hermitian, there exists a unitary matrix $\mathbf{L}$ of size n by n such that $\mathbf{D} = \mathbf{L}\operatorname{diag}_n(\mu_1, \ldots, \mu_n)\mathbf{L}^*$.

For any $i \in \{1, \ldots, n\}$, let the i-th column vector of $\mathbf{L}$ be denoted by $\mathbf{L}^{<i>}$. Let $\mathcal{L}$ be the submatrix of $\mathbf{L}$, of size n by $n-d$, whose column vectors are $\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<n>}$, in this order. Let $\mathbf{P} = \mathcal{L}\operatorname{diag}_{n-d}(\mu_{d+1}^{-1/2}, \ldots, \mu_n^{-1/2})$ and $\mathbf{Q} = \mathbf{P}^*\mathbf{N}\mathbf{P}$. The matrix $\mathbf{Q}$ is hermitian, of size $n-d$ by $n-d$. Thus, its eigenvalues are real. Let $\kappa_{\max}$ be the largest eigenvalue of $\mathbf{Q}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{Q}$. For any $\mathbf{x} \in \mathbb{C}^n$, we have

$$\kappa_{\min}\, \mathbf{x}^*\mathbf{D}\mathbf{x} \leqslant \mathbf{x}^*\mathbf{N}\mathbf{x} \leqslant \kappa_{\max}\, \mathbf{x}^*\mathbf{D}\mathbf{x}. \tag{23}$$

Moreover,

- we have $\mathbf{x}^*\mathbf{N}\mathbf{x} = \kappa_{\max}\, \mathbf{x}^*\mathbf{D}\mathbf{x}$ if $\mathbf{x} = \mathbf{P}\mathbf{w}$, where $\mathbf{w}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\max}$;
- we have $\mathbf{x}^*\mathbf{N}\mathbf{x} = \kappa_{\min}\, \mathbf{x}^*\mathbf{D}\mathbf{x}$ if $\mathbf{x} = \mathbf{P}\mathbf{w}$, where $\mathbf{w}$ is an eigenvector of $\mathbf{Q}$ associated with $\kappa_{\min}$; and
- the matrix $\mathbf{R} = \mathcal{L}^*\mathbf{N}\mathcal{L}\operatorname{diag}_{n-d}(\mu_{d+1}^{-1}, \ldots, \mu_n^{-1})$ has only real eigenvalues, $\kappa_{\max}$ is the largest eigenvalue of $\mathbf{R}$ and $\kappa_{\min}$ the smallest eigenvalue of $\mathbf{R}$.

Proof: This is a direct consequence of Theorem 3. □

Observation 5. A result similar to Corollary 3 was obtained in Theorem 3 and Theorem 5 of [21] and in Theorem 7 of [15]. Thus, Theorem 1 and Corollary 3 are not new. However, Theorem 2, Theorem 3, Corollary 1, Corollary 2 and Corollary 4 seem to be new.

III. EXAMPLES OF GENERALIZED RAYLEIGH RATIOS

A. USE OF AN EXTREMUM-SEEKING ALGORITHM

An extremum-seeking algorithm can be used to approximate the least upper bound (LUB) and greatest lower bound (GLB) of $r(\mathbf{x})$ for $\mathbf{x} \in D_r$, instead of computing them as eigenvalues according to Theorem 3, or as eigenvalues according to Theorem 1 if $\mathbf{D}$ is positive definite.

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It follows from Observation 2 that, to approximate the maximum and minimum values of $r(\mathbf{x})$, an extremum-seeking algorithm may posit $\mathbf{x} \in D_r \cap \mathbb{S}_n$, and further assume that one of the entries of $\mathbf{x}$ is real and nonnegative. These observations lead to simple parametrizations using only $2(n-1)$ real parameters. For instance, for $n = 4$, and under the assumption $\mathbb{S}_n \subset D_r$, the numerical algorithm can use

$$\mathbf{x} = \begin{pmatrix} \sin\theta_1 \sin\theta_2 \sin\theta_3 \exp j\phi_3 \\ \sin\theta_1 \sin\theta_2 \cos\theta_3 \exp j\phi_2 \\ \sin\theta_1 \cos\theta_2 \exp j\phi_1 \\ \cos\theta_1 \end{pmatrix} \tag{24}$$

where $\theta_1 \in [0, \pi/2]$, $\theta_2 \in [0, \pi/2]$, $\theta_3 \in [0, \pi/2]$, $\phi_1 \in [-\pi, \pi]$, $\phi_2 \in [-\pi, \pi]$ and $\phi_3 \in [-\pi, \pi]$. Thus, for $n = 4$, to estimate the bounds of $r(\mathbf{x})$ for $\mathbf{x} \in D_r$, an extremum-seeking algorithm may solve a problem having only 6 real unknowns each lying in a bounded interval. Here, we use 6 real parameters to obtain all vectors of a subset C of $\mathbb{S}_n$, C being the set of the vectors of $\mathbb{C}^4$, of unit length and having a real and nonnegative last entry. The set C can be also be viewed as the intersection of the unit hypersphere of $\mathbb{R}^8$ and a closed half-hyperplane of $\mathbb{R}^8$.

B. FIRST EXAMPLE

In a first example, using the polar decomposition of two arbitrary matrices, we obtain two arbitrary unitary matrices $\mathbf{U}_{N1}$ and $\mathbf{U}_{D1}$. These unitary matrices are used to create an arbitrary hermitian matrix $\mathbf{N}$ having known and arbitrary integer eigenvalues, and an arbitrary positive definite matrix $\mathbf{D}$ having known and arbitrary positive integer eigenvalues. More precisely, we use

$$\mathbf{N} = \mathbf{U}_{N1} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \mathbf{U}_{N1}^* \tag{25}$$

and

$$\mathbf{D} = \mathbf{U}_{D1} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \mathbf{U}_{D1}^* \,. \tag{26}$$

The computed values of $\mathbf{N}$ and $\mathbf{D}$ are very approximately

$$\mathbf{N} \simeq \begin{pmatrix} -1.4 & -0.8-0.0j & 2.5+0.3j & -2.1+1.5j \\ -0.8+0.0j & 5.2 & -0.7+0.2j & 1.2+0.1j \\ 2.5-0.3j & -0.7-0.2j & 6.2 & 1.1+0.3j \\ -2.1-1.5j & 1.2-0.1j & 1.1-0.3j & 6.1 \end{pmatrix} \tag{27}$$

and

$$\mathbf{D} \simeq \begin{pmatrix} 2.8 & 0.3+0.3j & -0.9-0.2j & -0.2-0.6j \\ 0.3-0.3j & 5.0 & 0.1-0.2j & 1.0+0.7j \\ -0.9+0.2j & 0.1+0.2j & 4.5 & 0.5+0.6j \\ -0.2+0.6j & 1.0-0.7j & 0.5-0.6j & 3.7 \end{pmatrix} . \tag{28}$$

Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$. The LUB and GLB of $r(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{0}$ have been computed using three independent methods: a numerical method based on extremum-seeking, using 6 real parametrers according to (24); a computation of eigenvalues according to Theorem 1; and a computation of eigenvalues according to Theorem 3. In the methods based on Theorem 1 and Theorem 3, we only used the computed values of $\mathbf{N}$ and $\mathbf{D}$, that is, we ignored (25) and (26). The three methods give exactly the same values, as shown in Table 1.

TABLE 1. Results for the first example, according to 3 methods.

Quantity	Numerical	Theorem 1	Theorem 3
LUB of $r(\mathbf{x})$	2.675391	2.675391	2.675391
GLB of $r(\mathbf{x})$	-0.880882	-0.880882	-0.880882

The method based on Theorem 3 using a diagonalization of $\mathbf{D}$ and a computation of the eigenvalues of $\mathbf{Q}$ given by (11), it may seem much more involved than the method based on Theorem 1. This is not actually the case, because the method based on Theorem 1 includes a computation of $\mathbf{D}^{1/2}$, which typically requires a diagonalization, and a computation of the eigenvalues of $\mathbf{M}$ given by (3).

Before using Theorem 1, it is a good practice to check that $\mathbf{D}$ is accurately invertible. Since $\mathbf{D}$ is positive semidefinite, this is best achieved by computing its condition number for matrix inversion with respect to the spectral norm [16, Sec. 5.8], which is the ratio of its largest eigenvalue to its smallest eigenvalue by [16, Sec. 7.3.P1].

C. SECOND EXAMPLE

In a second example, to obtain an arbitrary positive semidefinite matrix $\mathbf{D}$ of nullity $d = 2$, we use

$$\mathbf{D} = \mathbf{U}_{D1} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{U}_{D1}^* \,. \tag{29}$$

The computed value of $\mathbf{D}$ is very approximately

$$\mathbf{D} \simeq \begin{pmatrix} 4.3 & -0.9-0.8j & 1.6+0.2j & 0.3+1.3j \\ -0.9+0.8j & 0.9 & 0.2+0.2j & -1.2-0.9j \\ 1.6-0.2j & 0.2-0.2j & 1.1 & -0.6-0.3j \\ 0.3-1.3j & -1.2+0.9j & -0.6+0.3j & 2.6 \end{pmatrix} . \tag{30}$$

By Corollary 1, to ensure that $r(D_r)$ is bounded, we need to make sure that $\ker \mathbf{D} \subset \ker \mathbf{N}$. Thus, the eigenvectors of $\mathbf{D}$ associated with the eigenvalue 0, that is to say the third and fourth column vectors of $\mathbf{U}_{D1}$, denoted by $\mathbf{U}_{D1}^{<3>}$ and $\mathbf{U}_{D1}^{<4>}$, respectively, must also be eigenvectors of $\mathbf{N}$ associated with the eigenvalue 0. Let $\mathbf{U}_{N1}^{<1>}$ and $\mathbf{U}_{N1}^{<2>}$ be the first and second column vectors of $\mathbf{U}_{N1}$, respectively. The vectors $\mathbf{U}_{D1}^{<3>}$, $\mathbf{U}_{D1}^{<4>}$, $\mathbf{U}_{N1}^{<1>}$ and $\mathbf{U}_{N1}^{<2>}$ being linearly independent, we use a Gram-Schmidt orthonormalization process to obtain an arbitrary unitary matrix $\mathbf{U}_{N2}$ whose first and second column vectors are $\mathbf{U}_{D1}^{<3>}$ and $\mathbf{U}_{D1}^{<4>}$, respectively. We posit

$$\mathbf{N} = \mathbf{U}_{N2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \mathbf{U}_{N2}^* \,. \tag{31}$$

The arbitrary positive semidefinite matrix $\mathbf{D}$ defined by (29) and the arbitrary hermitian matrix $\mathbf{N}$ defined by (31) have known and arbitrary integer eigenvalues, and are such

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that $\ker \mathbf{D} \subset \ker \mathbf{N}$. The computed value of $\mathbf{N}$ is very approximately

$$\mathbf{N} \simeq \begin{pmatrix} 6.2 & -1.3-0.3j & 2.4+1.2j & 1.6+0.6j \\ -1.3+0.3j & -0.2 & -1.1-0.1j & 0.5+0.6j \\ 2.4-1.2j & -1.1+0.1j & 0.7 & 1.4+0.7j \\ 1.6-0.6j & 0.5-0.6j & 1.4-0.7j & -1.6 \end{pmatrix} \quad (32)$$

Let r be the generalized Rayleigh ratio of $\mathbf{N}$ to $\mathbf{D}$. The LUB and GLB of $r(\mathbf{x})$ for $\mathbf{x} \neq \mathbf{0}$ have been computed using two independent methods: a numerical method based on extremum-seeking, using 6 real parametrers according to (24); and a computation of eigenvalues according to Theorem 3. In the method based on Theorem 3, we only used the assumption $\ker \mathbf{D} \subset \ker \mathbf{N}$ and the computed values of $\mathbf{N}$ and $\mathbf{D}$. That is, we ignored (29) and (31). Both methods give exactly the same values, as shown in Table 2.

TABLE 2. Results for the second example, according to 2 methods.

Quantity	Numerical	Theorem 3
LUB of $r(\mathbf{x})$	1.584227	1.584227
GLB of $r(\mathbf{x})$	-0.841630	-0.841630

In the method based on Theorem 3, none of the computed eigenvalues of $\mathbf{D}$ was exactly zero. We have of course defined how small an eigenvalue of $\mathbf{D}$ must be, compared to the largest eigenvalue of $\mathbf{D}$, to be considered as equal to zero in the computation of $\mathbf{P}$ given by (10). Inaccuracies in computed values is also the reason why $\ker \mathbf{D} \subset \ker \mathbf{N}$ is an assumption of the method based on Theorem 3.

IV. ASSUMPTIONS AND AVAILABLE POWER

Fig. 1 shows the configuration considered in this article, in which an LTI MG having N ports is coupled to an LTI and passive MAA having N ports. If an actual setup comprises a transmitter having N antenna ports, feeders and N antennas, the feeders may be regarded as parts of the MG, or as parts of the MAA. Likewise, if an actual setup comprises a MIMO matching and/or decoupling network, or a MIMO antenna tuner, it may be regarded as a part of the MG, or as a part of the MAA.

The configuration shown in Fig. 1 operates in the harmonic steady state, at a given frequency f_G. We assume that: the ports of the MG are numbered from 1 to N; the ports of the MAA are numbered from 1 to N; and, for any integer $p \in \{1,\ldots,N\}$, port p of the MG is connected to port p of the MAA (positive terminal to positive terminal and negative terminal to negative terminal).

We assume that, at the frequency f_G, the MG has an impedance matrix (which may also be referred to as "internal impedance matrix"), denoted by $\mathbf{Z}_G$. This complex matrix is of size N by N.

Let $\mathbf{M}$ be a square complex matrix. The hermitian part of $\mathbf{M}$, denoted by $H(\mathbf{M})$, is the matrix given by

$$H(\mathbf{M}) = \frac{\mathbf{M}+\mathbf{M}^*}{2} . \quad (33)$$

The available power is defined as the greatest average power that can be drawn from the generator by an arbitrary

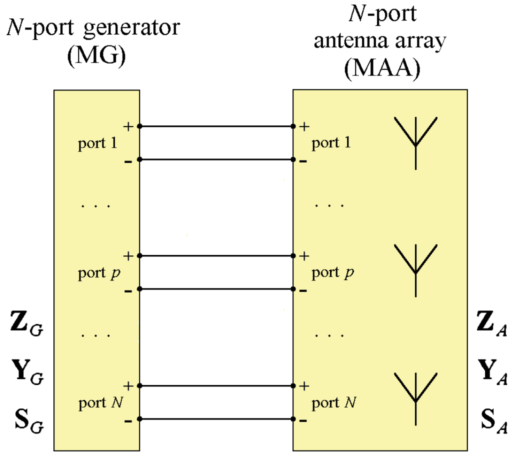

FIGURE 1. The MG and the MAA. $\mathbf{Z}_A$ and $\mathbf{Y}_A$ need not exist.

LTI and passive load [22, Sec. 3-8]. We assume that $H(\mathbf{Z}_G)$ is positive definite, so that, according to the maximum power transfer theorem for N-ports [23], the available power of the MG is defined, finite and nonnegative. We use P_{AVG} to denote the available power of the MG. Ignoring noise power contributions, and using the fact that, $H(\mathbf{Z}_G)$ being positive definite, it is invertible by [16, Sec. 7.1.7], we get [23]:

$$P_{AVG} = \mathbf{V}_{OG}^* \, \mathbf{Y}_{AVGO} \, \mathbf{V}_{OG} \, , \quad (34)$$

where $\mathbf{V}_{OG}$ is the column vector of the rms open-circuit voltages at ports 1 to N of the MG, and where the admittance matrix

$$\mathbf{Y}_{AVGO} = \frac{1}{2} \left(\mathbf{Z}_G + \mathbf{Z}_G^*\right)^{-1} \quad (35)$$

is positive definite.

Since $H(\mathbf{Z}_G)$ is positive definite, it follows from Lemma 1 of [24]-[25] and [21] that $\mathbf{Z}_G$ is invertible, $\mathbf{Y}_G = \mathbf{Z}_G^{-1}$ being such that $H(\mathbf{Y}_G)$ is positive definite. $\mathbf{Y}_G$ is the admittance matrix (which may also be referred to as "internal admittance matrix") of the MG. It also follows from this Lemma 1 that, instead of assuming that $\mathbf{Z}_G$ exists and that $H(\mathbf{Z}_G)$ is positive definite, we could equivalently have assumed that $\mathbf{Y}_G$ exists and that $H(\mathbf{Y}_G)$ is positive definite.

$H(\mathbf{Y}_G)$ being positive definite, it is invertible, so that, ignoring noise power contributions, P_{AVG} is also given by

$$P_{AVG} = \mathbf{I}_{SG}^* \, \mathbf{Z}_{AVGS} \, \mathbf{I}_{SG} \, , \quad (36)$$

where $\mathbf{I}_{SG} = \mathbf{Y}_G \mathbf{V}_{OG}$ is the column vector of the rms short-circuit currents at ports 1 to N of the MG (we use reference directions that are associated at each port of the MAA), and where the impedance matrix

$$\mathbf{Z}_{AVGS} = \frac{1}{2} \left(\mathbf{Y}_G + \mathbf{Y}_G^*\right)^{-1} \quad (37)$$

is positive definite.

$\mathbf{V}_{OG}$ and $\mathbf{I}_{SG}$ may take on any value lying in $\mathbb{C}^N$. The MAA need neither have an impedance matrix nor an admittance matrix. If it has an impedance matrix, it is denoted by $\mathbf{Z}_A$. If it has an admittance matrix, it is denoted by $\mathbf{Y}_A$.

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V. USING OTHER VARIABLES

We have expressed the available power as a function of the variables $\mathbf{V}_{OG}$ and $\mathbf{I}_{SG}$. We now want to explore the possibility of using other variables: $\mathbf{V}$, $\mathbf{I}$, $\mathbf{a}$ and $\hat{\mathbf{a}}$ defined below. In what follows, we assume that there is no incident electromagnetic signal received by the MAA, and we ignore noise power contributions. To avoid unnecessary assumptions, we will use the theory of parallel-augmented multiports and series-augmented multiports presented in Section II of [24]-[25] and also in Section II of [21]. Appendix A presents alternative derivations which do not use this theory but require more assumptions.

