The Douglas Triangle, Khufu and Khafre

How π, φ and √3 connect the Khufu and Khafre pyramids

Ian Douglas, B.Sc
ian@zti.co.za

30 December 2021
Version 1.0.0
DOI: https://doi.org/10.5281/zenodo.5810430

Unqid: ddbeecdd0c82198ca956c4d9fc7e9ff9
Please check via DOI for latest version.
Best viewed and printed in colour.

Abstract

The Douglas triangle is a special right triangle using φ and $\sqrt{3}$. It appears that its proportions and peculiarities impressed the Giza planners so much that they used it, when scaled by 50π, for the base lengths of the Khufu and Khafre pyramids. This paper presents the triangle and discusses some of its interesting features.

Keywords: Egyptology, Giza, geometry, archaeogeometry, π, pi, φ, golden ratio, history of mathematics.

Contents

Revision history:
1.0.0 30 December 2021 : Initial version.

1. Introduction

"*The language of Giza is mathematics.*"
Robert Bauval

"*You will believe.*"
The architects of Giza

As with most of my discoveries at Giza, I stumbled into this accidentally. In essence, it shows how the base dimensions of Khufu and Khafre are linked via a special right triangle using π, φ and √3. This answers the question, "Why 411?", given that there are several other reasons why Khufu is 440₢.

I have put the proofs in a separate section at the end, to improve the flow, and cater to those not so interested in the mathematics.

2. Notation, accuracy and methodology

2.1 Notation

Giza is a construction site, so we need to take an engineering rather than mathematical approach in dealing with mathematical constants like π, φ, e, or square roots. While working on Zep Tepi Mathematics 101 (ZTM101) [1] in 2020 it became apparent they thought 3 or 4 decimals were accurate enough, so I have used that, and things "just work."

I take the royal cubit (henceforth "cubit", symbol ₢) as π/6 metres, to 4 decimal places. (*The Beautiful Cubit System*, Douglas 2019 [2]).

Symbols used in this and other papers:

Symbol	*Name*	*Engineering/Construction value*	*Date*	*% Accuracy*
π	Archimedes' constant	3.1416	250 BCE	99.9998
e	Euler's number	2.7183	1683	99.9993
φ	Golden ratio	1.618 $\varphi+1 = \varphi^2 = 2.618$	500 BCE?	99.9979
ρ	Plastic number or ratio	1.3247 $\rho+1 = \rho^3 = 2.3247$	1924	99.9986
₢	Royal cubit aka cubit	0.5236m (π/6)	55.5k BCE?	99.9998
⌣	360° divided by			

Table 1: Symbols, names and values

3. The Douglas Triangle

The Douglas Triangle is a special right triangle analogous to the classic Pythagorean 3 : 4 : 5 design used for Khafre, and the Kepler $1 : \sqrt{\varphi} : \varphi$ design used for Khufu.

The side ratios can be expressed in slightly different ways, depending on the use case. For now, we will use the simple case analogous to the Kepler triangle, where the short side is 1. The long side is then φ^2, and the hypotenuse $\sqrt{3}\varphi$. (See proof 1.)

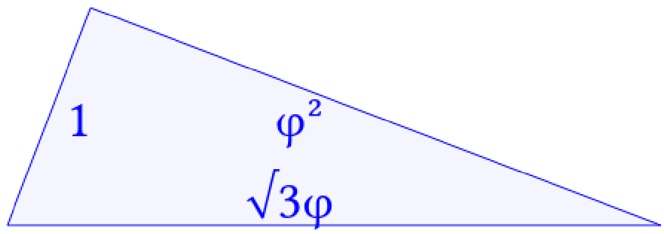


Figure 1: The Douglas Triangle

Alternatively, if we divide through by φ, then we have $\varphi^{-1} : \varphi : \sqrt{3}$.

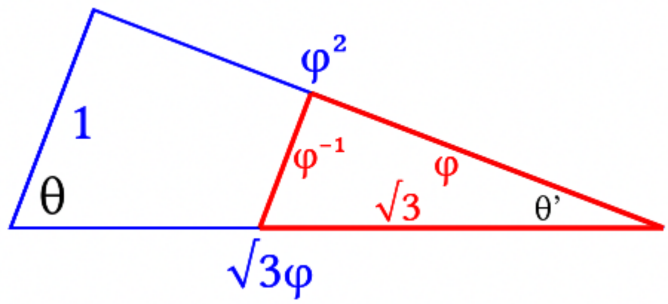


Figure 2: The Douglas Triangle, two versions

Dividing through again by φ takes us to $\varphi^{-2} : 1 : \frac{\sqrt{3}}{\varphi}$, which will be shown later.

This uses φ^{-2}, φ^{-1}, 1, φ, φ^2, and $\sqrt{3}$.

We can write the ratios more generally, for other sizes, as $k : k\varphi^2 : k\sqrt{3}\varphi$, where k is the scaling factor.

I find it curious that $\sqrt{3}$ appears here in a triangle based on φ, and that $\varphi^2 + \varphi^{-2} = 3$.

The large angle θ is arctan(φ^2) which is 69.09°, and the small angle θ' thus 20.91°.

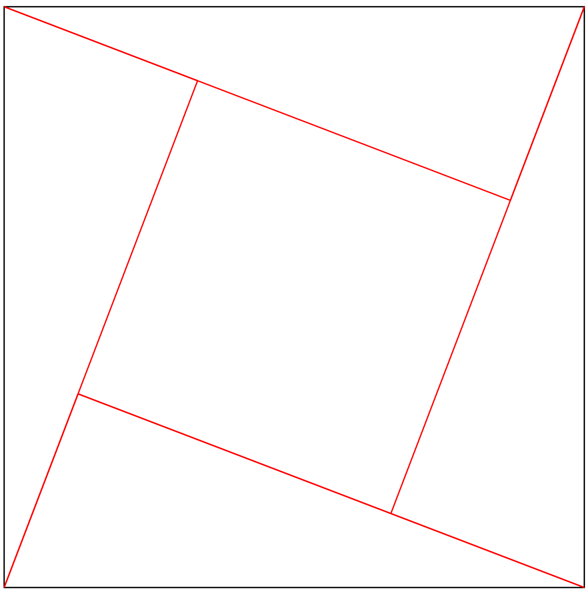
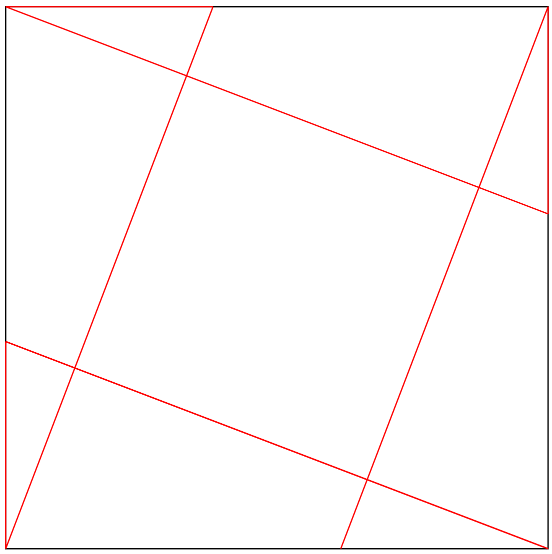
Comparing the basic trigonometric functions for the Kepler, Pythagorean, and Douglas triangles:

Trig Ratio	*Kepler*	*Pythagorean*	*Douglas*
sine(θ)	$1/\sqrt{\varphi^3}$	4/5	$\varphi/\sqrt{3}$
cosine(θ)	$1/\varphi^2$	3/5	$1/\sqrt{3}\varphi$
tangent(θ)	$\sqrt{\varphi}$	1⅓	φ^2

Trig Ratio	*Kepler*	*Pythagorean*	*Douglas*
cosecant(θ)	$\sqrt{\varphi^3}$	1¼	$\sqrt{3}/\varphi$
secant(θ)	φ^2	1⅔	$\sqrt{3}\varphi$
cotangent(θ)	$1/\sqrt{\varphi}$	¾	$1/\varphi^2$

Table 2: Trigonometric ratios for the three triangles

The triangle in turn leads to the Douglas Squares, which are explored in a separate paper, since they are more about mathematics than about Giza.

Douglas Square, base $\sqrt{3}\varphi$	*Douglas Square, base* φ^2
	

4. The Douglas Triangle at Giza

Giza is replete with “special” right triangles, including 3 : 4 : 5 (Pythagorean), 1:√φ : φ (Kepler), √2 : √3 : √5 (Legon), 1 : 2 : √5 (Golden), and possibly 1 : √2 : √3.

Some are used as design paradigms for the pyramids themselves, while others were used to lay out the site. Here is a site diagram adapted from ZTM101 showing where they are used.

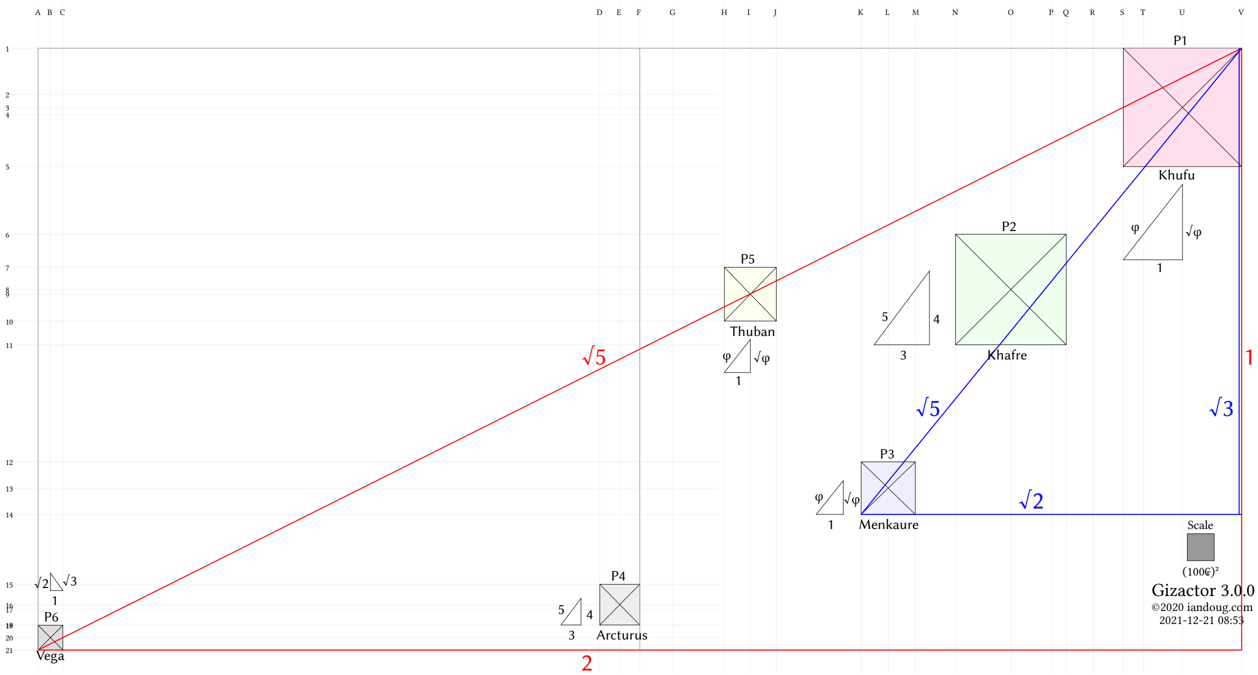


Figure 3: Giza plan showing special right triangles used.

The heights, and thus design triangles, for P4, P5 and P6 are based on my best guesses.

In order to compare the shapes, I have resized the various triangles to similar sizes, and added the Douglas Triangle.