Let us introduce a parallel-augmented multiport composed of the MAA (as original multiport), and of an N-port passive network of admittance matrix $\mathbf{Y}_G$. By Theorem 1 of [21], the MAA being passive, the parallel-augmented multiport has an impedance matrix $\mathbf{Z}_{PAM}$, of size N by N and such that $H(\mathbf{Z}_{PAM})$ is positive semidefinite. By inspection, we find that the column vector of the rms voltages at ports 1 to N of the MAA, denoted by $\mathbf{V}$, is given by

$$\mathbf{V} = \mathbf{Z}_{PAM}\,\mathbf{I}_{SG}\,. \tag{38}$$

By Corollary 1 of [21], $\mathbf{Z}_{PAM}$ is invertible if and only if the MAA has an admittance matrix $\mathbf{Y}_A$. In this case, we have

$$\mathbf{Z}_{PAM}^{-1} = \mathbf{Y}_A + \mathbf{Y}_G\,, \tag{39}$$

$\mathbf{V}$ can take on any value lying in $\mathbb{C}^N$, and (36)–(37) lead us to

$$P_{AVG} = \mathbf{V}^*\ \mathbf{Y}_{AVG}\,\mathbf{V}\,, \tag{40}$$

where the admittance matrix

$$\mathbf{Y}_{AVG} = \frac{1}{2}\mathbf{Z}_{PAM}^{-1*}\left(\mathbf{Y}_G + \mathbf{Y}_G^*\right)^{-1}\mathbf{Z}_{PAM}^{-1} \tag{41}$$

is positive definite.

To discuss the next variable, we introduce a series-augmented multiport, composed of the MAA (as original multiport), and of an N-port passive network of impedance matrix $\mathbf{Z}_G$. By Theorem 2 of [21], the MAA being passive, the series-augmented multiport has an admittance matrix $\mathbf{Y}_{SAM}$, of size N by N and such that $H(\mathbf{Y}_{SAM})$ is positive semidefinite. The column vector of the rms currents flowing into ports 1 to N of the MAA, denoted by $\mathbf{I}$, is given by

$$\mathbf{I} = \mathbf{Y}_{SAM}\,\mathbf{V}_{OG}\,. \tag{42}$$

By Corollary 2 of [21], $\mathbf{Y}_{SAM}$ is invertible if and only if the MAA has an impedance matrix $\mathbf{Z}_A$. In this case, we have

$$\mathbf{Y}_{SAM}^{-1} = \mathbf{Z}_A + \mathbf{Z}_G\,, \tag{43}$$

$\mathbf{I}$ can take on any value lying in $\mathbb{C}^N$, and (34)–(35) lead us to

$$P_{AVG} = \mathbf{I}^*\ \mathbf{Z}_{AVG}\,\mathbf{I}\,, \tag{44}$$

where the impedance matrix

$$\mathbf{Z}_{AVG} = \frac{1}{2}\mathbf{Y}_{SAM}^{-1*}\left(\mathbf{Z}_G + \mathbf{Z}_G^*\right)^{-1}\mathbf{Y}_{SAM}^{-1} \tag{45}$$

is positive definite.

There is a relationship between $\mathbf{Y}_{SAM}$ and $\mathbf{Z}_{PAM}$, since it follows from (38), (42), $\mathbf{V}_{OG} = \mathbf{Z}_G\mathbf{I}_{SG}$ and

$$\mathbf{I}_{SG} = \mathbf{I} + \mathbf{Y}_G\mathbf{V} \tag{46}$$

that, for any $\mathbf{I}_{SG} \in \mathbb{C}^N$, we have

$$\mathbf{I}_{SG} = (\mathbf{Y}_{SAM}\mathbf{Z}_G + \mathbf{Y}_G\mathbf{Z}_{PAM})\mathbf{I}_{SG}\,. \tag{47}$$

Consequently,

$$\mathbf{Y}_{SAM}\mathbf{Z}_G + \mathbf{Y}_G\mathbf{Z}_{PAM} = \mathbf{1}_N\,. \tag{48}$$

To address the next variable, we define the diagonal matrix $\mathbf{r}_0 = \mathrm{diag}_N(r_{01}, \ldots, r_{0N})$, of size N by N, the diagonal entries of which are N arbitrary positive reference resistances $r_{01}, \ldots, r_{0N}$. We have $\mathbf{r}_0^{1/2} = \mathrm{diag}_N(\sqrt{r_{01}}, \ldots, \sqrt{r_{0N}})$. For the reference resistances $r_{01}, \ldots, r_{0N}$, the column vector of the normalized rms incident voltages at ports 1 to N of the MAA is denoted by $\mathbf{a}$ and given by [26]:

$$\mathbf{a} = \mathbf{r}_0^{-1/2}\,\frac{\mathbf{V} + \mathbf{r}_0\mathbf{I}}{2}\,. \tag{49}$$

We can use (38), (42) and $\mathbf{V}_{OG} = \mathbf{Z}_G\mathbf{I}_{SG}$ to obtain

$$\mathbf{a} = \mathbf{r}_0^{-1/2}\,\mathbf{Z}_{PSG}\,\mathbf{I}_{SG}\,, \tag{50}$$

where

$$\mathbf{Z}_{PSG} = \frac{\mathbf{Z}_{PAM} + \mathbf{r}_0\mathbf{Y}_{SAM}\mathbf{Z}_G}{2}\,. \tag{51}$$

Using (48) in (51), we also get

$$\mathbf{Z}_{PSG} = \frac{\mathbf{Z}_{PAM} + \mathbf{r}_0(\mathbf{1}_N - \mathbf{Y}_G\mathbf{Z}_{PAM})}{2}\,. \tag{52}$$

Let $\mathbf{b}$ be the column vector of the normalized rms reflected voltages at ports 1 to N of the MAA, given by [26]

$$\mathbf{b} = \mathbf{r}_0^{-1/2}\,\frac{\mathbf{V} - \mathbf{r}_0\mathbf{I}}{2}\,. \tag{53}$$

As explained in Appendix B, for the reference resistances $r_{01}, \ldots, r_{0N}$, the scattering matrix $\mathbf{S}_A$ of the MAA exists. Thus, $\mathbf{a} = \mathbf{0}$ entails $\mathbf{b} = \mathbf{S}_A\,\mathbf{a} = \mathbf{0}$. Thus, for any column vector $\mathbf{I}_{SG} \in \mathbb{C}^N$ such that $\mathbf{a}$ given by (50) satisfies $\mathbf{a} = \mathbf{0}$, it follows from $\mathbf{V} = \mathbf{r}_0^{1/2}(\mathbf{a} + \mathbf{b})$ that $\mathbf{V} = \mathbf{0}$, it follows from $\mathbf{I} = \mathbf{r}_0^{-1/2}(\mathbf{a} - \mathbf{b})$ that $\mathbf{I} = \mathbf{0}$, and it follows from (46) that $\mathbf{I}_{SG} = \mathbf{0}$. Thus, $\mathbf{Z}_{PSG}$ is invertible and $\mathbf{a}$ can take on any value lying in $\mathbb{C}^N$. It follows that we can define the admittance matrix $\mathbf{Y}_{PSG} = \mathbf{Z}_{PSG}^{-1}$. By (36)–(37) and (50), we get

$$P_{AVG} = \mathbf{a}^*\ \mathbf{\Lambda}_{AVG}\,\mathbf{a}\,, \tag{54}$$

where the dimensionless matrix

$$\mathbf{\Lambda}_{AVG} = \frac{1}{2}\mathbf{r}_0^{1/2}\,\mathbf{Y}_{PSG}^*\left(\mathbf{Y}_G + \mathbf{Y}_G^*\right)^{-1}\mathbf{Y}_{PSG}\,\mathbf{r}_0^{1/2} \tag{55}$$

is positive definite.

In (55), we can replace $\mathbf{Y}_G$ and $\mathbf{Y}_{PSG}$ with their values expressed using the scattering matrix $\mathbf{S}_G$ of the MG, and $\mathbf{S}_A$. Since $\mathbf{Y}_G$ exists, $\mathbf{1}_N + \mathbf{S}_G$ is invertible and we can use [26]

$$\mathbf{Y}_G = \mathbf{r}_0^{-1/2}(\mathbf{1}_N - \mathbf{S}_G)(\mathbf{1}_N + \mathbf{S}_G)^{-1}\mathbf{r}_0^{-1/2} \tag{56}$$

to replace $\mathbf{Y}_G$. A more general version of this question is strictly covered in Appendix C, and the classical formula (56) is a consequence of (223), for $\mathbf{z}_0 = \mathbf{r}_0$.

Using (46), (49) and (53), we get

$$\mathbf{I}_{SG} = \left[\mathbf{r}_0^{-1/2}(\mathbf{1}_N - \mathbf{S}_A) + \mathbf{Y}_G\, \mathbf{r}_0^{1/2}(\mathbf{1}_N + \mathbf{S}_A)\right] \mathbf{a}\,. \quad (57)$$

For any $\mathbf{I}_{SG} \in \mathbb{C}^N$, by (50) we have $\mathbf{I}_{SG} = \mathbf{Y}_{PSG}\, \mathbf{r}_0^{1/2}\mathbf{a}$, and we also have (57). This leads us to

$$\mathbf{Y}_{PSG} = \left[\mathbf{r}_0^{-1/2}(\mathbf{1}_N - \mathbf{S}_A) + \mathbf{Y}_G\, \mathbf{r}_0^{1/2}(\mathbf{1}_N + \mathbf{S}_A)\right] \mathbf{r}_0^{-1/2}\,. \quad (58)$$

Using (56), we get

$$\mathbf{Y}_{PSG} = \mathbf{r}_0^{-1/2}\Big[\mathbf{1}_N - \mathbf{S}_A + (\mathbf{1}_N - \mathbf{S}_G)(\mathbf{1}_N + \mathbf{S}_G)^{-1}(\mathbf{1}_N + \mathbf{S}_A)\Big]\mathbf{r}_0^{-1/2}\,. \quad (59)$$

We may use (56) and (59) to replace $\mathbf{Y}_G$ and $\mathbf{Y}_{PSG}$ in (55), and obtain $\mathbf{\Lambda}_{AVG}$ as a function of $\mathbf{S}_G$ and $\mathbf{S}_A$. The resulting equation is cumbersome and ugly:

$$\begin{aligned}\mathbf{\Lambda}_{AVG} = &\frac{1}{2}\left[\mathbf{1}_N - \mathbf{S}_A^* + (\mathbf{1}_N + \mathbf{S}_A^*)(\mathbf{1}_N + \mathbf{S}_G^*)^{-1}(\mathbf{1}_N - \mathbf{S}_G^*)\right] \\ &\times \left[(\mathbf{1}_N - \mathbf{S}_G)(\mathbf{1}_N + \mathbf{S}_G)^{-1} + (\mathbf{1}_N + \mathbf{S}_G^*)^{-1}(\mathbf{1}_N - \mathbf{S}_G^*)\right]^{-1} \\ &\times \left[\mathbf{1}_N - \mathbf{S}_A + (\mathbf{1}_N - \mathbf{S}_G)(\mathbf{1}_N + \mathbf{S}_G)^{-1}(\mathbf{1}_N + \mathbf{S}_A)\right]. \end{aligned} \quad (60)$$

However, in the special case where $\mathbf{Y}_G = \mathbf{r}_0^{-1}$, (52) or (58) lead us to $\mathbf{Y}_{PSG} = 2\mathbf{r}_0^{-1}$, (55) to

$$\mathbf{\Lambda}_{AVG} = \mathbf{1}_N\,, \quad (61)$$

and (54) to the classical result

$$P_{AVG} = \mathbf{a}^*\, \mathbf{a}\,. \quad (62)$$

Let $\mathrm{Re}(z)$ denote the real part of a complex number z. To treat the last variable, let $\mathbf{z}_0 = \mathrm{diag}_N(z_{01}, \ldots, z_{0N})$, where the N arbitrary reference impedances $z_{01}, \ldots, z_{0N}$ are such that, for any $p \in \{1, \ldots, N\}$, $r_{0p} = \mathrm{Re}(z_{0p})$ is positive. Let $\mathbf{r}_0 = \mathrm{diag}_N(r_{01}, \ldots, r_{0N})$. For the reference impedances $z_{01}, \ldots, z_{0N}$, the column vector of the rms power waves incident at ports 1 to N of the MAA, denoted by $\hat{\mathbf{a}}$, is [27]:

$$\hat{\mathbf{a}} = \mathbf{r}_0^{-1/2}\, \frac{\mathbf{V} + \mathbf{z}_0\mathbf{I}}{2}\,. \quad (63)$$

$\hat{\mathbf{a}}$ is also the column vector of the rms pseudo-waves incident at ports 1 to N of the MAA [28], denoted by $\breve{\mathbf{a}}$. Using (38), (42) and $\mathbf{V}_{OG} = \mathbf{Z}_G\mathbf{I}_{SG}$, we get

$$\hat{\mathbf{a}} = \breve{\mathbf{a}} = \mathbf{r}_0^{-1/2}\, \hat{\mathbf{Z}}_{PSG}\, \mathbf{I}_{SG}\,, \quad (64)$$

where

$$\hat{\mathbf{Z}}_{PSG} = \frac{\mathbf{Z}_{PAM} + \mathbf{z}_0\mathbf{Y}_{SAM}\mathbf{Z}_G}{2}\,. \quad (65)$$

Using (48) in (65), we obtain

$$\hat{\mathbf{Z}}_{PSG} = \frac{\mathbf{Z}_{PAM} + \mathbf{z}_0(\mathbf{1}_N - \mathbf{Y}_G\mathbf{Z}_{PAM})}{2}\,. \quad (66)$$

Let $\breve{\mathbf{b}}$ be the column vector of the rms pseudo-waves reflected at ports 1 to N of the MAA, defined by [28]:

$$\breve{\mathbf{b}} = \mathbf{r}_0^{-1/2}\, \frac{\mathbf{V} - \mathbf{z}_0\mathbf{I}}{2}\,. \quad (67)$$

As shown in Appendix B, the pseudo-wave scattering matrix $\breve{\mathbf{S}}_A$ of the MAA exists, and $\breve{\mathbf{b}} = \breve{\mathbf{S}}\,\breve{\mathbf{a}}$. Thus, for any column vector $\mathbf{I}_{SG} \in \mathbb{C}^N$ such that $\hat{\mathbf{a}}$ given by (64) satisfies $\hat{\mathbf{a}} = \breve{\mathbf{a}} = \mathbf{0}$, we have $\breve{\mathbf{b}} = \mathbf{0}$, it follows from $\mathbf{V} = \mathbf{r}_0^{1/2}(\breve{\mathbf{a}} + \breve{\mathbf{b}})$ that $\mathbf{V} = \mathbf{0}$, it follows from $\mathbf{I} = \mathbf{z}_0^{-1}\mathbf{r}_0^{1/2}(\breve{\mathbf{a}} - \breve{\mathbf{b}})$ that $\mathbf{I} = \mathbf{0}$, and it follows from (46) that $\mathbf{I}_{SG} = \mathbf{0}$. Thus, $\hat{\mathbf{Z}}_{PSG}$ is invertible and $\hat{\mathbf{a}}$ can take on any value lying in $\mathbb{C}^N$. Thus, by (36)–(37) and (64), we get

$$P_{AVG} = \hat{\mathbf{a}}^*\, \hat{\mathbf{\Lambda}}_{AVG}\, \hat{\mathbf{a}}\,, \quad (68)$$

where the dimensionless matrix

$$\hat{\mathbf{\Lambda}}_{AVG} = \frac{1}{2}\mathbf{r}_0^{1/2}\hat{\mathbf{Z}}_{PSG}^{-1*}\left(\mathbf{Y}_G + \mathbf{Y}_G^*\right)^{-1}\hat{\mathbf{Z}}_{PSG}^{-1}\mathbf{r}_0^{1/2} \quad (69)$$

is positive definite.

In the special case where $\mathbf{Y}_G = \mathbf{z}_0^{-1}$, (66) leads us to $\hat{\mathbf{Z}}_{PSG} = \mathbf{z}_0/2$, (69) to

$$\hat{\mathbf{\Lambda}}_{AVG} = 2\mathbf{r}_0^{1/2}\mathbf{z}_0^{-1*}\left(\mathbf{z}_0^{-1} + \mathbf{z}_0^{-1*}\right)^{-1}\mathbf{z}_0^{-1}\mathbf{r}_0^{1/2} = \mathbf{1}_N\,, \quad (70)$$

and (68) to the well-known result

$$P_{AVG} = \hat{\mathbf{a}}^*\, \hat{\mathbf{a}}\,. \quad (71)$$

Clearly, the most convenient variables to compute P_{AVG} in the general case are $\mathbf{V}_{OG}$ and $\mathbf{I}_{SG}$, because (35) and (37) contain only parameters of the MG: $\mathbf{Z}_G$ in the case of (35); or $\mathbf{Y}_G$ in the case of (37). In contrast, (41) for the variable $\mathbf{V}$, (45) for the variable $\mathbf{I}$, (55) for the variable $\mathbf{a}$ and (69) for the variable $\hat{\mathbf{a}}$ are more involved because they depend on parameters of the MG and of the MAA, even though P_{AVG} is a characteristic of the MG alone.

It is important to note that: (38), (42), (46)–(60), and (63)–(69) are only based on the assumptions of Section IV; the variables $\mathbf{V}_{OG}$, $\mathbf{I}_{SG}$, $\mathbf{a}$ and $\hat{\mathbf{a}}$ can each always be used to determine P_{AVG}; and they can each take on any value lying in $\mathbb{C}^N$. In contrast, our analysis allows us to assert that: the variable $\mathbf{V}$ can be used to compute P_{AVG} only if the MAA has an admittance matrix; in the opposite case, $\mathbf{V}$ cannot take on any value lying in $\mathbb{C}^N$, since $\mathbf{Z}_{PAM}$ is not invertible; the variable $\mathbf{I}$ can be used to compute P_{AVG} only if the MAA has an impedance matrix; in the opposite case, $\mathbf{I}$ cannot take on any value lying in $\mathbb{C}^N$, since $\mathbf{Y}_{SAM}$ is not invertible.