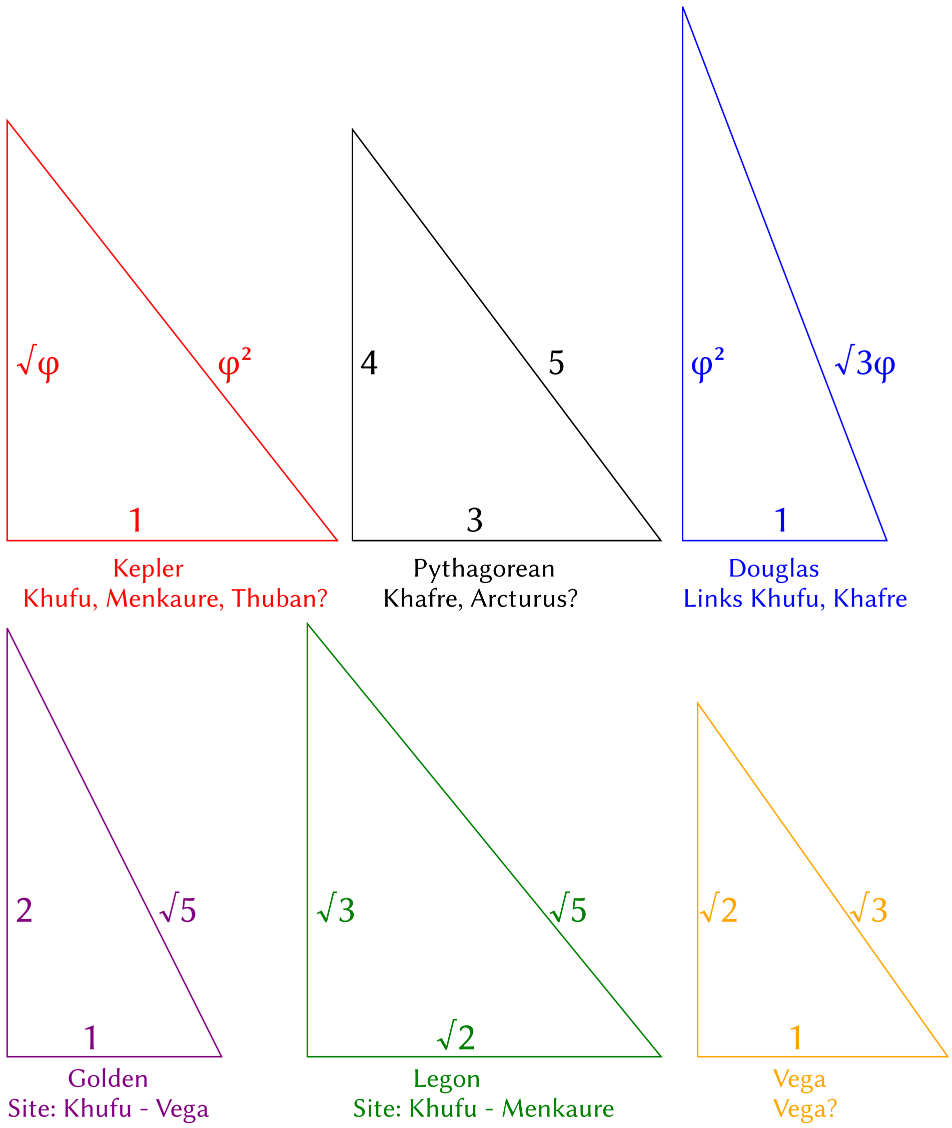


Figure 4: Comparison of shapes of Giza special right triangles.

The Douglas Triangle is a poor candidate for a pyramid, given its elongated shape. Nor can I find any use directly in the site plan.

If it is not used to design a pyramid, or to lay out the site, then how is it used?

One of the many puzzles at Giza is Khafre's base size of 411₢. A base of 414, or 408, with the same 3 : 4 : 5 ratios, would have looked almost identical, and avoided the awkward ½ cubit in the design (205.5 : 274 : 342.5). One explanation is that $440/411 \approx \sqrt{3}/\varphi$ (99.99% accurate). This is the cosecant in table 2 above. Another explanation is the desire to use multiples of 137 ($411 = 3 \times 137$, $274 = 2 \times 137$), which had no meaning for anyone except students of the Kabbalah until the Fine Structure Constant was discovered in 1916. A third possibility could be the desire to make the height π metres less than Khufu (6 cubits is π metres). From 274 and a 3 : 4 : 5 design, we can back-calculate the base of 411.

Or perhaps they just wanted to demonstrate that they totally understood Pythagoras, even from a weird base like 205.5Cr.

However, I stumbled across a different idea, which led to the Douglas Triangle.

I started by looking at the difference in their squares.

$440^2 - 411^2 = 24679 = 157.0955^2$.

This is basically $100\pi/2$, or 50π. Remember the builders rounded values to the nearest cubit for construction purposes.

These are then the three sides of a right-angled triangle, 157 : 411 : 440 (rounded), and suddenly the "why 411?" question was answered, and the Douglas Triangle was born.

The Douglas Triangle has the sides in ratio $k : k\varphi^2 : k\sqrt{3}\varphi$. If we set k to 50π, then

$k = 50\pi = 50 \text{ x } 3.1416 = 157.08$, rounded to 157,

$k\varphi^2 = 50\pi\varphi^2 = 50 \times 3.1416 \times 2.618 = 411.235$, rounded to 411,

$k\sqrt{3}\varphi = 50\pi\sqrt{3}\varphi = 50 \times 3.1416 \times 1.732 \times 1.618 = 440.197$, rounded to 440.

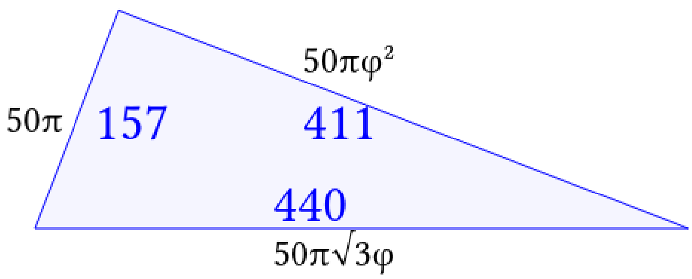


Figure 5: How the Douglas Triangle links Khufu and Khafre base sizes.

So Khufu and Khafre are related via the Douglas Triangle, with a multiplier of 50π, and encoding both π and φ.

Here is a diagram modified from my Diskerfery paper [3] showing another set of π, φ and $\sqrt{3}$ linkages between Khufu and Khafre. The symbol ⦣ means "360 divided by."

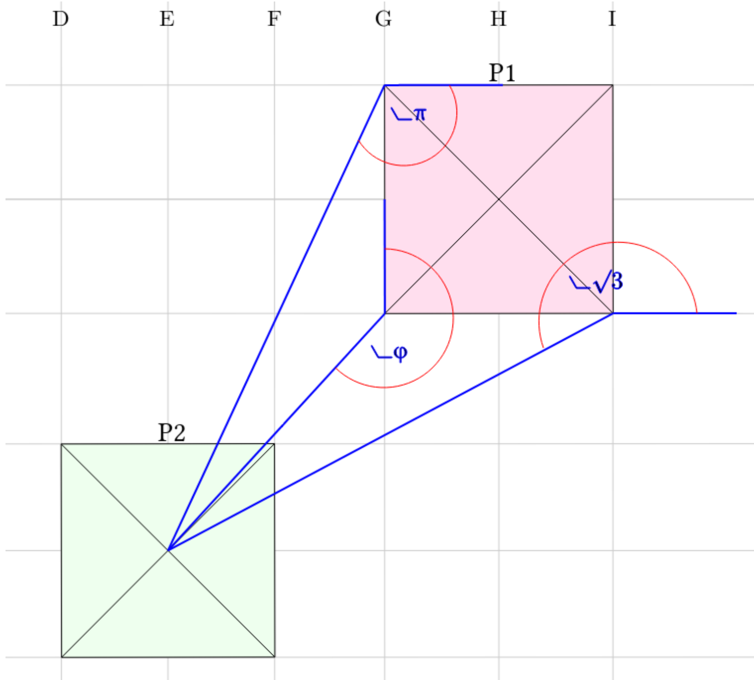


Figure 6: Another way that Khufu links to Khafre via π, φ and $\sqrt{3}$

At Giza, "Everything works together."

5. Constructing the Douglas Triangle

We can construct all three versions simultaneously.

Draw a vertical line and mark off the desired unit length of the short side.

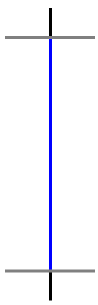


Figure 7: Constructing the Douglas Triangle, step 1

Extend the top and bottom markers horizontally as working lines.

Figure 8: Constructing the Douglas Triangle, step 2

We now need to mark off a length of φ^2 times the short side, on the top line. To construct this, we go via $\sqrt{5}$, as follows:

1. $\varphi^2 = \varphi + 1$
2. $\varphi = 1 + \varphi^{-1}$
3. $\sqrt{5} = 1 + \varphi^{-1} + \varphi^{-1}$

We already have the unit length 1. We will use this to get $\sqrt{5}$.
Then subtracting 1 will give us $2 \times \varphi^{-1}$, which we can halve to get φ^{-1}. Then $1 + 1 + \varphi^{-1} = \varphi^2$.