VI. THE RADIATED POWER

In this Section VI, a passive LTI multiport having N ports numbered from 1 to N, of admittance matrix $\mathbf{Y}_G$ at the frequency f_G, is referred to as “added network”. For any integer $p \in \{1, \ldots, N\}$, we can consider the single-port antenna number p, designated as SPA-p and shown in Fig. 2, made up of the MAA and the added network, each port of the added network being coupled to the port of same number of the MAA (positive terminal to positive terminal and negative terminal to negative terminal), the port of SPA-p being port p of the MAA coupled to port p of the added network [29].

A coordinate system having its origin close to the MAA being chosen, let $\mathbf{h}_{0\,p}$ be the vector effective length of SPA-p in a direction (θ, φ), as defined in [30, Sec. 5.2] and [31, Sec. 16.5] (a different definition also exists [32, Sec. 2.15]).

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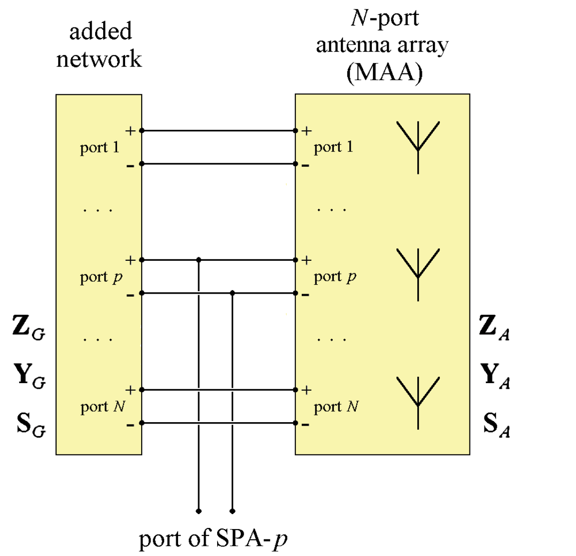

FIGURE 2. The single-port antenna SPA-p used in Section VI to define $\mathbf{E}_{0\,p}$.

Let $\mathbf{E}_{0\,p}$ be the rms electric field radiated by SPA-p used for emission, in the direction (θ, φ). At a large distance r of the origin, $\mathbf{E}_{0\,p}$ is given by

$$\mathbf{E}_{0\,p} = j\eta \frac{I_{0\,p}\, k\, e^{-jkr}}{4\pi r}\, \mathbf{h}_{0\,p} \tag{72}$$

where k is the wave number in the relevant medium, η is the intrinsic impedance of this medium, and $I_{0\,p}$ is a current flowing into the port of SPA-p.

If we now consider the configuration shown in Fig. 1, we find that the rms electric field radiated by the MAA used for emission, in the direction (θ, φ), denoted by $\mathbf{E}$, is given by

$$\mathbf{E} = j\eta \frac{k\, e^{-jkr}}{4\pi r} \sum_{p=1}^{N} I_{0\,p} \mathbf{h}_{0\,p}\,, \tag{73}$$

where $I_{0\,1}, \ldots, I_{0\,N}$ are the rms short-circuit currents at ports 1 to N of the MG, that is the entries of $\mathbf{I}_{SG}$. To derive (73), we have used a superposition of SPA-1 to SPA-N excited by the currents $I_{0\,1}$ to $I_{0\,N}$, respectively. This is possible because we know that $\mathbf{I}_{SG}$ may take on any value lying in $\mathbb{C}^N$.

The average power radiated by the MAA, denoted by P_{RAD}, being given by

$$P_{RAD} = \frac{1}{\eta} \iint \mathbf{E}^* \mathbf{E}\; r^2 \sin\theta \, d\theta d\varphi\,, \tag{74}$$

we get

$$P_{RAD} = \frac{\eta k^2}{16\pi^2} \sum_{p=1}^{N} \sum_{q=1}^{N} \bar{I}_{0\,p} I_{0\,q} \iint \mathbf{h}^*_{0\,p} \mathbf{h}_{0\,q}\; \sin\theta \, d\theta d\varphi\,. \tag{75}$$

Thus, we have

$$P_{RAD} = \mathbf{I}^*_{SG}\; \mathbf{Z}_{RADS}\; \mathbf{I}_{SG}\,, \tag{76}$$

where $\mathbf{Z}_{RADS}$ is a complex matrix of size N by N such that, for any integers p and q lying in $\{1, \ldots, N\}$, the entry of row p and column q of $\mathbf{Z}_{RAD}$ is

$$Z_{RADS\,pq} = \frac{\eta k^2}{16\pi^2} \iint \mathbf{h}^*_{0\,p} \mathbf{h}_{0\,q}\; \sin\theta \, d\theta d\varphi\,. \tag{77}$$

We see that $\mathbf{Z}_{RADS}$ is hermitian. Moreover, for any nonzero $\mathbf{I}_{SG} \in \mathbb{C}^N$, we know that P_{RADS} given by (77) must be nonnegative. It follows that $\mathbf{Z}_{RADS}$ is positive semidefinite. It follows from (76) that

$$P_{RAD} = \mathbf{V}^*_{OG}\; \mathbf{Y}_{RADO}\; \mathbf{V}_{OG}\,, \tag{78}$$

where the admittance matrix

$$\mathbf{Y}_{RADO} = \mathbf{Y}^*_G\; \mathbf{Z}_{RADS}\; \mathbf{Y}_G \tag{79}$$

is positive semidefinite.

In the case where the MAA has an admittance matrix, it follows from (38)-(39) and (76) that

$$P_{RAD} = \mathbf{V}^*\; \mathbf{Y}_{RAD}\; \mathbf{V}\,, \tag{80}$$

where the admittance matrix

$$\mathbf{Y}_{RAD} = \mathbf{Z}^{-1*}_{PAM}\; \mathbf{Z}_{RADS}\; \mathbf{Z}^{-1}_{PAM} \tag{81}$$

is positive semidefinite and depends only on the parameters of the MAA, since $\mathbf{V}$ and the MAA determine $\mathbf{I}$ and P_{RAD}.

In the case where the MAA has an impedance matrix, it follows from (42)-(43) and (78) that

$$P_{RAD} = \mathbf{I}^*\; \mathbf{Z}_{RAD}\; \mathbf{I}\,, \tag{82}$$

where the impedance matrix

$$\mathbf{Z}_{RAD} = \mathbf{Y}^{-1*}_{SAM}\; \mathbf{Y}_{RADO}\; \mathbf{Y}^{-1}_{SAM} \tag{83}$$

is positive semidefinite and depends only on the parameters of the MAA, since $\mathbf{I}$ and the MAA determine $\mathbf{V}$ and P_{RAD}. In Appendix D, we obtain an alternative derivation of (82), and we get (227), which may be used in the place of (83).

Since $\mathbf{Z}_{PSG}$ is invertible, it follows from (50) and (76) that

$$P_{RAD} = \mathbf{a}^*\; \mathbf{\Lambda}_{RAD}\; \mathbf{a}\,, \tag{84}$$

where the dimensionless matrix

$$\mathbf{\Lambda}_{RAD} = \mathbf{r}_0^{1/2}\; \mathbf{Y}^*_{PSG}\; \mathbf{Z}_{RADS}\; \mathbf{Y}_{PSG}\; \mathbf{r}_0^{1/2} \tag{85}$$

is positive semidefinite and depends only on the parameters of the MAA and $\mathbf{r}_0$, since the MAA, $\mathbf{r}_0$ and $\mathbf{a}$ determine $\mathbf{V}$ and $\mathbf{I}$, which determine P_{RAD}. In the special case where $\mathbf{Y}_G = \mathbf{r}_0^{-1}$, we have already observed that $\mathbf{Y}_{PSG} = 2\mathbf{r}_0^{-1}$, so that

$$\mathbf{\Lambda}_{RAD} = 4\, \mathbf{r}_0^{-1/2}\; \mathbf{Z}_{RADS}\; \mathbf{r}_0^{-1/2}\,. \tag{86}$$

Since $\hat{\mathbf{Z}}_{PSG}$ is invertible, it follows from (64) and (76) that

$$P_{RAD} = \hat{\mathbf{a}}^*\; \hat{\mathbf{\Lambda}}_{RAD}\; \hat{\mathbf{a}}\,, \tag{87}$$

where the dimensionless matrix

$$\hat{\mathbf{\Lambda}}_{RAD} = \mathbf{r}_0^{1/2}\; \hat{\mathbf{Z}}^{-1*}_{PSG}\; \mathbf{Z}_{RADS}\; \hat{\mathbf{Z}}^{-1}_{PSG}\; \mathbf{r}_0^{1/2} \tag{88}$$

is positive semidefinite and depends only on the parameters of the MAA and $\mathbf{z}_0$, since the MAA, $\mathbf{z}_0$ and $\hat{\mathbf{a}}$ determine $\mathbf{V}$ and $\mathbf{I}$, which determine P_{RAD}. In the special case where $\mathbf{Y}_G = \mathbf{z}_0^{-1}$, we have already seen that $\hat{\mathbf{Z}}_{PSG} = \mathbf{z}_0/2$, so that

$$\hat{\mathbf{\Lambda}}_{RAD} = 4\, \mathbf{r}_0^{1/2} \mathbf{z}_0^{-1*}\; \mathbf{Z}_{RADS}\; \mathbf{z}_0^{-1} \mathbf{r}_0^{1/2}\,. \tag{89}$$

We observe that P_{RAD} is only a function of the excitation and of the characteristics of the MAA, if the excitation is specified using any one of the variables $\mathbf{a}$ (for a fixed $\mathbf{r}_0$), or $\hat{\mathbf{a}}$ (for a fixed $\mathbf{z}_0$), or $\mathbf{V}$ if $\mathbf{Y}_A$ exists, or $\mathbf{I}$ if $\mathbf{Z}_A$ exists.

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VII. THE TRANSDUCER EFFICIENCY

In this Section VII, we mainly want to study the radiated power to available power ratio

$$e_T = \frac{P_{RAD}}{P_{AVG}} \,. \tag{90}$$

By analogy with the definitions of the transducer power gain [22, Sec. 21-18] and of the radiation efficiency [33], we will refer to e_T as the transducer efficiency of the MAA.

Let $\mathbf{X}$ denote one of the variables $\mathbf{V}_{OG}$, $\mathbf{I}_{SG}$, $\mathbf{V}$, $\mathbf{I}$, $\mathbf{a}$, or $\hat{\mathbf{a}}$. Based on the results of Sections IV-V about P_{AVG} and of Section VI about P_{RAD}, we find that, if the variable $\mathbf{X}$ is applicable, e_T is given by

$$e_T = \frac{\mathbf{X}^* \, \mathbf{M}_{RAD} \, \mathbf{X}}{\mathbf{X}^* \, \mathbf{M}_{AVG} \, \mathbf{X}} \,, \tag{91}$$

where the matrices $\mathbf{M}_{RAD}$ and $\mathbf{M}_{AVG}$ are both of size N by N, and given in Table 3. We note that $\mathbf{M}_{RAD}$ is always positive semidefinite, and $\mathbf{M}_{AVG}$ always positive definite.

TABLE 3. Variable $\mathbf{X}$ and associated $\mathbf{M}_{RAD}$ and $\mathbf{M}_{AVG}$.

Variable $\mathbf{X}$	Applicability	$\mathbf{M}_{RAD}$	$\mathbf{M}_{AVG}$
$\mathbf{V}_{OG}$	always	$\mathbf{Y}_{RADO}$	$\mathbf{Y}_{AVGO}$
$\mathbf{I}_{SG}$	always	$\mathbf{Z}_{RADS}$	$\mathbf{Z}_{AVGS}$
$\mathbf{V}$	if $\mathbf{Y}_A$ exists	$\mathbf{Y}_{RAD}$	$\mathbf{Y}_{AVG}$
$\mathbf{I}$	if $\mathbf{Z}_A$ exists	$\mathbf{Z}_{RAD}$	$\mathbf{Z}_{AVG}$
$\mathbf{a}$	always	$\mathbf{\Lambda}_{RAD}$	$\mathbf{\Lambda}_{AVG}$
$\hat{\mathbf{a}}$	always	$\hat{\mathbf{\Lambda}}_{RAD}$	$\hat{\mathbf{\Lambda}}_{AVG}$

The transducer efficiency of the MAA is given by (91) in the form of a generalized Rayleigh ratio of $\mathbf{M}_{RAD}$ to $\mathbf{M}_{AVG}$, in the variable $\mathbf{X}$. Since $\mathbf{M}_{AVG}$ is positive definite, e_T is defined for any nonzero $\mathbf{X} \in \mathbb{C}^N$.

Importantly, (91) shows that the transducer efficiency depends on the excitation. Also, we must keep in mind that, among the 6 variables listed in Table 3 to define the excitation, only $\mathbf{V}_{OG}$, $\mathbf{I}_{SG}$, $\mathbf{a}$, and $\hat{\mathbf{a}}$ are always applicable.

More generally, for any variable $\mathbf{X}$ given by $\mathbf{X} = \mathbf{C}\,\mathbf{I}_{SG}$, where $\mathbf{C}$ is an invertible matrix of size N by N defining the change of variable, we have

$$e_T = \frac{\mathbf{X}^* \, \mathbf{C}^{-1*} \, \mathbf{Z}_{RADS} \, \mathbf{C}^{-1} \, \mathbf{X}}{\mathbf{X}^* \, \mathbf{C}^{-1*} \, \mathbf{Z}_{AVGS} \, \mathbf{C}^{-1} \, \mathbf{X}} \,, \tag{92}$$

where, for instance, the matrices $\mathbf{C}$ corresponding to some possible variables $\mathbf{X}$ are shown in Table 4. To build Table 4, we have used (38), (42), (50) and (64).

TABLE 4. Variable $\mathbf{X}$ and corresponding $\mathbf{C}$.

Variable $\mathbf{X}$	Applicability	$\mathbf{C}$
$\mathbf{V}_{OG}$	always	$\mathbf{Z}_G$
$\mathbf{I}_{SG}$	always	$\mathbf{1}_N$
$\mathbf{V}$	if $\mathbf{Y}_A$ exists	$\mathbf{Z}_{PAM}$
$\mathbf{I}$	if $\mathbf{Z}_A$ exists	$\mathbf{Y}_{SAM}\mathbf{Z}_G$
$\mathbf{a}$	always	$\mathbf{r}_0^{-1/2}\mathbf{Z}_{PSG}$
$\hat{\mathbf{a}}$	always	$\mathbf{r}_0^{-1/2}\hat{\mathbf{Z}}_{PSG}$

The total active reflection coefficient (TARC), defined in the special case where $\mathbf{Y}_G = \mathbf{r}_0^{-1}$, as a function of the variable $\mathbf{a}$, and denoted by $\Gamma^t_{\mathbf{a}}$, satisfies [34]–[36]:

$$\Gamma^t_{\mathbf{a}} = \sqrt{\frac{P_{AVG} - P_{RAD}}{P_{AVG}}} = \sqrt{1 - e_T} \,, \tag{93}$$

However, an alternative definition of the TARC, which does not satisfies (93), is also being used by some authors [15, Sec. VII.B]. To avoid this imbroglio, we suggest to use the transducer efficiency of the MAA in place of the TARC, since the definition of the transducer efficiency of the MAA is more general and unambiguous.

It follows from Observation 2 that the set of the values of e_T obtained for all $\mathbf{X} \in \mathbb{C}^N$ such that $\mathbf{X} \neq \mathbf{0}$ is equal to the set of the values of e_T obtained for all $\mathbf{X} \in \mathbb{S}_N$. $\mathbf{C}$ being assumed to be an invertible matrix, we can assert that: the set of the values of e_T, obtained by applying (92) to any nonzero $\mathbf{X} = \mathbf{C}\,\mathbf{I}_{SG} \in \mathbb{C}^N$ does not depend on the selected $\mathbf{C}$. Thus, for any choice of applicable variable $\mathbf{X}$ and the associated $\mathbf{M}_{RAD}$ and $\mathbf{M}_{AVG}$ according to Table 3, the set of the values of e_T, obtained by applying (91) to any nonzero $\mathbf{X} \in \mathbb{C}^N$, or to any $\mathbf{X} \in \mathbb{S}_N$, is independent of the chosen applicable variable.

For any choice of applicable variable $\mathbf{X}$ and the associated $\mathbf{M}_{RAD}$ and $\mathbf{M}_{AVG}$ according to Table 3, $\mathbf{M}_{AVG}$ being positive definite, we can apply Theorem 1 to e_T regarded as a generalized Rayleigh ratio of $\mathbf{M}_{RAD}$ to $\mathbf{M}_{AVG}$, in the variable $\mathbf{X}$. It follows that:

(a) the set of the values of the transducer efficiency e_T, obtained for all nonzero $\mathbf{X} \in \mathbb{C}^N$, has a least element referred to as "minimum value" and denoted by $e_{T\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $e_{T\,MAX}$;

(b) we have

$$0 \leqslant e_{T\,MIN} \leqslant e_{T\,MAX} \leqslant 1 \,; \tag{94}$$

and

(c) the eigenvalues of $\mathbf{M}_{RAD}\,\mathbf{M}_{AVG}^{-1}$ are real and nonnegative, the largest of these eigenvalues is $e_{T\,MAX}$, and the smallest of these eigenvalues is $e_{T\,MIN}$.

The values of $e_{T\,MAX}$ and $e_{T\,MIN}$ computed using (c) are of course independent of the chosen applicable variable. For any choice of applicable variable $\mathbf{X}$ and the associated $\mathbf{M}_{RAD}$ and $\mathbf{M}_{AVG}$ according to Table 3, we can also apply Theorem 3 to e_T regarded as a generalized Rayleigh ratio of $\mathbf{M}_{RAD}$ to $\mathbf{M}_{AVG}$, in the variable $\mathbf{X}$.