To get $\sqrt{5}$, we construct a double unit square as follows. Draw the vesica pisces.

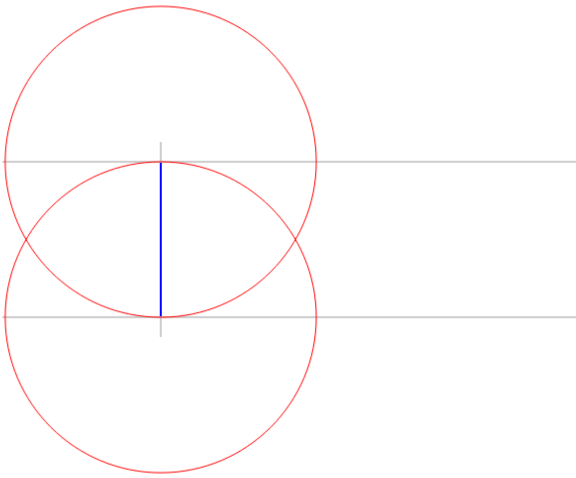


Figure 9: Constructing the Douglas Triangle, step 3

Complete the squares and draw the diagonal.

By Pythagoras, the diagonal is $\sqrt{(2^2+1^2)} = \sqrt{5}$.

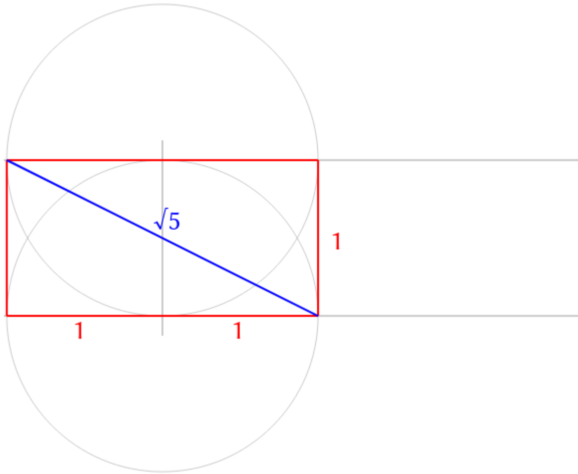


Figure 10: Constructing the Douglas Triangle, step 4

Transfer the $\sqrt{5}$ length to the top line.

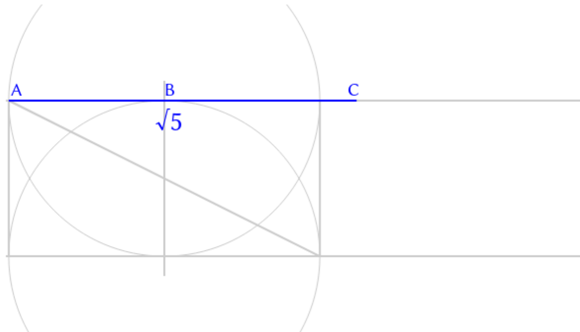


Figure 11: Constructing the Douglas Triangle, step 5

$\sqrt{5} = 1 + \varphi^{-1} + \varphi^{-1}$, and line AB is 1. So line BC is $\varphi^{-1} + \varphi^{-1}$. Bisecting that will give us φ^{-1}. So bisect BC at D.

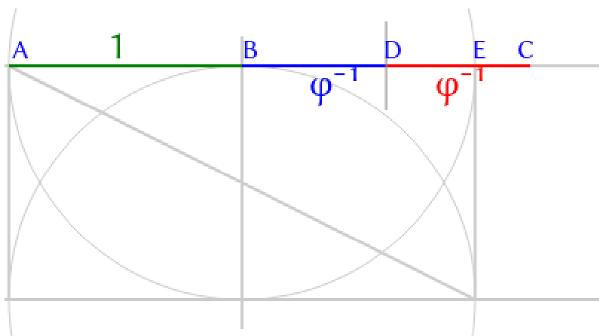


Figure 12: Constructing the Douglas Triangle, step 6

Now extend line AE, which is 2, by DC, which is φ^{-1}. That gives us a line φ^2 long.

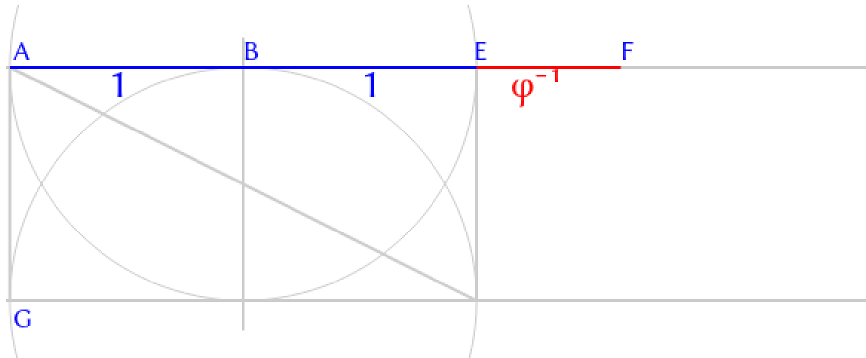


Figure 13: Constructing the Douglas Triangle, step 7

Now complete triangle AFG to get all three versions.

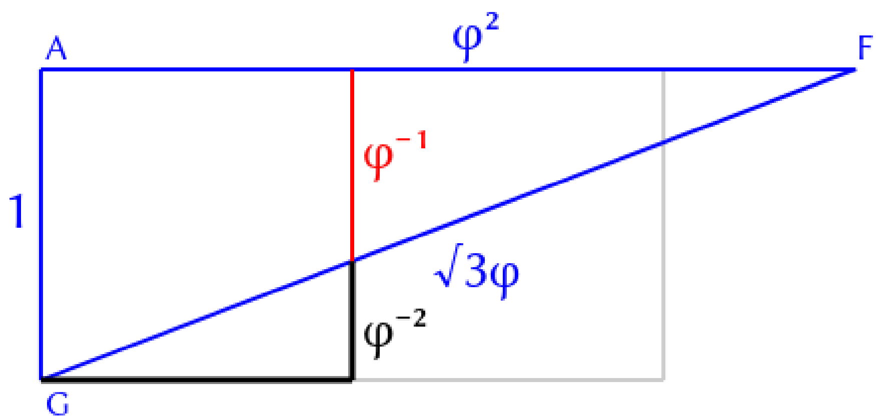


Figure 14: Constructing the Douglas Triangle, step 8

6. Interesting Features

Some ratios for the first two versions of the triangle:

Item	$1 : \varphi^2 : \sqrt{3\varphi}$	$\varphi^{-1} : \varphi : \sqrt{3}$	*Proof #*
Triangle Area	$\varphi^2/2$	½	2, 3
Square on the hypotenuse	$3\varphi^2$	3	
Ratio of Square on hypotenuse to Triangle Area	6	6	4, 5
Square on long side	φ^4	φ^2	
Ratio of Square on long side to Triangle Area	5.236...	5.236...	6, 7

Table 3: Area ratios

5.236 is 10 times the ꞓ:m ratio.