If $\mathbf{X}/||\mathbf{X}||_2$ is constant and known, we can use (91) or (92) to compute e_T, which lies in $[e_{T\,MIN}, e_{T\,MAX}]$. In the opposite case, to obtain a suitable metric of the transducer efficiency, we can consider the worst-case value, that is the lowest transducer efficiency. According to this idea, suitable metrics are $e_{T\,MIN}$ and the transducer efficiency figure of the MAA, denoted by F_{TE} and defined by:

$$F_{TE} = \sqrt{1 - e_{T\,MIN}} \,. \tag{95}$$

Using (94) and (95), we get

$$0 \leqslant F_{TE} \leqslant 1 \,. \tag{96}$$

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If the TARC is given by (93), we see that F_{TE} is the maximum value of the TARC, for all nonzero $\mathbf{a} \in \mathbb{C}^N$.

$e_{T\,MINdB} = 10 \log e_{T\,MIN}$ is $e_{T\,MIN}$ expressed in decibels. F_{TE} expressed in decibels is $F_{TEdB} = 20 \log F_{TE}$. We have $F_{TE} = 0$, or $e_{T\,MIN} = 1$, or $e_{T\,MINdB} = 0\,$dB, if and only if $P_{RAD} = P_{AVG}$ for any nonzero excitation. We have $F_{TE} = 1$, or $F_{TEdB} = 0\,$dB, or $e_{T\,MIN} = 0$, if and only if there exists at least one nonzero excitation such that $P_{RAD} = 0$.

The use of F_{TE} or $e_{T\,MIN}$ as a design parameter is relevant to situations in which the transducer efficiency is important but the location of $\mathbf{X}/||\mathbf{X}||_2$ on $\mathbb{S}_N$ is not constant or not known. In such situations, we could also consider $\mathbf{X}/||\mathbf{X}||_2$ as a random complex unit vector. In this case, if we had suitable information on the statistics of $\mathbf{X}/||\mathbf{X}||_2$, we could derive an expectation $\langle e_T \rangle$ of e_T, which would be such that $\langle e_T \rangle \in [e_{T\,MIN}, e_{T\,MAX}]$.

e_T is a function of: the characteristics of the MAA; of $\mathbf{Z}_G$ or equivalently $\mathbf{Y}_G$ or $\mathbf{S}_G$; and of a specified excitation $\mathbf{X}$. In contrast, F_{TE} and $e_{T\,MIN}$ are determined without reference to a specified excitation, and depends only on: the MAA; and $\mathbf{Z}_G$ or equivalently $\mathbf{Y}_G$ or $\mathbf{S}_G$.

VIII. POWER RECEIVED BY THE MAA

We now look at the average power received by the ports of the MAA during emission, denoted by P_{RPA}, which may also be referred to as "average power delivered by the MG during emission" or "average power accepted by the ports of the MAA during emission". We want to compute it without unnecessary assumptions. To this end, as in Section V, we will use the theory of parallel-augmented multiports and series-augmented multiports presented in Section II of [21]. Some of the formulas for computing P_{RPA} derived in this Section VIII can yet also be obtained without this theory. As already said, we assume that there is no incident electromagnetic signal received by the MAA, and we ignore noise power contributions.

P_{RPA} being given by

$$P_{RPA} = \frac{\mathbf{I}^*\mathbf{V} + \mathbf{V}^*\mathbf{I}}{2}\,, \tag{97}$$

we can use (38), (42) and $\mathbf{I}_{SG} = \mathbf{Y}_G \mathbf{V}_{OG}$ to obtain

$$P_{RPA} = \mathbf{V}_{OG}^* \frac{\mathbf{Y}_G^* \mathbf{Z}_{PAM}^* \mathbf{Y}_{SAM} + \mathbf{Y}_{SAM}^* \mathbf{Z}_{PAM} \mathbf{Y}_G}{2} \mathbf{V}_{OG}\,. \tag{98}$$

It follows from (38), (42), $\mathbf{I}_{SG} = \mathbf{Y}_G \mathbf{V}_{OG}$ and

$$\mathbf{V}_{OG} = \mathbf{V} + \mathbf{Z}_G \mathbf{I} \tag{99}$$

that, for any $\mathbf{V}_{OG} \in \mathbb{C}^N$, we have

$$\mathbf{V}_{OG} = (\mathbf{Z}_{PAM} \mathbf{Y}_G + \mathbf{Z}_G \mathbf{Y}_{SAM}) \mathbf{V}_{OG}\,. \tag{100}$$

Consequently,

$$\mathbf{Z}_{PAM} \mathbf{Y}_G + \mathbf{Z}_G \mathbf{Y}_{SAM} = \mathbf{1}_N\,. \tag{101}$$

Using (98) and (101), we obtain

$$P_{RPA} = \mathbf{V}_{OG}^* \mathbf{Y}_{RPAO} \mathbf{V}_{OG}\,, \tag{102}$$

where the admittance matrix

$$\mathbf{Y}_{RPAO} = \frac{\mathbf{Y}_{SAM} + \mathbf{Y}_{SAM}^* - \mathbf{Y}_{SAM}^* (\mathbf{Z}_G + \mathbf{Z}_G^*) \mathbf{Y}_{SAM}}{2} \tag{103}$$

is positive semidefinite because the MAA is passive and $\mathbf{V}_{OG}$ can take on any value lying in $\mathbb{C}^N$.

We can use (38), (42), (97) and $\mathbf{V}_{OG} = \mathbf{Z}_G \mathbf{I}_{SG}$ to obtain

$$P_{RPA} = \mathbf{I}_{SG}^* \frac{\mathbf{Z}_G^* \mathbf{Y}_{SAM}^* \mathbf{Z}_{PAM} + \mathbf{Z}_{PAM}^* \mathbf{Y}_{SAM} \mathbf{Z}_G}{2} \mathbf{I}_{SG}\,, \tag{104}$$

Using (48) and (104), we get

$$P_{RPA} = \mathbf{I}_{SG}^* \, \mathbf{Z}_{RPAS} \, \mathbf{I}_{SG}\,, \tag{105}$$

where the impedance matrix

$$\mathbf{Z}_{RPAS} = \frac{\mathbf{Z}_{PAM} + \mathbf{Z}_{PAM}^* - \mathbf{Z}_{PAM}^* (\mathbf{Y}_G + \mathbf{Y}_G^*) \mathbf{Z}_{PAM}}{2} \tag{106}$$

is positive semidefinite because the MAA is passive and $\mathbf{I}_{SG}$ can take on any value lying in $\mathbb{C}^N$.

Note that (103) and (106) can also be obtained "by inspection", if we look at the effects of the terms of the numerators of their right-hand sides, when (103) is used in (102) and when (106) is used in (105): the first two terms produce twice the average power received by the augmented multiport, that is the series-augmented multiport in the case of (103) or the parallel-augmented multiport in the case of (106); and the last term produces the opposite of twice the part of this average power which is not delivered to the MAA.

We know that: the MAA has an admittance matrix $\mathbf{Y}_A$ if and only if $\mathbf{Z}_{PAM}$ is invertible. In this case, it follows from (39) that we have $\mathbf{Y}_G = \mathbf{Z}_{PAM}^{-1} - \mathbf{Y}_A$, which combined with (106) leads us to

$$\mathbf{Z}_{RPAS} = \mathbf{Z}_{PAM}^* \frac{\mathbf{Y}_A + \mathbf{Y}_A^*}{2} \mathbf{Z}_{PAM}\,. \tag{107}$$

Also, if the MAA has an admittance matrix $\mathbf{Y}_A$, we know that $\mathbf{V}$ can take on any value lying in $\mathbb{C}^N$, and it follows from (38), (105) and (107) that

$$P_{RPA} = \mathbf{V}^* \, \mathbf{Y}_{RPA} \, \mathbf{V}\,, \tag{108}$$

where the admittance matrix

$$\mathbf{Y}_{RPA} = \frac{\mathbf{Y}_A + \mathbf{Y}_A^*}{2} \tag{109}$$

is positive semidefinite and depends only on the parameters of the MAA, since $\mathbf{V}$ and the MAA determine $\mathbf{I}$ and P_{RPA}.

We know that: the MAA has an impedance matrix $\mathbf{Z}_A$ if and only if $\mathbf{Y}_{SAM}$ is invertible. In this case, it follows from (43) that we have $\mathbf{Z}_G = \mathbf{Y}_{SAM}^{-1} - \mathbf{Z}_A$, which combined with (103) leads us to

$$\mathbf{Y}_{RPAO} = \mathbf{Y}_{SAM}^* \frac{\mathbf{Z}_A + \mathbf{Z}_A^*}{2} \mathbf{Y}_{SAM}\,. \tag{110}$$

Also, if the MAA has an impedance matrix $\mathbf{Z}_A$, we know that $\mathbf{I}$ can take on any value lying in $\mathbb{C}^N$, and it follows from (42), (102) and (110) that

$$P_{RPA} = \mathbf{I}^* \, \mathbf{Z}_{RPA} \, \mathbf{I}\,, \tag{111}$$

where the impedance matrix

$$\mathbf{Z}_{RPA} = \frac{\mathbf{Z}_A + \mathbf{Z}_A^*}{2} \quad (112)$$

is positive semidefinite and depends only on the parameters of the MAA, because $\mathbf{I}$ and the MAA determine $\mathbf{V}$ and P_{RPA}. We observe that we could also have derived (107), (109), (110) and (112) directly from (97).

So far, we have computed P_{RPA} from the variables $\mathbf{V}_{OG}$, $\mathbf{I}_{SG}$, $\mathbf{V}$ and $\mathbf{I}$. As regards the variable $\mathbf{a}$, using (50) and (105)-(106), we get

$$P_{RPA} = \mathbf{a}^* \, \mathbf{\Lambda}_{RPA} \, \mathbf{a} \,, \quad (113)$$

where the dimensionless matrix $\mathbf{\Lambda}_{RPA}$, given by

$$\mathbf{\Lambda}_{RPA} = \mathbf{r}_0^{1/2} \, \mathbf{Y}_{PSG}^* \, \mathbf{Z}_{RPAS} \, \mathbf{Y}_{PSG} \, \mathbf{r}_0^{1/2} \,, \quad (114)$$

is positive semidefinite and depends only on the parameters of the MAA and $\mathbf{r}_0$, since the MAA, $\mathbf{r}_0$ and $\mathbf{a}$ determine $\mathbf{V}$ and $\mathbf{I}$, which determine P_{RPA}. A well-known formula, which is simpler than (114), can be directly obtained by computing $\mathbf{a}^*\mathbf{a} - \mathbf{b}^*\mathbf{b}$ using (49) and (53). This formula is [26]

$$\mathbf{\Lambda}_{RPA} = \mathbf{1}_N - \mathbf{S}_A^* \mathbf{S}_A \,. \quad (115)$$

As regards the variable $\hat{\mathbf{a}}$, using (64) and (105)-(106), we get

$$P_{RPA} = \hat{\mathbf{a}}^* \, \hat{\mathbf{\Lambda}}_{RPA} \, \hat{\mathbf{a}} \,, \quad (116)$$

where the dimensionless matrix $\hat{\mathbf{\Lambda}}_{RPA}$, given by

$$\hat{\mathbf{\Lambda}}_{RPA} = \mathbf{r}_0^{1/2} \, \hat{\mathbf{Z}}_{PSG}^{-1*} \, \mathbf{Z}_{RPAS} \, \hat{\mathbf{Z}}_{PSG}^{-1} \, \mathbf{r}_0^{1/2} \,, \quad (117)$$

is positive semidefinite and depends only on the parameters of the MAA and $\mathbf{z}_0$, since the MAA, $\mathbf{z}_0$ and $\hat{\mathbf{a}}$ determine $\mathbf{V}$ and $\mathbf{I}$, which determine P_{RPA}.

We observe that P_{RPA} is only a function of the excitation and of the characteristics of the MAA, if the excitation is specified using any one of the variables $\mathbf{a}$ (for a fixed $\mathbf{r}_0$), or $\hat{\mathbf{a}}$ (for a fixed $\mathbf{z}_0$), or $\mathbf{V}$ if $\mathbf{Y}_A$ exists, or $\mathbf{I}$ if $\mathbf{Z}_A$ exists.

IX. THE RADIATION EFFICIENCY

According to the standard vocabulary of the IEEE, the radiation efficiency is given by [29], [32, Sec. 16.6]–[33]:

$$e_R = \frac{P_{RAD}}{P_{RPA}} \,. \quad (118)$$

As in Section VII, we use $\mathbf{X}$ to denote one of the variables $\mathbf{V}_{OG}$, $\mathbf{I}_{SG}$, $\mathbf{V}$, $\mathbf{I}$, $\mathbf{a}$, or $\hat{\mathbf{a}}$. Based on the results of Section VI about P_{RAD} and of Section VIII about P_{RPA}, we find that, if the variable $\mathbf{X}$ is applicable, e_R is given by

$$e_R = \frac{\mathbf{X}^* \, \mathbf{M}_{RAD} \, \mathbf{X}}{\mathbf{X}^* \, \mathbf{M}_{RPA} \, \mathbf{X}} \,, \quad (119)$$

where the matrices $\mathbf{M}_{RAD}$ and $\mathbf{M}_{RPA}$ are both of size N by N, and given in Table 5. We note that $\mathbf{M}_{RAD}$ and $\mathbf{M}_{RPA}$ are always positive semidefinite.

The radiation efficiency of the MAA is given by (119) in the form of a generalized Rayleigh ratio of $\mathbf{M}_{RAD}$ to $\mathbf{M}_{RPA}$, in the variable $\mathbf{X}$. According to the explanations provided in Section II.A, it is defined for $\mathbf{X} \in D(\mathbf{M}_{RPA})$ where

$$D(\mathbf{M}_{RPA}) = \{\mathbf{X} \in \mathbb{C}^N : \mathbf{X} \notin \ker \mathbf{M}_{RPA}\} \,. \quad (120)$$

TABLE 5. Variable $\mathbf{X}$ and associated $\mathbf{M}_{RAD}$ and $\mathbf{M}_{RPA}$.

Variable $\mathbf{X}$	Applicability	$\mathbf{M}_{RAD}$	$\mathbf{M}_{RPA}$
$\mathbf{V}_{OG}$	always	$\mathbf{Y}_{RADO}$	$\mathbf{Y}_{RPAO}$
$\mathbf{I}_{SG}$	always	$\mathbf{Z}_{RADS}$	$\mathbf{Z}_{RPAS}$
$\mathbf{V}$	if $\mathbf{Y}_A$ exists	$\mathbf{Y}_{RAD}$	$\mathbf{Y}_{RPA}$
$\mathbf{I}$	if $\mathbf{Z}_A$ exists	$\mathbf{Z}_{RAD}$	$\mathbf{Z}_{RPA}$
$\mathbf{a}$	always	$\mathbf{\Lambda}_{RAD}$	$\mathbf{\Lambda}_{RPA}$
$\hat{\mathbf{a}}$	always	$\hat{\mathbf{\Lambda}}_{RAD}$	$\hat{\mathbf{\Lambda}}_{RPA}$

Importantly, (119) shows that the radiation efficiency depends on the excitation. Also, we must keep in mind that, among the 6 variables listed in Table 5 to define the excitation, only $\mathbf{V}_{OG}$, $\mathbf{I}_{SG}$, $\mathbf{a}$, and $\hat{\mathbf{a}}$ are always applicable.

More generally, for any variable $\mathbf{X}$ given by $\mathbf{X} = \mathbf{C}\,\mathbf{I}_{SG}$, where $\mathbf{C}$ is an invertible matrix of size N by N defining the change of variable, we have

$$e_R = \frac{\mathbf{X}^* \, \mathbf{C}^{-1*} \, \mathbf{Z}_{RADS} \, \mathbf{C}^{-1} \, \mathbf{X}}{\mathbf{X}^* \, \mathbf{C}^{-1*} \, \mathbf{Z}_{RPAS} \, \mathbf{C}^{-1} \, \mathbf{X}} \,, \quad (121)$$

where, for instance, the matrices $\mathbf{C}$ corresponding to some possible variables $\mathbf{X}$ are shown in Table 4 above.

It follows from Observation 2 that the set of the values of e_R obtained for all $\mathbf{X} \in D(\mathbf{M}_{RPA})$, in which $\mathbf{M}_{RPA}$ is associated with the variable $\mathbf{X}$ in Table 5, is equal to the set of the values of e_R obtained for all $\mathbf{X} \in D(\mathbf{M}_{RPA}) \cap \mathbb{S}_N$. $\mathbf{C}$ being assumed to be an invertible matrix, we can assert that: the set of the values of e_R, obtained by applying (121) to any $\mathbf{X} = \mathbf{C}\,\mathbf{I}_{SG} \in D(\mathbf{M}_{RPA})$ does not depend on the selected $\mathbf{C}$. Thus, for any choice of applicable variable $\mathbf{X}$ and the associated $\mathbf{M}_{RAD}$ and $\mathbf{M}_{RPA}$ according to Table 5, the set of the values of e_R, obtained by applying (119) to any $\mathbf{X} \in D(\mathbf{M}_{RPA})$, or to any $\mathbf{X} \in D(\mathbf{M}_{RPA}) \cap \mathbb{S}_N$, is independent of the chosen applicable variable.