The Douglas Triangle leads to the usual favourite irrationals, using precise values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$, φ and φ^2, good approximations for $\sqrt{5}$, π and ₲, and approximations for $\sqrt{\varphi}$, e and ρ. If we were in "Engineering mode" rather than "Mathematics mode," and treated the values as lengths, then they would be "close enough", except for $\sqrt{\varphi}$, e and ρ which are only correct to two decimals. I have seen other places where they appear to be happy with e at 2.72 rather than 2.7183.

In the triangle, we have 1, φ, φ^2 and $\sqrt{3}$. Remember also that $\varphi^2 - 1 = \varphi$.

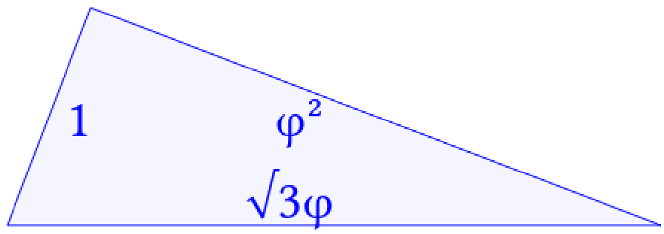


Figure 15: The Douglas Triangle

Adding the squares on each side gives us $\sqrt{2}$, $\sqrt{6}$, and approximations for π and ₲, with ₲ being used as a value rather than a unit of length. See proof 14 for the diagonal of the square on the long side.

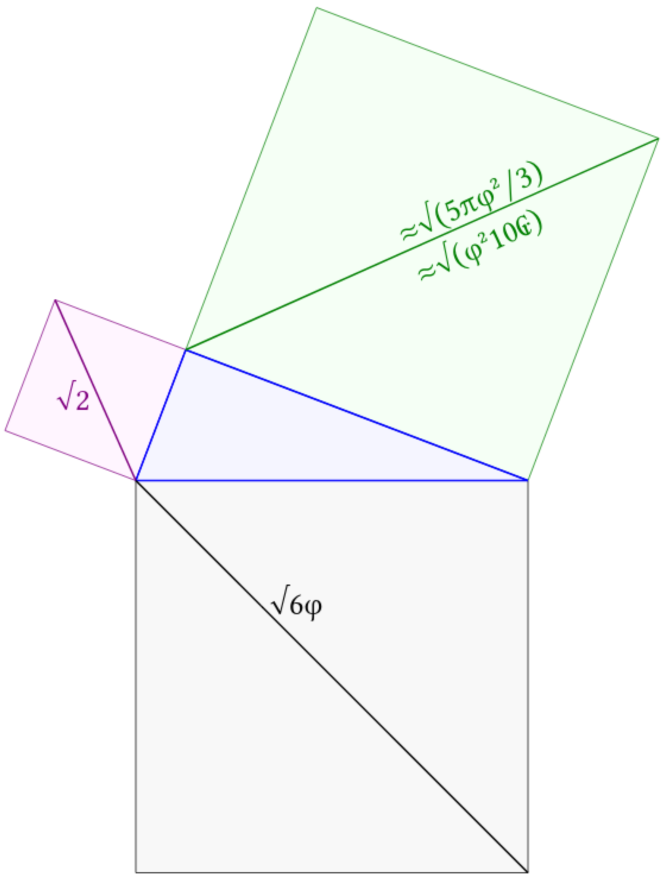


Figure 16: Diagonals of the squares.

Back-calculating the value for π above gives a value of 3.141640786.

Similarly, the ₢ comes out as 0.523606797. These calculations are done using mathematical rather than engineering values for $\sqrt{2}$ and φ.

Summing the squares gives $1 + (\varphi^2)^2 + (\sqrt{3}\varphi)^2 = 15.70820393$ which is $\approx 5\pi$. 5π is 15.70796327 (mathematical) or 15.708 (engineering).

Thus, a circle with the same area would have a radius 2.236085107 (mathematical), or 2.236 (engineering), which $\approx \sqrt{5}$. $\sqrt{5}$ is 2.236067977.

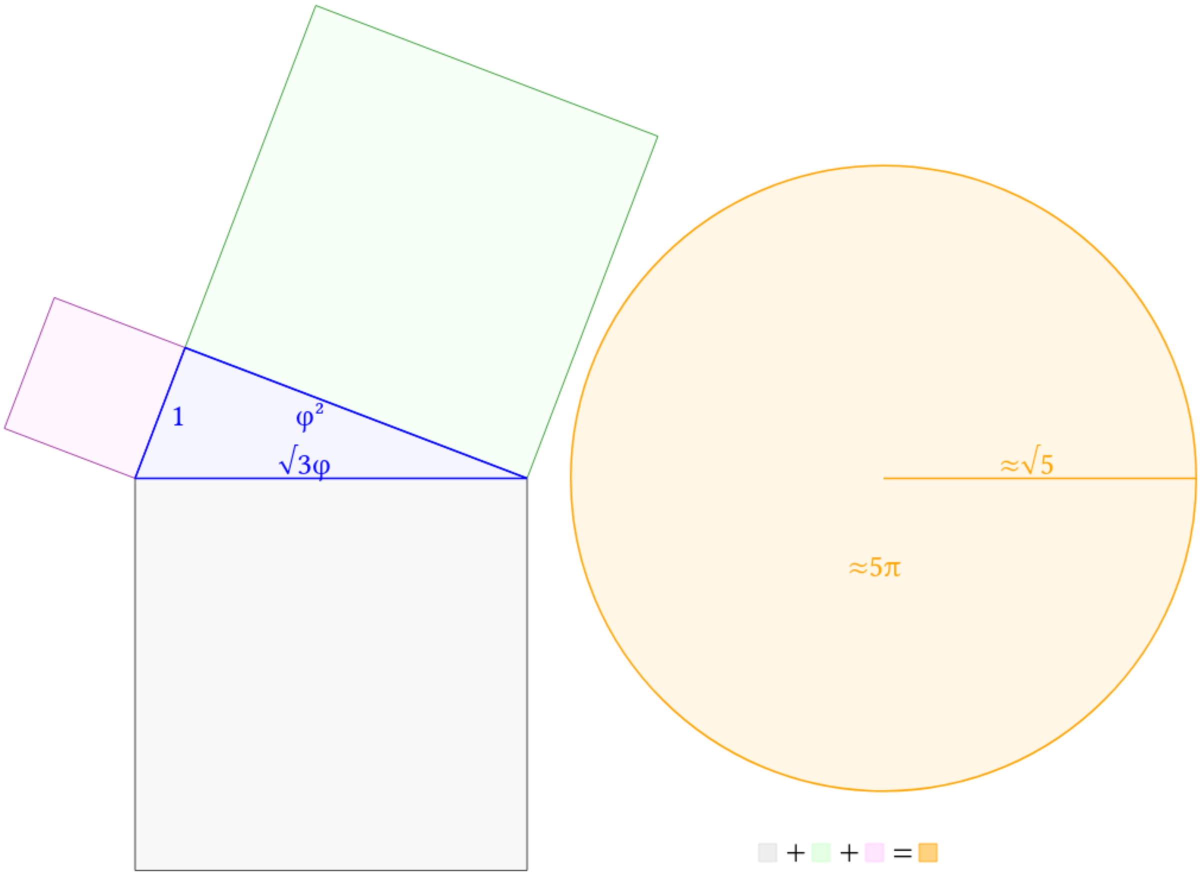


Figure 17: Sum of the squares is equal to the circle.