We observe that power conservation entails $e_R \leqslant 1$. We now assume $D(\mathbf{M}_{RPA}) \neq \varnothing$, since otherwise studying e_R is not interesting. For any choice of applicable variable $\mathbf{X}$ and the associated $\mathbf{M}_{RAD}$ and $\mathbf{M}_{RPA}$ according to Table 5, $\mathbf{M}_{RAD}$ and $\mathbf{M}_{RPA}$ being positive semidefinite, we can apply Corollary 1 to e_R regarded as a generalized Rayleigh ratio of $\mathbf{M}_{RAD}$ to $\mathbf{M}_{RPA}$. It follows that $\ker \mathbf{M}_{RPA} \subset \ker \mathbf{M}_{RAD}$. Thus, the assumptions of Theorem 3 applied to e_R regarded as a generalized Rayleigh ratio of $\mathbf{M}_{RAD}$ to $\mathbf{M}_{RPA}$, in the variable $\mathbf{X}$, are satisfied. In the special case where $\mathbf{M}_{RPA}$ is positive definite, we can also apply Theorem 1 to e_R regarded in this manner. It follows that:

(a) the set of the values of the radiation efficiency e_R, obtained for all $\mathbf{X} \in D(\mathbf{M}_{RPA})$, has a least element referred to as "minimum value" and denoted by $e_{R\,MIN}$, and a greatest element referred to as "maximum value" and denoted by $e_{R\,MAX}$;

(b) we have

$$0 \leqslant e_{R\,MIN} \leqslant e_{R\,MAX} \leqslant 1 \,; \quad (122)$$

(c) to compute $e_{R\,MIN}$ and $e_{R\,MAX}$, we can compute the eigenvalues of $\mathbf{M}_{RPA}$, label them according to a non-

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decreasing order $\mu_1, \ldots, \mu_N$, compute a unitary matrix $\mathbf{L}$ of size N by N such that

$$\mathbf{M}_{RPA} = \mathbf{L}\,\mathrm{diag}_N(\mu_1, \ldots, \mu_N)\,\mathbf{L}^* \tag{123}$$

and perform the following steps;

(d) d being the nullity of $\mathbf{M}_{RPA}$, we have $d \leqslant N-1$, so that $0 < \mu_{d+1} \leqslant \ldots \leqslant \mu_N$, so that we can build a matrix $\mathcal{L}$ defined as the submatrix of $\mathbf{L}$, of size N by $N-d$, whose column vectors are $\mathbf{L}^{<d+1>}, \ldots, \mathbf{L}^{<N>}$, in this order;

(e) we can compute

$$\mathbf{R} = \mathcal{L}^* \mathbf{M}_{RAD} \mathcal{L}\,\mathrm{diag}_{N-d}\left(\frac{1}{\mu_{d+1}}, \ldots, \frac{1}{\mu_N}\right) \tag{124}$$

(f) the eigenvalues of $\mathbf{R}$ are real and nonnegative, the largest of these eigenvalues is $e_{R\,MAX}$, and the smallest of these eigenvalues is $e_{R\,MIN}$; and

(g) if $\mathbf{M}_{RPA}$ is positive definite, then the eigenvalues of $\mathbf{M}_{RAD}\,\mathbf{M}_{RPA}^{-1}$ are real and nonnegative, the largest of these eigenvalues is $e_{R\,MAX}$, and the smallest of these eigenvalues is $e_{R\,MIN}$.

The values of $e_{R\,MAX}$ and $e_{R\,MIN}$ computed using (f) or (g) are of course independent of the chosen applicable variable.

If $\mathbf{X}/||\mathbf{X}||_2$ is constant and known, we can use (119) to compute e_R. In the opposite case, to obtain a suitable metric of the radiation efficiency, we can consider the worst-case value, that is the lowest radiation efficiency. According to this idea, suitable metrics are $e_{R\,MIN}$ and the radiation efficiency figure of the MAA, denoted by F_{RE} and defined by:

$$F_{RE} = \sqrt{1 - e_{R\,MIN}}\,. \tag{125}$$

Using (122) and (125), we get

$$0 \leqslant F_{RE} \leqslant 1\,. \tag{126}$$

$e_{R\,MINdB} = 10 \log e_{R\,MIN}$ is $e_{R\,MIN}$ expressed in decibels. F_{RE} expressed in decibels is $F_{REdB} = 20 \log F_{RE}$. We have $F_{RE} = 0$, or $e_{R\,MIN} = 1$, or $e_{R\,MINdB} = 0\,\mathrm{dB}$, if and only if, for any excitation such that e_R is defined, we have $P_{RAD} = P_{RPA}$. We have $F_{RE} = 1$, or $F_{REdB} = 0\,\mathrm{dB}$, or $e_{R\,MIN} = 0$, if and only if there exists at least one excitation such that e_R is defined and $P_{RAD} = 0$. It follows that $F_{RE} = 1$ entails $F_{TE} = 1$.

The use of F_{RE} or $e_{R\,MIN}$ as a design parameter is relevant to situations in which the radiation efficiency is important but the location of $\mathbf{X}/||\mathbf{X}||_2$ on $\mathbb{S}_N$ is not constant or not known. In such situations, we could also consider $\mathbf{X}/||\mathbf{X}||_2$ as a random complex unit vector. In this case, we might be able to derive an expectation $\langle e_R \rangle$ of e_R, which would be such that $\langle e_R \rangle \in [e_{R\,MIN}, e_{R\,MAX}]$.

We have seen in Section VI and Section VIII that P_{RAD} and P_{RPA} are only functions of the excitation and of the characteristics of the MAA, if the excitation is specified using any one of the variables $\mathbf{a}$, or $\hat{\mathbf{a}}$, or $\mathbf{V}$ if $\mathbf{Y}_A$ exists, or $\mathbf{I}$ if $\mathbf{Z}_A$ exists. Consequently, e_R is only a function of an excitation specified using any one of these variables, and of the characteristics of the MAA. It follows that, in contrast, F_{RE} and $e_{R\,MIN}$ being determined without reference to a specified excitation, they depend only on the MAA.

X. THE POWER TRANSFER RATIOS

According to [15, Sec. II], the power transfer ratio during emission is given by

$$t_E = \frac{P_{RPA}}{P_{AVG}}\,. \tag{127}$$

This quantity is relevant to our investigation of the transducer efficiency and the radiation efficiency because, by (90), (118) and (127), we have

$$e_T = e_R\, t_E\,. \tag{128}$$

The properties of t_E were studied in detail in [15], but only as a function of the variable $\mathbf{I}_{SG}$. To broaden our perspective on t_E, we use $\mathbf{X}$ to denote one of the variables $\mathbf{V}_{OG}$, $\mathbf{I}_{SG}$, $\mathbf{V}$, $\mathbf{I}$, $\mathbf{a}$, or $\hat{\mathbf{a}}$. Based on the results of Sections IV-V about P_{AVG} and of Section VIII about P_{RPA}, we find that, if the variable $\mathbf{X}$ is applicable, t_E is given by

$$t_E = \frac{\mathbf{X}^*\,\mathbf{M}_{RPA}\,\mathbf{X}}{\mathbf{X}^*\,\mathbf{M}_{AVG}\,\mathbf{X}}\,, \tag{129}$$

where the matrices $\mathbf{M}_{RPA}$ and $\mathbf{M}_{AVG}$ are both of size N by N, and given in Table 3 and Table 5 above. We note that $\mathbf{M}_{RPA}$ is always positive semidefinite, and $\mathbf{M}_{AVG}$ always positive definite.

The power transfer ratio during emission is given by (129) in the form of a generalized Rayleigh ratio of $\mathbf{M}_{RPA}$ to $\mathbf{M}_{AVG}$, in the variable $\mathbf{X}$. Since $\mathbf{M}_{AVG}$ is positive definite, t_E is defined for any nonzero $\mathbf{X} \in \mathbb{C}^N$.

Importantly, (129) shows that the power transfer ratio during emission depends on the excitation. Also, we must keep in mind that, among the 6 variables listed in Table 3 and Table 5 to define the excitation, only $\mathbf{V}_{OG}$, $\mathbf{I}_{SG}$, $\mathbf{a}$, and $\hat{\mathbf{a}}$ are always applicable.

More generally, for any variable $\mathbf{X}$ given by $\mathbf{X} = \mathbf{C}\,\mathbf{I}_{SG}$, where $\mathbf{C}$ is an invertible matrix of size N by N defining the change of variable, we have

$$t_E = \frac{\mathbf{X}^*\,\mathbf{C}^{-1*}\,\mathbf{Z}_{RPAS}\,\mathbf{C}^{-1}\,\mathbf{X}}{\mathbf{X}^*\,\mathbf{C}^{-1*}\,\mathbf{Z}_{AVGS}\,\mathbf{C}^{-1}\,\mathbf{X}}\,, \tag{130}$$

where, for instance, the matrices $\mathbf{C}$ corresponding to some possible variables $\mathbf{X}$ are shown in Table 4 above.

It follows from Observation 2 that the set of the values of t_E obtained for all $\mathbf{X} \in \mathbb{C}^N$ such that $\mathbf{X} \neq \mathbf{0}$ is equal to the set of the values of t_E obtained for all $\mathbf{X} \in \mathbb{S}_N$. $\mathbf{C}$ being assumed to be an invertible matrix, we can assert that: the set of the values of t_E, obtained by applying (130) to any nonzero $\mathbf{X} = \mathbf{C}\,\mathbf{I}_{SG} \in \mathbb{C}^N$ does not depend on the selected $\mathbf{C}$. Thus, for any choice of applicable variable $\mathbf{X}$ and the associated $\mathbf{M}_{RPA}$ and $\mathbf{M}_{AVG}$ according to Table 3 and Table 5, the set of the values of t_E, obtained by applying (129) to any nonzero $\mathbf{X} \in \mathbb{C}^N$, or to any $\mathbf{X} \in \mathbb{S}_N$, is independent of the chosen applicable variable.

For any choice of applicable variable $\mathbf{X}$ and the associated $\mathbf{M}_{RPA}$ and $\mathbf{M}_{AVG}$ according to Table 3 and Table 5, $\mathbf{M}_{AVG}$ being positive definite, we can apply Theorem 1 to t_E regarded as a generalized Rayleigh ratio of $\mathbf{M}_{RPA}$ to $\mathbf{M}_{AVG}$, in the variable $\mathbf{X}$. It follows that:

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(a) the set of the values of the power transfer ratio during emission t_E, obtained for all nonzero $\mathbf{X} \in \mathbb{C}^N$, has a least element referred to as "minimum value" and denoted by t_{MIN}, and a greatest element referred to as "maximum value" and denoted by t_{MAX};
(b) we have

$$0 \leqslant t_{MIN} \leqslant t_{MAX} \leqslant 1\,; \tag{131}$$

and
(c) the eigenvalues of $\mathbf{M}_{RPA}\,\mathbf{M}_{AVG}^{-1}$ are real and nonnegative, the largest of these eigenvalues is t_{MAX}, and the smallest of these eigenvalues is t_{MIN}.

The values of t_{MAX} and t_{MIN} computed using (c) are of course independent of the chosen applicable variable. For any choice of applicable variable $\mathbf{X}$ and the associated $\mathbf{M}_{RPA}$ and $\mathbf{M}_{AVG}$ according to Table 3 and Table 5, we can also apply Theorem 3 to t_E regarded as a generalized Rayleigh ratio of $\mathbf{M}_{RPA}$ to $\mathbf{M}_{AVG}$, in the variable $\mathbf{X}$.

If $\mathbf{X}/||\mathbf{X}||_2$ is constant and known, we can use (129) or (130) to compute t_E, which lies in $[t_{MIN},\, t_{MAX}]$. In the opposite case, to obtain a suitable metric of the power transfer ratio during emission, we can consider the worst-case value, that is the lowest power transfer ratio during emission. According to this idea, suitable metrics are t_{MIN} and the power match figure of the MAA, denoted by F_M and defined by [15, Sec. V]:

$$F_M = \sqrt{1 - t_{MIN}}\,. \tag{132}$$

Using (131) and (132), we get

$$0 \leqslant F_M \leqslant 1\,. \tag{133}$$

$t_{MINdB} = 10 \log t_{MIN}$ is t_{MIN} expressed in decibels. F_M expressed in decibels is $F_{TEdB} = 20 \log F_M$. We have $F_M = 0$, or $t_{MIN} = 1$, or $t_{MINdB} = 0\,\text{dB}$, if and only if, for any nonzero excitation, we have $P_{RPA} = P_{AVG}$. We have $F_M = 1$, or $F_{TEdB} = 0\,\text{dB}$, or $t_{MIN} = 0$, if and only if there exists at least one nonzero excitation such that $P_{RPA} = 0$.

It follows from the maximum power transfer theorem for multiports [23] that $t_{MIN} = 1$ and $F_M = 0$ if and only if $\mathbf{Z}_A = \mathbf{Z}_G^*$. Consequently, t_{MIN} and F_M are matching metrics that measure the closeness of $\mathbf{Z}_A$ to the wanted value $\mathbf{Z}_G^*$, for which maximum power transfer occurs with any excitation (this is one of the possible meanings of "matched") [15, Sec. V.C].

The use of F_M or t_{MIN} as a design parameter is relevant to situations in which the power transfer ratio during emission is important, but the location of $\mathbf{X}/||\mathbf{X}||_2$ on $\mathbb{S}_N$ is not constant or not known. In such situations, we could also consider $\mathbf{X}/||\mathbf{X}||_2$ as a random complex unit vector. In this case, we might be able to derive an expectation $\langle t_E \rangle$ of t_E, which would be such that $\langle t_E \rangle \in [t_{MIN},\, t_{MAX}]$.

t_E is a function of: the characteristics of the MAA; of $\mathbf{Z}_G$ or equivalently $\mathbf{Y}_G$ or $\mathbf{S}_G$; and of a specified excitation $\mathbf{X}$. In contrast, F_M and t_{MIN} are determined without reference to a specified excitation, and depends only on: the MAA; and $\mathbf{Z}_G$ or equivalently $\mathbf{Y}_G$ or $\mathbf{S}_G$.

Several inequalities can be derived from (128), among which:

$$e_{T\,MIN} \leqslant \min\{e_{R\,MIN} t_{MAX},\, e_{R\,MAX} t_{MIN}\}\,, \tag{134}$$

$$e_{R\,MIN}\, t_{MIN} \leqslant e_{T\,MIN} \leqslant \min\{e_{R\,MIN},\, t_{MIN}\}\,, \tag{135}$$

and

$$\max\{F_{RE},\, F_M\} \leqslant F_{TE}\,, \tag{136}$$

A fundamental property of t_{MAX} and t_{MIN} is that they are an upper bound and a lower bound, respectively, of the set of the values of the power transfer ratio during reception, obtained for all possible incident field configurations [15, Sec. II-III]. Thus, F_M and t_{MIN} are also relevant to reception. We will not discuss this aspect further, since it is not directly related to the properties of the transducer efficiency and the radiation efficiency.

XI. SINGLE PORT EXCITATIONS

For any integer $p \in \{1, \ldots, N\}$, we can use $\mathbf{E}_p$ to denote a unit column vector of size N, the entries of which are zero except the entry of row p which is equal to 1. If we assume that the value of the variable $\mathbf{X}$ is given by $\mathbf{X} = X\mathbf{E}_p$, where $X \in \mathbb{C}$ is nonzero, we obtain a single-port excitation of the MAA, at port p, for the variable $\mathbf{X}$.

There are several ways of looking at single-port excitations of the MAA. According to a first perspective, we use the configuration presented in Section IV and shown in Fig. 1, and we determine how a given single-port excitation may be produced. For instance, if we assume that we control the short-circuit current $\mathbf{I}_{SG}$, we find that, by order of increasing complexity:

- the single-port excitations at port p for the variable $\mathbf{I}_{SG}$ correspond to $\mathbf{I}_{SG} = X\mathbf{E}_p$;
- the single-port excitations at port p for the variable $\mathbf{V}_{OG}$ correspond to $\mathbf{I}_{SG} = X\mathbf{Y}_G\mathbf{E}_p$;
- the single-port excitations at port p for the variable $\mathbf{a}$ correspond to $\mathbf{I}_{SG} = X\mathbf{Y}_{PSG}\,\mathbf{r}_0^{1/2}\,\mathbf{E}_p$;
- the single-port excitations at port p for the variable $\hat{\mathbf{a}}$ correspond to $\mathbf{I}_{SG} = X\hat{\mathbf{Z}}_{PSG}^{-1}\,\mathbf{r}_0^{1/2}\,\mathbf{E}_p$;
- if $\mathbf{Y}_A$ exists, we get the single-port excitations at port p for the variable $\mathbf{V}$ using $\mathbf{I}_{SG} = \mathbf{Z}_{PAM}^{-1}\mathbf{E}_p$; and
- if $\mathbf{Z}_A$ exists, we get the single-port excitations at port p for the variable $\mathbf{I}$ using $\mathbf{I}_{SG} = \mathbf{Y}_G\mathbf{Y}_{SAM}^{-1}\mathbf{E}_p$.

According to a second perspective, we look for a simple configuration using only one single-port generator, which can be used to obtain a single-port excitation of the MAA, at port p, for the variable $\mathbf{X}$. We find out that, by order of increasing complexity:

- if $\mathbf{Z}_A$ exists, the single-port excitations at port p for the variable $\mathbf{I}$ correspond to a configuration in which each port of the MAA, except port p, is open-circuited, a single-port generator being coupled to port p;
- if $\mathbf{Y}_A$ exists, the single-port excitations at port p for the variable $\mathbf{V}$ correspond to a configuration in which each port of the MAA, except port p, is short-circuited, a single-port generator being coupled to port p;

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- the single-port excitations at port p for the variable $\mathbf{a}$ correspond to a setup in which, for each $q \in \{1, \ldots, N\}$ such that $q \neq p$, port q of the MAA is coupled to a resistance $r_{0\,q}$, port p of the MAA being coupled to a single-port generator;
- the single-port excitations at port p for the variable $\hat{\mathbf{a}}$ correspond to a setup in which, for each $q \in \{1, \ldots, N\}$ such that $q \neq p$, port q of the MAA is coupled to an impedance $z_{0\,q}$, port p of the MAA being coupled to a single-port generator;
- the single-port excitations at port p for the variable $\mathbf{I}_{SG}$ correspond to the setup shown in Fig. 2, comprising the "added network" defined in Section VI, in which, for each $q \in \{1, \ldots, N\}$ such that $q \neq p$, port q of the MAA is coupled to port q of the added network, port p of the MAA being coupled to a current source connected in parallel with port p of the added network; and
- the single-port excitations at port p for the variable $\mathbf{V}_{OG}$ correspond to a configuration comprising the same added network, in which, for each $q \in \{1, \ldots, N\}$ such that $q \neq p$, port q of the MAA is coupled to port q of the added network, port p of the MAA being coupled to a voltage source connected in series with port p of the added network.

The second perspective allows us to see that setups that are equivalent to the ones producing, for any $p \in \{1, \ldots, N\}$, single-port excitations at port p, for the variable $\mathbf{I}$, or the variable $\mathbf{V}$, or the variable $\mathbf{a}$, can be easily realized in the laboratory.

Single-port excitations for the variable $\mathbf{I}_{SG}$ are of special interest, because they were used in Section VI to directly compute P_{RAD} using (75), and the entries of $\mathbf{Z}_{RADS}$ using (77). By (38), they can also be used to directly compute the entries of $\mathbf{Z}_{PAM}$, from which we can derive $\mathbf{Z}_{RPAS}$ for a known $\mathbf{Y}_G$, using (106).