I showed earlier (Table 3) that the square on the hypotenuse is 6 times the area of the triangle. Since this is equal to the other two squares, the sum of all the squares, and thus also the circle, is equal to 12 times the area of the triangle.

Finally, the circumcircle around the triangle has a circumference of $\sqrt{3}\varphi\pi$ or 8.80436706, which is $\approx 2\varphi e$. $2\varphi e$ is 8.796544779, or back-calculating e from 8.80436706 gives 2.720699046.

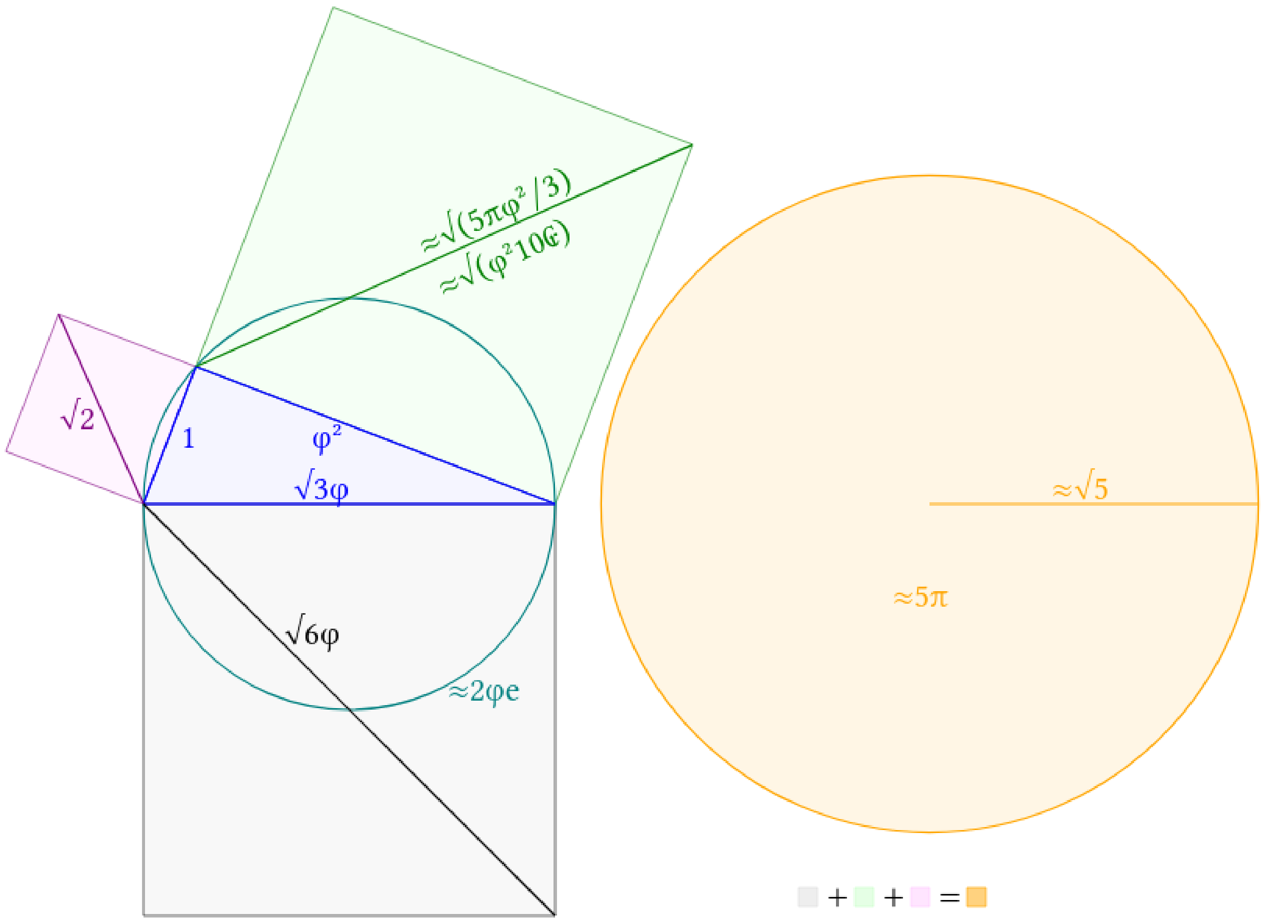


Figure 18: The circumcircle versus the sum of the squares.

The ratio of the area of the large circle (or sum of squares) to the circumcircle is $\approx 2\sqrt{\varphi}$ (proof 8).

The ratio of the area of the large circle (or sum of squares) to the square on the long side is $\approx \sqrt{3}\rho$ (proof 9).

If we draw a circle on the short side, then the ratio of area of the square on the hypotenuse to the area of the circle is 10 (proof 10).

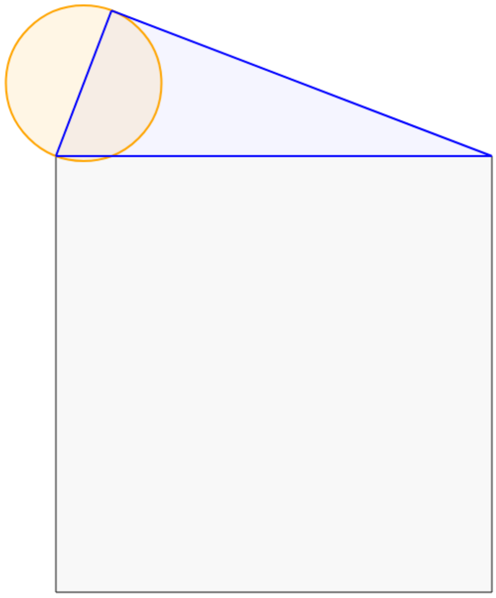


Figure 19: Circle on short side versus square on hypotenuse.

Similarly, the ratio of the area of the square on the long side to the area of the circle on the short side is $3\frac{1}{3}\varphi^2$ (proof 11).