If $\mathbf{Z}_A$ exists, the single-port excitations for the variable $\mathbf{I}$ can of course be used to directly compute the entries of $\mathbf{Z}_A$, from which we can derive $\mathbf{Z}_{RPA}$, using (112). We show in Appendix D that these single-port excitations can also be used to directly compute P_{RAD} using (226), and the entries of $\mathbf{Z}_{RAD}$ using (227). As noted in Section VI and Section VIII, $\mathbf{Z}_{RAD}$ and $\mathbf{Z}_{RPA}$ only depend on the parameters of the MAA.

Thus, to compute e_T and e_R for any specified excitation, as well as $e_{T\,MIN}$, F_{TE}, $e_{R\,MIN}$ and F_{RE} for unspecified excitations, it is convenient to use single-port excitations for the variable $\mathbf{I}_{SG}$, or, if $\mathbf{Z}_A$ exists, single-port excitations for the variable $\mathbf{I}$. However, the other variables listed in Table 3 to Table 5 can also be used. Single-port excitations for the variable $\mathbf{a}$ can of course be used to directly compute the entries of $\mathbf{S}_A$. They can also be used to compute the entries of $\mathbf{\Lambda}_{RAD}$. An advantage of the variable $\mathbf{a}$ is that $\mathbf{S}_A$ and $\mathbf{\Lambda}_{RAD}$ always exist, and only depend on the MAA. A drawback of the variable $\mathbf{a}$ is that general formulas for the computation of $\mathbf{\Lambda}_{AVG}$, such as (52) and (55), or (60), are not simple.

In the case of a single-port excitation of the MAA, at port p, for the variable $\mathbf{a}$, the radiation efficiency is sometimes referred to as "radiation efficiency at port p", or "embedded radiation efficiency at port p", and the transducer efficiency as "total radiation efficiency at port p", or "total embedded radiation efficiency at port p" [37, Sec. 3.3.3].

One of the variables $\mathbf{X}$ considered in Table 3 being chosen, we observe that:

- it follows from (91) that knowing the values of e_T for the single-port excitations for the chosen variable $\mathbf{X}$ does not allow us to compute e_T for arbitrary values of this variable; and
- it follows from (119) that knowing the values of e_R for the single-port excitations for the chosen variable $\mathbf{X}$ does not allow us to compute e_T for arbitrary values of this variable.

Consequently, the "embedded radiation efficiency at port p" and the "total embedded radiation efficiency at port p" are not very useful, because they do not allow us to compute e_T or e_R for any specified excitation, or $e_{T\,MIN}$, F_{TE}, $e_{R\,MIN}$ or F_{RE} for unspecified excitations.

XII. SOME EFFICIENCY METRICS COMPUTATIONS

A. FIRST CONFIGURATION

In this example, $\mathbf{Y}_A$ exists, so that $\mathbf{Y}_{RAD}$ is defined and given by (81). Using also (85), we get

$$\mathbf{Y}_{RAD} = \mathbf{Z}_{PAM}^{-1*} \mathbf{Z}_{PSG}^{*} \mathbf{r}_0^{-1/2} \mathbf{\Lambda}_{RAD} \mathbf{r}_0^{-1/2} \mathbf{Z}_{PSG} \mathbf{Z}_{PAM}^{-1} \,. \tag{137}$$

Using (39) and (52), we also get

$$\mathbf{Z}_{PSG} \mathbf{Z}_{PAM}^{-1} = \frac{1}{2} \left(\mathbf{1}_N + \mathbf{r}_0 \mathbf{Y}_A \right), \tag{138}$$

in which $\mathbf{Y}_G$ disappeared. It follows that

$$\mathbf{Y}_{RAD} = \frac{1}{4} (\mathbf{1}_N + \mathbf{r}_0 \mathbf{Y}_A)^* \mathbf{r}_0^{-1/2} \mathbf{\Lambda}_{RAD} \mathbf{r}_0^{-1/2} (\mathbf{1}_N + \mathbf{r}_0 \mathbf{Y}_A) \,, \tag{139}$$

which is consistent with the fact that, as explained in Section VI, $\mathbf{\Lambda}_{RAD}$ and $\mathbf{Y}_{RAD}$ depend only on the parameters of the MAA. Here, $\mathbf{Y}_A$ and $\mathbf{Y}_{RAD}$ were computed from $\mathbf{S}_A$ and $\mathbf{\Lambda}_{RAD}$, for $\mathbf{r}_0 = r_0\, \mathbf{1}_2$ where $r_0 = 50\,\Omega$, so that

$$\mathbf{Y}_{RAD} = \frac{1}{4 r_0} (\mathbf{1}_N + r_0 \mathbf{Y}_A)^* \mathbf{\Lambda}_{RAD} (\mathbf{1}_N + r_0 \mathbf{Y}_A) \,. \tag{140}$$

At the frequency $f_G = 2200\,\text{MHz}$, we obtained:

$$\mathbf{Y}_A \simeq \begin{pmatrix} 11.141 - 10.910j & 8.978 + 17.447j \\ 8.978 + 17.447j & 18.562 + 7.676j \end{pmatrix} \text{mS} \tag{141}$$

and

$$\mathbf{Y}_{RAD} \simeq \begin{pmatrix} 10.171 & 9.110 + 0.110j \\ 9.110 - 0.110j & 17.565 \end{pmatrix} \text{mS} \tag{142}$$

We now assume an MG having uncoupled ports, such that

$$\mathbf{Z}_G = \mathbf{r}_0 = \begin{pmatrix} 25 & 0 \\ 0 & 20 \end{pmatrix} \Omega \,. \tag{143}$$

Among the matrices defined in Section IV and Section V, we find:

$$\mathbf{Y}_{AVGO} = \begin{pmatrix} 10.000 & 0.000 \\ 0.000 & 12.500 \end{pmatrix} \text{mS} \,, \tag{144}$$

$$\mathbf{Z}_{AVGS} = \begin{pmatrix} 6.250 & 0.000 \\ 0.000 & 5.000 \end{pmatrix} \Omega\,, \tag{145}$$

$$\mathbf{Z}_{PAM} \simeq \begin{pmatrix} 17.099 + 5.011j & -1.506 - 4.839j \\ -1.506 - 4.839j & 13.496 - 0.494j \end{pmatrix} \Omega\,, \tag{146}$$

$$\mathbf{Y}_{AVG} \simeq \begin{pmatrix} 19.015 & 5.428 + 0.552j \\ 5.428 - 0.552j & 26.204 \end{pmatrix} \mathrm{mS}\,, \tag{147}$$

$$\mathbf{Y}_{SAM} \simeq \begin{pmatrix} 12.642 - 8.017j & 3.011 + 9.677j \\ 3.011 + 9.677j & 16.260 + 1.235j \end{pmatrix} \mathrm{mS}\,, \tag{148}$$

$$\mathbf{Z}_{AVG} \simeq \begin{pmatrix} 32.868 & -0.961 + 2.457j \\ -0.961 - 2.457j & 31.913 \end{pmatrix} \Omega\,, \tag{149}$$

$$\mathbf{Y}_{PSG} = \begin{pmatrix} 80.000 & 0.000 \\ 0.000 & 100.000 \end{pmatrix} \mathrm{mS}\,, \tag{150}$$

and

$$\mathbf{\Lambda}_{AVG} = \begin{pmatrix} 1.000 & 0.000 \\ 0.000 & 1.000 \end{pmatrix}. \tag{151}$$

Among the matrices defined in Section VI, we find:

$$\mathbf{Z}_{RADS} \simeq \begin{pmatrix} 2.786 & 1.499 - 0.275j \\ 1.499 + 0.275j & 3.124 \end{pmatrix} \Omega\,, \tag{152}$$

$$\mathbf{Y}_{RADO} \simeq \begin{pmatrix} 4.457 & 2.997 - 0.551j \\ 2.997 + 0.551j & 7.809 \end{pmatrix} \mathrm{mS}\,, \tag{153}$$

$$\mathbf{Z}_{RAD} \simeq \begin{pmatrix} 12.110 & 7.093 - 0.247j \\ 7.093 + 0.247j & 21.723 \end{pmatrix} \Omega\,, \tag{154}$$

and

$$\mathbf{\Lambda}_{RAD} \simeq \begin{pmatrix} 0.446 & 0.268 - 0.049j \\ 0.268 + 0.049j & 0.625 \end{pmatrix}. \tag{155}$$

Among the matrices defined in Section VIII, we find:

$$\mathbf{Y}_{RPAO} \simeq \begin{pmatrix} 4.985 & 2.781 - 0.590j \\ 2.781 + 0.590j & 8.374 \end{pmatrix} \mathrm{mS}\,, \tag{156}$$

$$\mathbf{Z}_{RPAS} \simeq \begin{pmatrix} 3.116 & 1.390 - 0.295j \\ 1.390 + 0.295j & 3.350 \end{pmatrix} \Omega\,, \tag{157}$$

$$\mathbf{Y}_{RPA} \simeq \begin{pmatrix} 11.141 & 8.978 \\ 8.978 & 18.562 \end{pmatrix} \mathrm{mS}\,, \tag{158}$$

$$\mathbf{Z}_{RPA} \simeq \begin{pmatrix} 13.886 & 6.425 \\ 6.425 & 23.183 \end{pmatrix} \Omega\,, \tag{159}$$

and

$$\mathbf{\Lambda}_{RPA} \simeq \begin{pmatrix} 0.449 & 0.249 - 0.053j \\ 0.249 + 0.053j & 0.670 \end{pmatrix}. \tag{160}$$

The MAA being reciprocal in this example, $\mathbf{Y}_A$ is symmetric so that $\mathbf{Y}_{RPA}$ and $\mathbf{Z}_{RPA}$ are real. Using (142), (144)–(145), (147), (149) and (151)–(160), we computed:

- the eigenvalues of $\mathbf{M}_{RAD}\mathbf{M}_{AVG}^{-1}$ for the variables $\mathbf{V}_{OG}, \mathbf{I}_{SG}, \mathbf{V}, \mathbf{I}$ and $\mathbf{a}$, to obtain 5 times the same values $e_{T\,MIN} \simeq 0.248309$ and $F_{TE} \simeq 0.867001$;
- the eigenvalues of $\mathbf{M}_{RAD}\mathbf{M}_{RPA}^{-1}$ for these 5 variables, to obtain 5 times the same results $e_{R\,MIN} \simeq 0.785088$ and $F_{RE} \simeq 0.463586$; and
- the eigenvalues of $\mathbf{M}_{RPA}\mathbf{M}_{AVG}^{-1}$ for these 5 variables, to obtain 5 times the same results $t_{MIN} \simeq 0.315917$ and $F_M \simeq 0.827093$.

In contrast, the following results show that the values of e_T, e_R and t_E for single-port excitations depend on the selected variable and on the selected port:

- the single-port excitations for the variable $\mathbf{V}_{OG}$ provide $e_T \simeq 0.445706$, $e_R \simeq 0.894023$ and $t_E \simeq 0.498539$ at port 1, and $e_T \simeq 0.624706$, $e_R \simeq 0.932529$ and $t_E \simeq 0.669905$ at port 2;
- the single-port excitations for the variable $\mathbf{I}_{SG}$ provide $e_T \simeq 0.445706$, $e_R \simeq 0.894023$ and $t_E \simeq 0.498539$ at port 1, and $e_T \simeq 0.624706$, $e_R \simeq 0.932529$ and $t_E \simeq 0.669905$ at port 2;
- the single-port excitations for the variable $\mathbf{V}$ provide $e_T \simeq 0.534886$, $e_R \simeq 0.912929$ and $t_E \simeq 0.585901$ at port 1, and $e_T \simeq 0.670301$, $e_R \simeq 0.946291$ and $t_E \simeq 0.708346$ at port 2;
- the single-port excitations for the variable $\mathbf{I}$ provide $e_T \simeq 0.368450$, $e_R \simeq 0.872129$ and $t_E \simeq 0.422472$ at port 1, and $e_T \simeq 0.680702$, $e_R \simeq 0.937036$ and $t_E \simeq 0.726442$ at port 2; and
- the single-port excitations for the variable $\mathbf{a}$ provide $e_T \simeq 0.445706$, $e_R \simeq 0.894023$ and $t_E \simeq 0.498539$ at port 1, and $e_T \simeq 0.624706$, $e_R \simeq 0.932529$ and $t_E \simeq 0.669905$ at port 2.

We can use Theorem 1 to compute excitations such that $e_T = e_{T\,MIN}$ or $e_R = e_{R\,MIN}$, that is to say the worst excitations for the efficiency metrics $e_{T\,MIN}$ and $e_{R\,MIN}$, respectively. For instance, possible excitations providing $e_T = e_{T\,MIN}$ are

$$\mathbf{I}_{SG} \simeq \begin{pmatrix} -1.990 + 0.373j \\ 1.639 - 0.006j \end{pmatrix} \mathrm{A\ rms} \tag{161}$$

and

$$\mathbf{a} \simeq \begin{pmatrix} -4.975 + 0.933j \\ 3.666 - 0.013j \end{pmatrix} \mathrm{V}\Omega^{-1/2}\,\mathrm{rms}\,, \tag{162}$$

for both of which we obtain $e_T \simeq 0.248309$, $e_R \simeq 0.785753$ and $t_E \simeq 0.316015$. For instance, possible excitations providing $e_R = e_{R\,MIN}$ are

$$\mathbf{I}_{SG} \simeq \begin{pmatrix} -2.715 + 0.192j \\ 2.255 + 0.083j \end{pmatrix} \mathrm{A\ rms} \tag{163}$$

and

$$\mathbf{a} \simeq \begin{pmatrix} -6.786 + 0.480j \\ 5.041 + 0.187j \end{pmatrix} \mathrm{V}\Omega^{-1/2}\,\mathrm{rms}\,, \tag{164}$$

for both of which we obtain $e_T \simeq 0.249097$, $e_R \simeq 0.785088$ and $t_E \simeq 0.317286$.

The numerical values of $e_{T\,MIN}$, F_{TE}, $e_{R\,MIN}$, F_{RE}, t_{MIN}, F_M, e_T, e_R and t_E shown in this Section XII-A are consistent with (128) and (134)–(136).

B. SECOND CONFIGURATION

In this example, we use the same MAA as in the first configuration, so that $\mathbf{Y}_A$ and $\mathbf{Y}_{RAD}$ are again given by (141) and (142), respectively, at f_G. We assume a different MG, such that

$$\mathbf{Z}_G = \begin{pmatrix} 20 - 30j & 10 + 30j \\ 10 + 30j & 30 \end{pmatrix} \Omega\,. \tag{165}$$

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The ports of the MG are coupled. Moreover, since (141) gives

$$\mathbf{Z}_A \simeq \begin{pmatrix} 13.886+26.541j & 6.425-28.547j \\ 6.425-28.547j & 23.183-1.818j \end{pmatrix} \Omega\,, \quad (166)$$

we may consider that the value of $\mathbf{Z}_A$ is not very far from $\mathbf{Z}_G^*$ providing maximum power transfer for any excitation.

We posit

$$\mathbf{z}_0 = \begin{pmatrix} 20+30j & 0 \\ 0 & 30 \end{pmatrix} \Omega\,, \quad (167)$$

and $\mathbf{r}_0 = \mathrm{Re}(\mathbf{z}_0)$.

Among the matrices defined in Section IV and Section V, we find:

$$\mathbf{Y}_{AVGO} = \begin{pmatrix} 15.000 & -5.000 \\ -5.000 & 10.000 \end{pmatrix} \mathrm{mS}\,, \quad (168)$$

$$\mathbf{Z}_{AVGS} = \begin{pmatrix} 36.500 & -15.500 \\ -15.500 & 21.000 \end{pmatrix} \Omega\,, \quad (169)$$

$$\mathbf{Z}_{PAM} \simeq \begin{pmatrix} 69.812+14.153j & -31.554-9.799j \\ -31.554-9.799j & 41.463+2.322j \end{pmatrix} \Omega\,, \quad (170)$$

$$\mathbf{Y}_{AVG} \simeq \begin{pmatrix} 11.536 & 9.170+0.151j \\ 9.170-0.151j & 19.089 \end{pmatrix} \mathrm{mS}\,, \quad (171)$$

$$\mathbf{Y}_{SAM} \simeq \begin{pmatrix} 33.785+5.294j & -10.191-2.907j \\ -10.191-2.907j & 21.805+1.921j \end{pmatrix} \mathrm{mS}\,, \quad (172)$$

$$\mathbf{Z}_{AVG} \simeq \begin{pmatrix} 14.607 & 6.581-0.092j \\ 6.581+0.092j & 23.687 \end{pmatrix} \Omega\,, \quad (173)$$

$$\mathbf{Y}_{PSG} \simeq \begin{pmatrix} 29.444-1.402j & 16.708-3.889j \\ 17.574-7.740j & 41.908-3.046j \end{pmatrix} \mathrm{mS}\,, \quad (174)$$

$$\mathbf{\Lambda}_{AVG} \simeq \begin{pmatrix} 0.462 & 0.243+0.046j \\ 0.243-0.046j & 0.772 \end{pmatrix}\,, \quad (175)$$

$$\hat{\mathbf{Z}}_{PSG}^{-1} \simeq \begin{pmatrix} 20.316-7.624j & 21.287-7.881j \\ 10.569-9.388j & 43.759-6.544j \end{pmatrix} \mathrm{mS}\,, \quad (176)$$

and

$$\hat{\mathbf{\Lambda}}_{AVG} \simeq \begin{pmatrix} 0.250 & 0.240+0.057j \\ 0.240-0.057j & 0.883 \end{pmatrix}\,. \quad (177)$$

Among the matrices defined in Section VI, we find:

$$\mathbf{Z}_{RADS} \simeq \begin{pmatrix} 28.171 & -10.533-0.233j \\ -10.533+0.233j & 17.069 \end{pmatrix} \Omega\,, \quad (178)$$

$$\mathbf{Y}_{RADO} \simeq \begin{pmatrix} 11.478 & -3.222-0.099j \\ -3.222+0.099j & 8.559 \end{pmatrix} \mathrm{mS}\,, \quad (179)$$

$$\mathbf{Z}_{RAD} \simeq \begin{pmatrix} 12.110 & 7.093-0.247j \\ 7.093+0.247j & 21.723 \end{pmatrix} \Omega\,, \quad (180)$$

$$\mathbf{\Lambda}_{RAD} \simeq \begin{pmatrix} 0.391 & 0.257+0.037j \\ 0.257-0.037j & 0.704 \end{pmatrix}\,, \quad (181)$$

and

$$\hat{\mathbf{\Lambda}}_{RAD} \simeq \begin{pmatrix} 0.212 & 0.240+0.059j \\ 0.240-0.059j & 0.819 \end{pmatrix}\,. \quad (182)$$

Among the matrices defined in Section VIII, we find:

$$\mathbf{Y}_{RPAO} \simeq \begin{pmatrix} 14.220 & -4.755+0.076j \\ -4.755-0.076j & 9.740 \end{pmatrix} \mathrm{mS}\,, \quad (183)$$

$$\mathbf{Z}_{RPAS} \simeq \begin{pmatrix} 34.746 & -14.709-0.278j \\ -14.709+0.278j & 20.271 \end{pmatrix} \Omega\,, \quad (184)$$

$$\mathbf{Y}_{RPA} \simeq \begin{pmatrix} 11.141 & 8.978 \\ 8.978 & 18.562 \end{pmatrix} \mathrm{mS}\,, \quad (185)$$

$$\mathbf{Z}_{RPA} \simeq \begin{pmatrix} 13.886 & 6.425 \\ 6.425 & 23.183 \end{pmatrix} \Omega\,, \quad (186)$$

$$\mathbf{\Lambda}_{RPA} \simeq \begin{pmatrix} 0.440 & 0.237+0.040j \\ 0.237-0.040j & 0.754 \end{pmatrix}\,, \quad (187)$$

and

$$\hat{\mathbf{\Lambda}}_{AVG} \simeq \begin{pmatrix} 0.239 & 0.234+0.053j \\ 0.234-0.053j & 0.863 \end{pmatrix}\,. \quad (188)$$

In this second configuration, $\mathbf{Y}_{RAD}$, $\mathbf{Z}_{RAD}$, $\mathbf{Y}_{RPA}$ and $\mathbf{Z}_{RPA}$ are the same as the ones applicable to the first configuration covered in Section XII.A. We also note that $\mathbf{Y}_{PSG}$ given by (174) and $\hat{\mathbf{Z}}_{PSG}^{-1}$ given by (176) are neither hermitian nor symmetric, unlike all other matrices shown in this Section XII, and in contrast to $\mathbf{Y}_{PSG}$ given by (150) for the first configuration, which is diagonal and real. Using (142), (168)–(169), (171), (173), (175) and (177)–(188), we computed:

- the eigenvalues of $\mathbf{M}_{RAD}\mathbf{M}_{AVG}^{-1}$ for the variables $\mathbf{V}_{OG}$, $\mathbf{I}_{SG}$, $\mathbf{V}$, $\mathbf{I}$, $\mathbf{a}$ and $\hat{\mathbf{a}}$, to obtain 6 times the same values $e_{T\,MIN} \simeq 0.748022$ and $F_{TE} \simeq 0.501975$;
- the eigenvalues of $\mathbf{M}_{RAD}\mathbf{M}_{RPA}^{-1}$ for these 6 variables, to obtain six times the same results $e_{R\,MIN} \simeq 0.785088$ and $F_{RE} \simeq 0.463586$; and
- the eigenvalues of $\mathbf{M}_{RPA}\mathbf{M}_{AVG}^{-1}$ for these 6 variables, to obtain six times the same results $t_{MIN} \simeq 0.946499$ and $F_M \simeq 0.231302$.

The MAA being the same in the first and second configurations, the values of $e_{R\,MIN}$ and F_{RE} obtained here are identical to the ones obtained in Section XII.A.

The following results show that the values of e_T, e_R and t_E for single-port excitations depend on the selected variable and on the selected port:

- the single-port excitations for the variable $\mathbf{V}_{OG}$ provide $e_T \simeq 0.765190$, $e_R \simeq 0.807137$ and $t_E \simeq 0.948029$ at port 1, and $e_T \simeq 0.855895$, $e_R \simeq 0.878712$ and $t_E \simeq 0.974034$ at port 2;
- the single-port excitations for the variable $\mathbf{I}_{SG}$ provide $e_T \simeq 0.771795$, $e_R \simeq 0.810748$ and $t_E \simeq 0.951955$ at port 1, and $e_T \simeq 0.812787$, $e_R \simeq 0.842010$ and $t_E \simeq 0.965293$ at port 2;
- the single-port excitations for the variable $\mathbf{V}$ provide $e_T \simeq 0.881657$, $e_R \simeq 0.912929$ and $t_E \simeq 0.965745$ at port 1, and $e_T \simeq 0.920160$, $e_R \simeq 0.946291$ and $t_E \simeq 0.972385$ at port 2;
- the single-port excitations for the variable $\mathbf{I}$ provide $e_T \simeq 0.829092$, $e_R \simeq 0.872129$ and $t_E \simeq 0.950653$

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at port 1, and $e_T \simeq 0.917103$, $e_R \simeq 0.937036$ and $t_E \simeq 0.978728$ at port 2;

- the single-port excitations for the variable $\mathbf{a}$ provide $e_T \simeq 0.846873$, $e_R \simeq 0.888009$ and $t_E \simeq 0.953676$ at port 1, and $e_T \simeq 0.911934$, $e_R \simeq 0.934199$ and $t_E \simeq 0.976167$ at port 2; and
- the single-port excitations for the variable $\hat{\mathbf{a}}$ provide $e_T \simeq 0.846873$, $e_R \simeq 0.888009$ and $t_E \simeq 0.953676$ at port 1, and $e_T \simeq 0.927752$, $e_R \simeq 0.949032$ and $t_E \simeq 0.977577$ at port 2.

As in Section XII.A, we can use Theorem 1 to compute the worst excitations for the efficiency metrics $e_{T\,MIN}$ and $e_{R\,MIN}$. For instance, possible excitations such that we get $e_T = e_{T\,MIN}$ are

$$\mathbf{I}_{SG} \simeq \begin{pmatrix} -5.953 + 1.030j \\ 4.821 + 0.218j \end{pmatrix} \text{A rms} \tag{189}$$

and

$$\mathbf{a} \simeq \begin{pmatrix} -74.757 + 21.494j \\ 44.408 - 14.455j \end{pmatrix} \text{V}\Omega^{-1/2}\,\text{rms}\,, \tag{190}$$

for both of which we obtain $e_T \simeq 0.748022$, $e_R \simeq 0.785657$ and $t_E \simeq 0.952096$. For instance, possible excitations providing $e_R = e_{R\,MIN}$ are

$$\mathbf{I}_{SG} \simeq \begin{pmatrix} -5.664 + 0.027j \\ 4.978 + 0.052j \end{pmatrix} \text{A rms} \tag{191}$$

and

$$\mathbf{a} \simeq \begin{pmatrix} -73.584 + 12.283j \\ 45.884 - 11.741j \end{pmatrix} \text{V}\Omega^{-1/2}\,\text{rms}\,, \tag{192}$$

for both of which we obtain $e_T \simeq 0.748614$, $e_R \simeq 0.785088$ and $t_E \simeq 0.953542$.

The numerical values of $e_{T\,MIN}$, F_{TE}, $e_{R\,MIN}$, F_{RE}, t_{MIN}, F_M, e_T, e_R and t_E shown in this Section XII-B are consistent with (128) and (134)–(136).

XIII. CONCLUSION

We have derived some general properties of generalized Rayleigh ratios. We have used these properties to present a detailed theory of 2 generalized Rayleigh ratios which are efficiency metrics relevant to a specified excitation of an LTI MAA by an LTI MG. These efficiency metrics are the transducer efficiency e_T and the radiation efficiency e_R. The article only assumes that $\mathbf{Z}_G$ exists and has a positive definite hermitian part, or, equivalently, that $\mathbf{Y}_G$ exists and has a positive definite hermitian part. We have neither assumed that the MAA is reciprocal, nor that $\mathbf{Z}_G$ or $\mathbf{Y}_G$ are symmetric.

The power transfer ratio during emission t_E satisfies $e_T = e_R\, t_E$. We have provided formulas for computing e_T, e_R and t_E, by utilizing different variables to define the excitation. The variables $\mathbf{I}_{SG}$, $\mathbf{V}_{OG}$, $\mathbf{a}$, and $\hat{\mathbf{a}}$ are always applicable, while the variable $\mathbf{V}$ is applicable if and only if $\mathbf{Y}_A$ exists, and the variable $\mathbf{I}$ is applicable if and only if $\mathbf{Z}_A$ exists (as shown in Table 3 or Table 5). We have not assumed any relationship between $\mathbf{Z}_G$ and the diagonal resistance matrix $\mathbf{r}_0$ used to define the variable $\mathbf{a}$, or the diagonal impedance matrix $\mathbf{z}_0$ used to define the variable $\hat{\mathbf{a}}$.

We have defined and investigated 4 new efficiency metrics relevant to unspecified excitations: the minimum transducer efficiency $e_{T\,MIN}$, the transducer efficiency figure F_{TE}, the minimum radiation efficiency $e_{R\,MIN}$, and the radiation efficiency figure F_{RE}. They are connected with the minimum power transfer ratio during emission t_{MIN} and the power match figure F_M, which are matching metrics. We have seen that $e_{T\,MIN}$, F_{TE}, $e_{R\,MIN}$, F_{RE}, t_{MIN} and F_M do not depend on the applicable variable used to compute them.

We have noted that e_R has a special property: it is only a function of the characteristics of the MAA and of the excitation, if the excitation is specified using any one of the variables $\mathbf{a}$, or $\hat{\mathbf{a}}$, or $\mathbf{V}$ if $\mathbf{Y}_A$ exists, or $\mathbf{I}$ if $\mathbf{Z}_A$ exists. Consequently, F_{RE} and $e_{R\,MIN}$ depend only on the MAA.

During the design of an MAA, we can use appropriate simulation programs to determine the new metrics, for any arbitrary $\mathbf{Z}_G$ of interest. At the MAA characterization stage, the instruments used to perform the measurements are typically such that the results are the ones which would be obtained with an ideal LTI MG having a well-defined impedance matrix, which need not be diagonal. For instance, if the setup used to characterize the MAA comprises a MIMO matching and decoupling network (MDN) considered in [1]–[2] and a 2-port 50-Ω vector network analyzer (VNA), each of the N output ports of the MDN being connected to one of the N ports of the MAA through a coaxial cable, two input ports of the MDN being coupled to the measurement ports of the VNA, the remaining input ports of the MDN being coupled to 50-Ω loads, then the measurement results are the ones which would be obtained with an LTI MG presenting an impedance matrix that is typically not diagonal.

In a radio transmitter using a single antenna, the antenna output need not behave like an LTI port, and the nominal impedance of the antenna output is typically defined as the complex conjugate of a load impedance for which the antenna output was designed, somewhat like the large-signal output impedance of a radio-frequency power device is the complex conjugate of a load impedance for which the device produces the largest radio-frequency output power [38]–[39]. Even in the case where the final stage of the transmitter's radio-frequency power amplifier operates in class A, often regarded as corresponding to a “linear amplifier”, this nominal impedance is typically not related to an impedance presented by the antenna output of the transmitter [40, Sec. 4.4].

We now consider an actual configuration comprising the MAA, possibly some LTI components, and a radio transmitter having N antenna outputs. If the transmitter's antenna outputs behaved like an LTI multiport, we could directly apply all results presented in this article. However, based on what we said about antenna outputs, we will not assume that the transmitter's antenna outputs behave like an LTI multiport. For a specified excitation, we can nevertheless determine e_R, using any one of the variables for which e_R may be defined without reference to the characteristics of an LTI MG, that is the variables $\mathbf{a}$, or $\hat{\mathbf{a}}$, or $\mathbf{V}$ if $\mathbf{Y}_A$ exists, or $\mathbf{I}$ if $\mathbf{Z}_A$ exists. For unspecified excitations, we can also use $e_{R\,MIN}$ and F_{RE}, which only depend on the MAA.

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In the actual configuration, let $\mathbf{Z}_N$ be the load impedance matrix for which the antenna outputs of the transmitter are specified. We assume that the MAA has an impedance matrix $\mathbf{Z}_A$, and that, if $\mathbf{Z}_A$ was equal to a wanted impedance matrix $\mathbf{Z}_W$, then the transmitter's antenna outputs would see the impedance matrix $\mathbf{Z}_N$. We of course have $\mathbf{Z}_W = \mathbf{Z}_N$ if each antenna output of the transmitter is directly connected to the appropriate port of the MAA. We now also consider the theoretical configuration of Section IV, under the assumption that the LTI MG is such that $\mathbf{Z}_G = \mathbf{Z}_W^*$. We see that t_{MIN} and F_M are matching metrics that measure the closeness of $\mathbf{Z}_A$ to $\mathbf{Z}_W$, since $t_{MIN} = 1$ and $F_M = 0$ if and only if $\mathbf{Z}_A = \mathbf{Z}_G^* = \mathbf{Z}_W$. It follows that the theoretical configuration has some relevance to the operation of the transmitter with the MAA in the actual configuration. For this reason, in addition to e_R, $e_{R\,MIN}$ and F_{RE}, which are fully relevant to the actual configuration, we see that: e_T, t_E and the TARC given by (93), defined for the theoretical configuration, have some relevance to a specified excitation in the actual configuration; and $e_{T\,MIN}$, F_{TE}, t_{MIN} and F_M, defined for the theoretical configuration, have some relevance to unspecified excitations in the actual configuration.

If we now come back to the context of the design of the MAA, we see that using $\mathbf{Z}_G = \mathbf{Z}_W^*$ is an interesting choice, for which maximizing $e_{R\,MIN}$ or minimizing F_{RE} combines the aim of a good radiation efficiency for any excitation, and the aim of getting the transmitter's antenna outputs to see an impedance matrix close to $\mathbf{Z}_N$.

APPENDIX A

In this Appendix A, we derive the main results of Section V, without using a parallel-augmented multiport or a series-augmented multiport, but we need to assume that $\mathbf{Y}_A$ exists. The MAA being passive, $H(\mathbf{Y}_A)$ is positive semidefinite. It follows that $H(\mathbf{Y}_A + \mathbf{Y}_G)$ is positive definite, so that, by Lemma 1 of [21], $\mathbf{Y}_A + \mathbf{Y}_G$ is invertible. Thus, if there is no incident electromagnetic signal received by the MAA, the column vector of the rms voltages at ports 1 to N of the MAA, denoted by $\mathbf{V}$, is given by

$$\mathbf{V} = \mathbf{Y}_T^{-1}\,\mathbf{I}_{SG}\,, \tag{193}$$

where $\mathbf{Y}_T = \mathbf{Y}_A + \mathbf{Y}_G$, so that (36)–(37) lead us to (40) and

$$\mathbf{Y}_{AVG} = \frac{1}{2}\mathbf{Y}_T^*\,(\mathbf{Y}_G + \mathbf{Y}_G^*)^{-1}\,\mathbf{Y}_T\,. \tag{194}$$

It follows from (193) that the column vector of the rms currents flowing into ports 1 to N of the MAA, denoted by $\mathbf{I}$, is given by

$$\mathbf{I} = \mathbf{Y}_A\mathbf{Y}_T^{-1}\,\mathbf{I}_{SG}\,. \tag{195}$$

$\mathbf{Z}_A$ exists if and only if $\mathbf{Y}_A$ is invertible. In this case, (36)–(37) lead us to (44) and

$$\mathbf{Z}_{AVG} = \frac{1}{2}\mathbf{Y}_A^{-1*}\mathbf{Y}_T^*\,(\mathbf{Y}_G + \mathbf{Y}_G^*)^{-1}\,\mathbf{Y}_T\mathbf{Y}_A^{-1}\,. \tag{196}$$

We define $\mathbf{r}_0 = \mathrm{diag}_N(r_{01}, \ldots, r_{0N})$, as in Section V. For the reference resistances $r_{01}, \ldots, r_{0N}$, the column vector $\mathbf{a}$ of the normalized rms incident voltages at ports 1 to N of the MAA is given by [26]:

$$\mathbf{a} = \mathbf{r}_0^{-1/2}\,\frac{\mathbf{V} + \mathbf{r}_0\mathbf{I}}{2} = \mathbf{r}_0^{-1/2}\,\frac{\mathbf{1}_N + \mathbf{r}_0\mathbf{Y}_A}{2}\,\mathbf{V}\,. \tag{197}$$

Since $H(\mathbf{Y}_A)$ is positive semidefinite, $H(\mathbf{r}_0^{1/2}\mathbf{Y}_A\mathbf{r}_0^{1/2})$ is positive semidefinite, so that $H(\mathbf{1}_N + \mathbf{r}_0^{1/2}\mathbf{Y}_A\mathbf{r}_0^{1/2})$ is positive definite. It follows that, by Lemma 1 of [21], $\mathbf{1}_N + \mathbf{r}_0^{1/2}\mathbf{Y}_A\mathbf{r}_0^{1/2}$ is invertible. This matrix being similar to $\mathbf{1}_N + \mathbf{r}_0\mathbf{Y}_A$, we may conclude that $\mathbf{1}_N + \mathbf{r}_0\mathbf{Y}_A$ is invertible. Thus, by (194) and (197), we get (54) and

$$\begin{aligned}\mathbf{\Lambda}_{AVG} = {}& 2\mathbf{r}_0^{1/2}(\mathbf{1}_N + \mathbf{r}_0\mathbf{Y}_A)^{-1*}\mathbf{Y}_T^* \\ & \times (\mathbf{Y}_G + \mathbf{Y}_G^*)^{-1}\,\mathbf{Y}_T(\mathbf{1}_N + \mathbf{r}_0\mathbf{Y}_A)^{-1}\mathbf{r}_0^{1/2}\,.\end{aligned} \tag{198}$$

In (198), we can replace $\mathbf{Y}_G$, $\mathbf{Y}_A$ and $\mathbf{Y}_T = \mathbf{Y}_A + \mathbf{Y}_G$ with their values expressed using the scattering matrix $\mathbf{S}_G$ of the MG and $\mathbf{S}_A$ of the MAA, by utilizing (56) and

$$\mathbf{Y}_A = \mathbf{r}_0^{-1/2}(\mathbf{1}_N - \mathbf{S}_A)(\mathbf{1}_N + \mathbf{S}_A)^{-1}\mathbf{r}_0^{-1/2}\,, \tag{199}$$

where the classical formula (199) is a consequence of (223), for $\mathbf{z}_0 = \mathbf{r}_0$. In the special case where $\mathbf{Y}_G = \mathbf{r}_0^{-1}$, (198) leads us to $\mathbf{\Lambda}_{AVG} = \mathbf{1}_N$, and (54) leads us to (62).