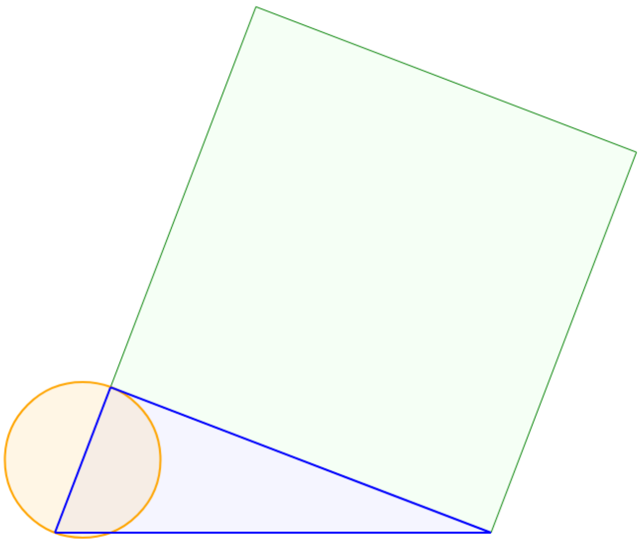


Figure 20: Circle on short side versus square on long side.

If we use the other sizes of the Douglas triangle as well, then we get this:

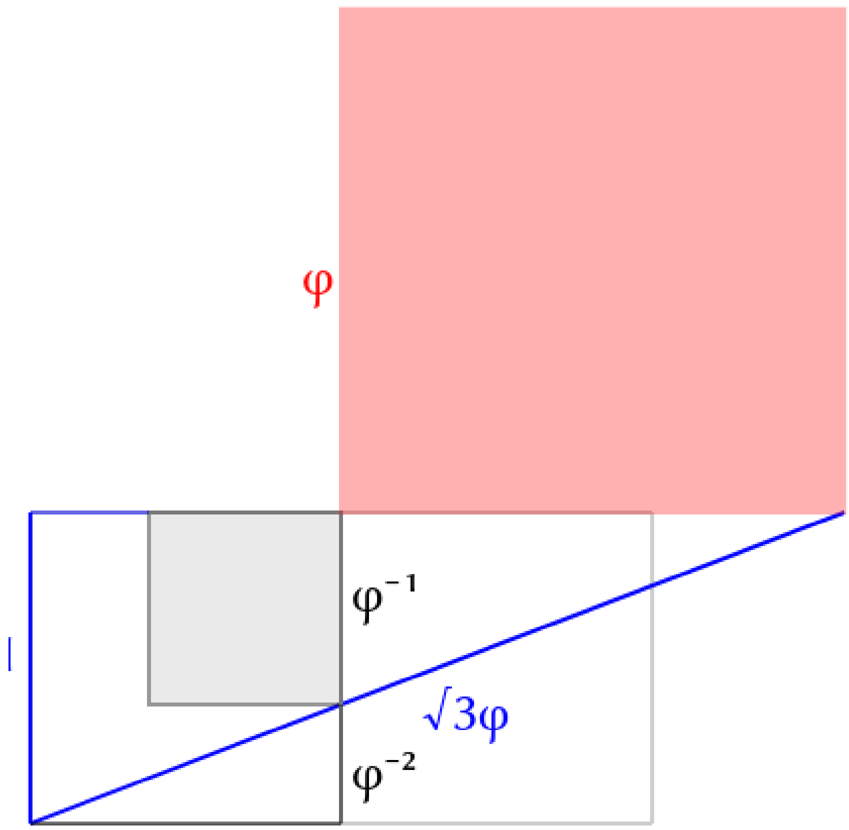


Figure 21: Squares on the smaller triangle.

Here,
1. The lengths $\varphi + \varphi^{-1} = \sqrt{5}$
2. Red square + grey square = $\varphi^2 + (\varphi^{-1})^2 = \varphi^2 + \varphi^{-2} = 3$
3. Red square - grey square = $\varphi^2 - (\varphi^{-1})^2 = \varphi^2 - \varphi^{-2} = \sqrt{5}$
4, The height $\varphi + \varphi^{-1} + \varphi^{-2} = \varphi^2$ = the width. We could draw a bounding square around this, and have multiple internal φ divisions (diagram below).

If we add the diagonal in the red square, we can construct an isosceles triangle, the base will be $\sqrt{2}\varphi$ and the height, $\sqrt{(2.5)}\varphi$. (Proof 12)

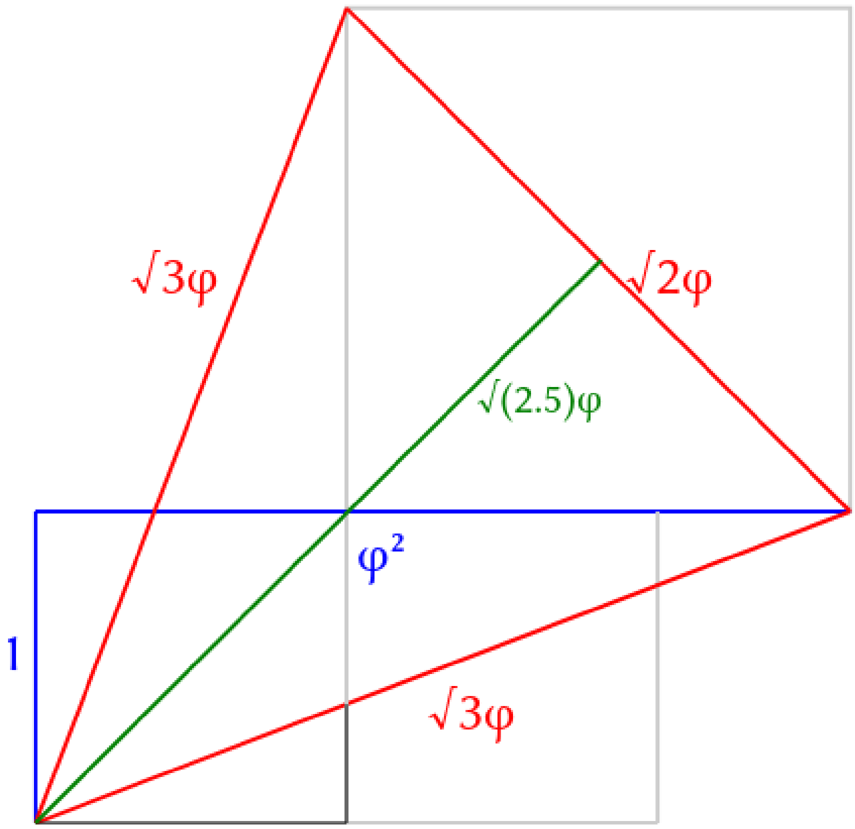


Figure 22: The isosceles triangle

Then the area of the red triangle, divided by the area of the Douglas triangle, is $\sqrt{5}$. (Proof 13)

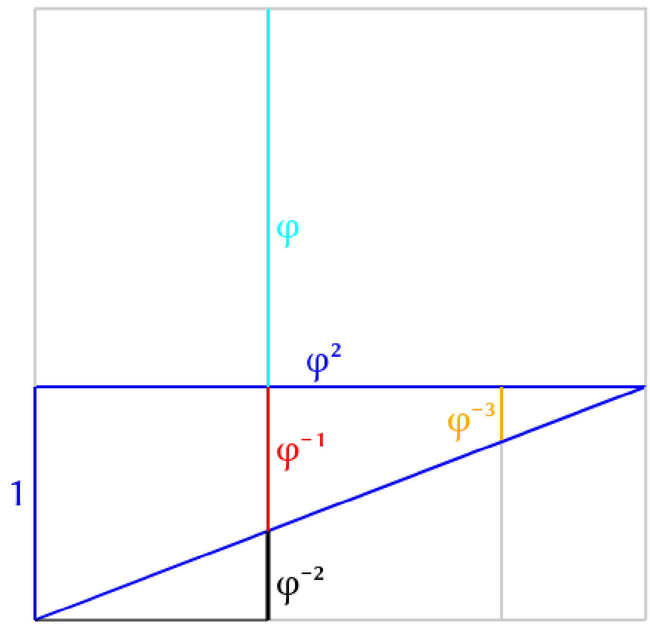


Figure 23: Phi² square showing some of the internal φ divisions.