We define $\mathbf{z}_0 = \mathrm{diag}_N(z_{01}, \ldots, z_{0N})$ as in Section V. For the reference impedances $z_{01}, \ldots, z_{0N}$, the column vector $\hat{\mathbf{a}}$ of the rms power waves incident at ports 1 to N of the MAA is given by [27]:

$$\hat{\mathbf{a}} = \mathbf{r}_0^{-1/2}\,\frac{\mathbf{V} + \mathbf{z}_0\mathbf{I}}{2} = \mathbf{r}_0^{-1/2}\,\frac{\mathbf{1}_N + \mathbf{z}_0\mathbf{Y}_A}{2}\,\mathbf{V}\,. \tag{200}$$

This vector may also be viewed as the column vector of the rms pseudo-waves incident at ports 1 to N of the MAA [28]. If we assume that $\mathbf{1}_N + \mathbf{z}_0\mathbf{Y}_A$ is invertible, by utilizing (194) and (200), we obtain (68) and

$$\begin{aligned}\hat{\mathbf{\Lambda}}_{AVG} = {}& 2\mathbf{r}_0^{1/2}(\mathbf{1}_N + \mathbf{z}_0\mathbf{Y}_A)^{-1*}\mathbf{Y}_T^* \\ & \times (\mathbf{Y}_G + \mathbf{Y}_G^*)^{-1}\,\mathbf{Y}_T(\mathbf{1}_N + \mathbf{z}_0\mathbf{Y}_A)^{-1}\mathbf{r}_0^{1/2}\,.\end{aligned} \tag{201}$$

In the special case where $\mathbf{Y}_G = \mathbf{z}_0^{-1}$, (201) leads us to $\hat{\mathbf{\Lambda}}_{AVG} = \mathbf{1}_N$, and (68) leads us to (71).

APPENDIX B

In this Appendix B, we provide proofs of the existence of the scattering matrix, the pseudo-wave scattering matrix and the power-wave scattering matrix of a device under study (DUS), at a specified frequency. We assume that the DUS is a passive n-port LTI device. The need for such proofs is discussed at the end of this appendix.

The ports of the DUS are numbered from 1 to n. We define $\mathbf{z}'_0 = \mathrm{diag}_n(z'_{01}, \ldots, z'_{0n})$, where the n arbitrary reference impedances $z'_{01}, \ldots, z'_{0n}$ are such that, for any integer $p \in \{1, \ldots, n\}$, $r'_{0p} = \mathrm{Re}(z'_{0p})$ is positive. Using the theory of parallel-augmented multiports and series-augmented multiports presented in [21, Sec. II], the DUS is regarded as the original multiport, and we introduce a series-augmented multiport composed of the DUS and of an added multiport, such that, for any integer $p \in \{1, \ldots, n\}$, port p of the DUS is connected in series with an impedor (i.e., a passive single-port LTI component) of impedance z'_{0p} at the specified frequency. The added mutiport has an impedance matrix, which is equal to $\mathbf{z}'_0$ at the specified frequency, and has therefore a positive definite hermitian part.

By Theorem 2 of [21], we can say that, at the specified frequency, the series-augmented multiport has an admittance matrix, denoted by $\mathbf{Y}'_{SAM}$, which depends on $\mathbf{z}'_0$ and has a positive semidefinite hermitian part. Let $\mathbf{V}'$ be the column vector of the rms voltages at ports 1 to n of the original multiport, and $\mathbf{I}'$ be the column vector of the rms currents flowing into ports 1 to n of the original multiport. We have

$$\mathbf{Y}'_{SAM}(\mathbf{V}' + \mathbf{z}'_0\mathbf{I}') = \mathbf{I}' \,. \tag{202}$$

Using

$$\hat{\mathbf{a}}' = \mathbf{r}'^{-1/2}_0 \frac{\mathbf{V}' + \mathbf{z}'_0\mathbf{I}'}{2} \,, \tag{203}$$

where $\mathbf{r}'_0 = \mathrm{Re}(\mathbf{z}'_0)$, we get

$$2\mathbf{Y}'_{SAM}\, \mathbf{r}'^{1/2}_0\, \hat{\mathbf{a}}' = \mathbf{I}' \,. \tag{204}$$

Thus, we have

$$\left(\mathbf{1}_n - 2\mathbf{r}'^{-1/2}_0\, \mathbf{z}'_0\mathbf{Y}'_{SAM}\, \mathbf{r}'^{1/2}_0\right) \hat{\mathbf{a}}' = \hat{\mathbf{a}}' - \mathbf{r}'^{-1/2}_0\, \mathbf{z}'_0\mathbf{I}' \,, \tag{205}$$

which leads us to

$$\left(\mathbf{1}_n - 2\mathbf{r}'^{-1/2}_0\, \mathbf{z}'_0\mathbf{Y}'_{SAM}\, \mathbf{r}'^{1/2}_0\right) \hat{\mathbf{a}}' = \check{\mathbf{b}}' \,, \tag{206}$$

where

$$\check{\mathbf{b}}' = \mathbf{r}'^{-1/2}_0 \frac{\mathbf{V}' - \mathbf{z}'_0\mathbf{I}'}{2} \,. \tag{207}$$

$\hat{\mathbf{a}}'$ being the column vector of the pseudo-waves incident at ports 1 to n of the original multiport, and $\check{\mathbf{b}}'$ being the column vector of the pseudo-waves reflected at ports 1 to n of the original multiport, we have proven that, for the reference impedances $z'_{01}, \ldots, z'_{0n}$, the DUS has a pseudo-wave scattering matrix, given by

$$\check{\mathbf{S}}' = \mathbf{1}_n - 2\mathbf{r}'^{-1/2}_0\, \mathbf{z}'_0\mathbf{Y}'_{SAM}\, \mathbf{r}'^{1/2}_0 \,, \tag{208}$$

at the specified frequency, and such that, for any $\hat{\mathbf{a}}' \in \mathbb{C}^N$, we have

$$\check{\mathbf{S}}'\hat{\mathbf{a}}' = \check{\mathbf{b}}' \,. \tag{209}$$

Using (204), we also get

$$\left(\mathbf{1}_n - \mathbf{r}'^{-1/2}_0\, (\mathbf{z}'_0 + \overline{\mathbf{z}'_0})\mathbf{Y}'_{SAM}\, \mathbf{r}'^{1/2}_0\right) \hat{\mathbf{a}}' = \hat{\mathbf{a}}' - \mathbf{r}'^{-1/2}_0 \frac{(\mathbf{z}'_0 + \overline{\mathbf{z}'_0})\mathbf{I}'}{2} \,, \tag{210}$$

where the horizontal bar above a vector represents the complex conjugate vector. Thus, for any $\hat{\mathbf{a}}' \in \mathbb{C}^N$, we have

$$\left(\mathbf{1}_n - \mathbf{r}'^{-1/2}_0\, (\mathbf{z}'_0 + \overline{\mathbf{z}'_0})\mathbf{Y}'_{SAM}\, \mathbf{r}'^{1/2}_0\right) \hat{\mathbf{a}}' = \hat{\mathbf{b}}' \,, \tag{211}$$

where

$$\hat{\mathbf{b}}' = \mathbf{r}'^{-1/2}_0 \frac{\mathbf{V}' - \overline{\mathbf{z}'_0}\mathbf{I}'}{2} \,. \tag{212}$$

$\hat{\mathbf{a}}'$ being the column vector of the power waves incident at ports 1 to n of the original multiport, and $\hat{\mathbf{b}}'$ being the column vector of the power waves reflected at ports 1 to n of the original multiport, we have proven that, for the reference impedances $z'_{01}, \ldots, z'_{0n}$, the DUS has a power-wave scattering matrix, given by

$$\hat{\mathbf{S}}' = \mathbf{1}_n - \mathbf{r}'^{-1/2}_0\, (\mathbf{z}'_0 + \overline{\mathbf{z}'_0})\mathbf{Y}'_{SAM}\, \mathbf{r}'^{1/2}_0 \,, \tag{213}$$

at the specified frequency, and such that, for any $\hat{\mathbf{a}}' \in \mathbb{C}^N$, we have

$$\hat{\mathbf{S}}'\hat{\mathbf{a}}' = \hat{\mathbf{b}}' \,. \tag{214}$$

The proof of the existence of a scattering matrix of the DUS, for n reference resistances, is a direct consequence of (208) or (213), in the special case where the reference impedances $z'_{01}, \ldots, z'_{0n}$ are real, since, in this special case, the scattering matrix is the pseudo-wave scattering matrix, and also the power-wave scattering matrix. For the reference resistances $r'_{01}, \ldots, r'_{0n}$, the scattering matrix is given by

$$\mathbf{S}' = \mathbf{1}_n - 2\mathbf{r}'^{1/2}_0\, \mathbf{Y}'_{SAM}\, \mathbf{r}'^{1/2}_0 \,, \tag{215}$$

at the specified frequency.

As explained in [21, Sec. III], the existence of the scattering matrix was discussed by several authors [26], [41] who regarded as obvious: the existence of the admittance matrix of a series-augmented multiport comprising n resistors added in series to the ports of the original multiport; or the existence of the impedance matrix of a parallel-augmented multiport comprising n resistors added in parallel to the ports of the original multiport. The existence of the scattering matrix is proven in a paper of Youla, Castriota and Carlin [42]–[43], by utilizing several assumptions, which, unfortunately, do not have a clear physical significance. In Section 3.3 of [44, Ch. 2], the existence of different "scattering matrices" was asserted, based on: the assumed existence of the admittance matrix of a particular series-augmented multiport comprising n impedors added in series to the ports of the original multiport; and the assumed existence of the impedance matrix of a particular parallel-augmented multiport comprising n impedors added in parallel to the ports of the original multiport. We are not aware of any sound proof of the existence of the pseudo-wave scattering matrix such that $\check{\mathbf{S}}'\hat{\mathbf{a}}' = \check{\mathbf{b}}'$, and of the power-wave scattering matrix such that $\hat{\mathbf{S}}'\hat{\mathbf{a}}' = \hat{\mathbf{b}}'$.

This background explains that this Appendix B was needed, to provide simple and direct proofs, firmly based on the results of [21], of the existence of the scattering matrix, the pseudo-wave scattering matrix and the power-wave scattering matrix of the DUS at the specified frequency.

APPENDIX C

In this Appendix C, using the same assumptions and notations as in Appendix B, we summarize the derivation of some classical results used elsewhere in this article. We use a parallel-augmented multiport composed of the DUS (as original multiport) and of an added multiport, such that, for any integer $p \in \{1, \ldots, n\}$, port p of the DUS is connected in parallel with an impedor of impedance z'_{0p} at the specified frequency. By Theorem 1 of [21], we can say that, at the specified frequency, the parallel-augmented multiport has an

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impedance matrix, denoted by $\mathbf{Z}'_{PAM}$, which depends on $\mathbf{z}'_0$ and has a positive semidefinite hermitian part. We have

$$\mathbf{Z}'_{PAM}(\mathbf{I}' + \mathbf{z}'^{-1}_0 \mathbf{V}') = \mathbf{V}' \,. \tag{216}$$

Using (203), we get

$$2\mathbf{Z}'_{PAM}\, \mathbf{z}'^{-1}_0 \mathbf{r}'^{1/2}_0\, \hat{\mathbf{a}}' = \mathbf{V}' \,, \tag{217}$$

which can be used to obtain

$$\left(2\mathbf{r}'^{-1/2}_0\, \mathbf{Z}'_{PAM}\, \mathbf{z}'^{-1}_0 \mathbf{r}'^{1/2}_0 - \mathbf{1}_n\right) \hat{\mathbf{a}}' = \check{\mathbf{b}}' \,, \tag{218}$$

for any $\hat{\mathbf{a}}' \in \mathbb{C}^N$. It follows that

$$\check{\mathbf{S}}' = 2\mathbf{r}'^{-1/2}_0\, \mathbf{Z}'_{PAM}\, \mathbf{z}'^{-1}_0 \mathbf{r}'^{1/2}_0 - \mathbf{1}_n \,, \tag{219}$$

at the specified frequency, which provides an alternative proof of the existence of the pseudo-wave scattering matrix.

We now assume that the DUS has an admittance matrix, denoted by $\mathbf{Y}'$. It follows from Corollary 1 of [21] that $\mathbf{Z}'_{PAM}$ is invertible and

$$\mathbf{Z}'^{-1}_{PAM} = \mathbf{Y}' + \mathbf{z}'^{-1}_0 \,. \tag{220}$$

Using (220) in (219), we get

$$\check{\mathbf{S}}' = 2\mathbf{r}'^{-1/2}_0 \left(\mathbf{Y}' + \mathbf{z}'^{-1}_0\right)^{-1} \mathbf{z}'^{-1}_0 \mathbf{r}'^{1/2}_0 - \mathbf{1}_n \,, \tag{221}$$

which entails that $\check{\mathbf{S}}' + \mathbf{1}_n$ is invertible. After some manipulations, we obtain

$$\begin{aligned} \check{\mathbf{S}}' &= \mathbf{r}'^{-1/2}_0 \left(\mathbf{1}_n + \mathbf{z}'_0 \mathbf{Y}'\right)^{-1} \left(\mathbf{1}_n - \mathbf{z}'_0 \mathbf{Y}'\right) \mathbf{r}'^{1/2}_0 \\ &= \mathbf{r}'^{-1/2}_0 \left(\mathbf{1}_n - \mathbf{z}'_0 \mathbf{Y}'\right) \left(\mathbf{1}_n + \mathbf{z}'_0 \mathbf{Y}'\right)^{-1} \mathbf{r}'^{1/2}_0 \end{aligned} \tag{222}$$

and

$$\begin{aligned} \mathbf{Y}' &= \mathbf{z}'^{-1}_0 \mathbf{r}'^{1/2}_0 \left(\mathbf{1}_n + \check{\mathbf{S}}'\right)^{-1} \left(\mathbf{1}_n - \check{\mathbf{S}}'\right) \mathbf{r}'^{-1/2}_0 \\ &= \mathbf{z}'^{-1}_0 \mathbf{r}'^{1/2}_0 \left(\mathbf{1}_n - \check{\mathbf{S}}'\right) \left(\mathbf{1}_n + \check{\mathbf{S}}'\right)^{-1} \mathbf{r}'^{-1/2}_0 \end{aligned} \tag{223}$$

for the pseudo-wave scattering matrix. In the case $\mathbf{z}'_0 = \mathbf{r}'_0$, we get the corresponding formulas for the scattering matrix.

APPENDIX D

This Appendix D provides an alternative derivation of (82), and a result which may be used in the place of (83).

For any integer $p \in \{1, \ldots, N\}$, we can consider a single-port antenna called SPAOC-p, obtained by using only port p of the MAA, the other ports of the MAA being left open-circuited. A coordinate system having its origin close to the MAA being chosen, let $\mathbf{h}_p$ be the vector effective length of SPAOC-p in a direction (θ, φ), as defined in [30, Sec. 5.2] and [31, Sec. 16.5]. Let $\mathbf{E}_p$ be the electric field radiated by SPAOC-p used for emission, in the direction (θ, φ). At a large distance r of the origin, $\mathbf{E}_p$ is given by

$$\mathbf{E}_p = j\eta \frac{I_p\, k\, e^{-jkr}}{4\pi r}\, \mathbf{h}_p \tag{224}$$

where k is the wave number in the relevant medium, η is the intrinsic impedance of this medium, and I_p is a current flowing into the port of SPAOC-p.

If we now consider the configuration shown in Fig. 1, the linearity of the MAA entails that the electric field radiated by the MAA used for emission in the direction (θ, φ), denoted $\mathbf{E}$, is given by

$$\mathbf{E} = j\eta \frac{k\, e^{-jkr}}{4\pi r} \sum_{p=1}^{N} I_p \mathbf{h}_p \,, \tag{225}$$

where $I_1, \ldots, I_N$ are the rms currents flowing into ports 1 to N of the MAA, that is the entries of $\mathbf{I}$. In the derivation of (225), we have used a superposition of SPAOC-1 to SPAOC-N excited by the currents I_1 to I_N, respectively. This is possible if and only if we assume that $\mathbf{I}$ may take on any value lying in $\mathbb{C}^N$. By (42), this is possible if and only if $\mathbf{Y}_{SAM}$ is invertible, that is to say if and only if the MAA has an impedance matrix.

The power radiated by the MAA, denoted by P_{RAD}, being given by (74), we get

$$P_{RAD} = \frac{\eta k^2}{16\pi^2} \sum_{p=1}^{N} \sum_{q=1}^{N} \bar{I}_p I_q \iint \mathbf{h}^*_p \mathbf{h}_q \;\sin\theta \, d\theta d\varphi \,. \tag{226}$$

Thus, we obtain (82), where $\mathbf{Z}_{RAD}$ is such that, for any integers p and q lying in $\{1, \ldots, N\}$, the entry of row p and column q of $\mathbf{Z}_{RAD}$ is

$$Z_{RAD\,pq} = \frac{\eta k^2}{16\pi^2} \iint \mathbf{h}^*_p \mathbf{h}_q \;\sin\theta \, d\theta d\varphi \,. \tag{227}$$

We see that $\mathbf{Z}_{RAD}$ is hermitian. Moreover, P_{RAD} being nonnegative for any nonzero $\mathbf{I} \in \mathbb{C}^N$, it follows that $\mathbf{Z}_{RAD}$ is positive semidefinite.

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