Figure 23 above shows some internal φ divisions. There are others, for example the various rectangles divided in the golden ratio.

7. Applications

The Douglas Triangle can be used in the same way as the familiar 30 : 60 : 90 triangle, to do orthographic perspective drawings. While beauty is in the eye of the beholder, it could be argued that the Douglas Perspective with foreshortening produces a more natural view of 3D objects.

Figure 24 shows a cube drawn in conventional 30° perspective, then the analogous view using the Douglas Perspective, and finally the Douglas Perspective with foreshortening, to be explained shortly.

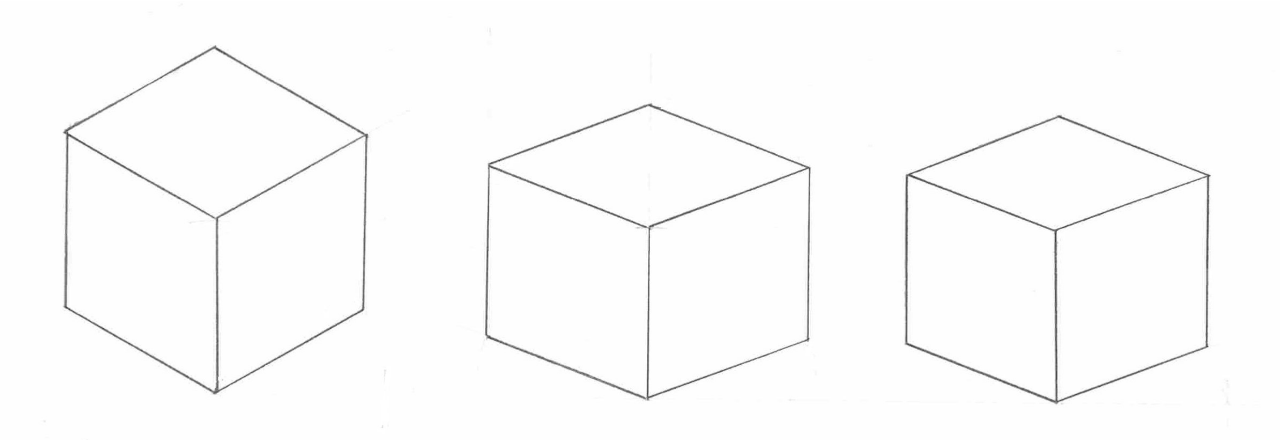


Figure 24: A cube drawn in conventional, Douglas, and foreshortened Douglas projection

All three have the same height, and the first two have the left and right sides the same length as the height. The conventional view presents a top that is unnaturally angled and visually too large. The standard Douglas perspective looks more rectangular than cubic, and somewhat too short for its widths. The third version, with foreshortening, looks like a cube, even though the sides are shorter than the height. It is arguably more natural looking than the conventional 30° view.

Figure 25 shows Khufu, Khafre and Menkaure, in conventional (top row) and foreshortened Douglas (bottom row) projections, hand-drawn at scale of 1cm:100₢. The height was measured from the intersection of the diagonals of the hidden base.

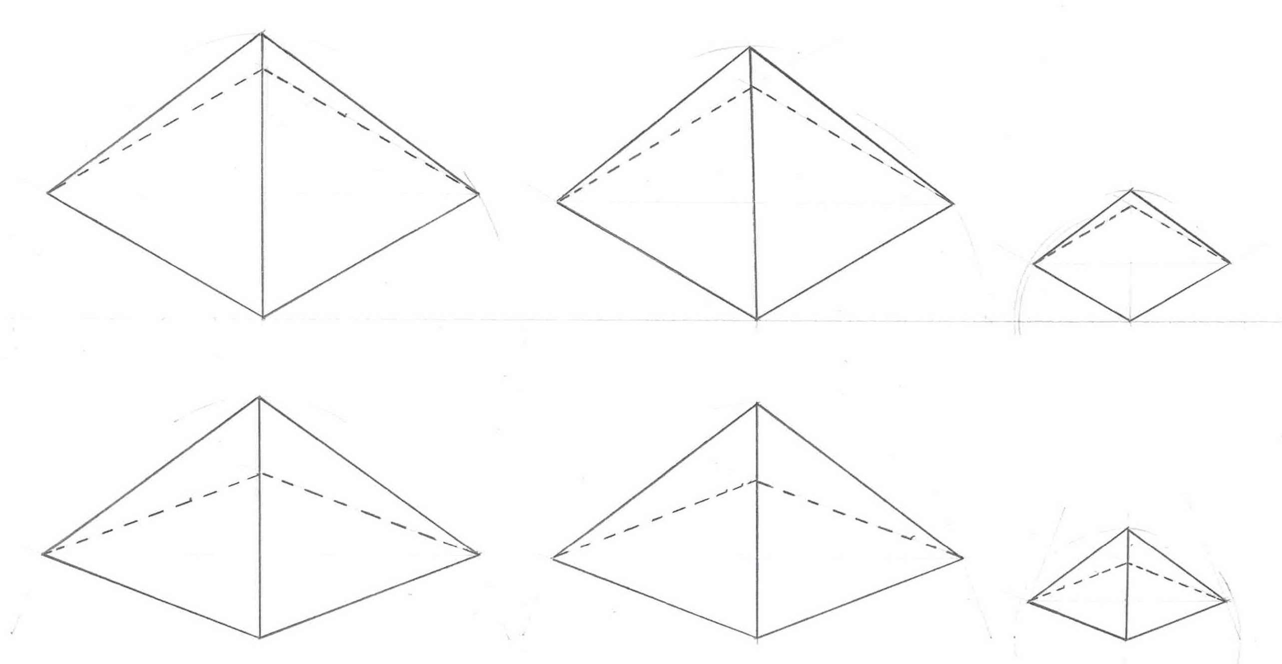


Figure 25: Khufu, Khafre and Menkaure drawn in conventional and foreshortened Douglas projection

To do the foreshortening, begin as usual by drawing an arc equal to the height, cutting the X axis to the left and right. From the X axis intersections, draw lines at angle θ back towards the origin. Draw in the usual θ' angles, and erect the verticals where the θ and θ' lines cross, rather than where the θ' line and arc cross. (That would give the regular Douglas projection.)

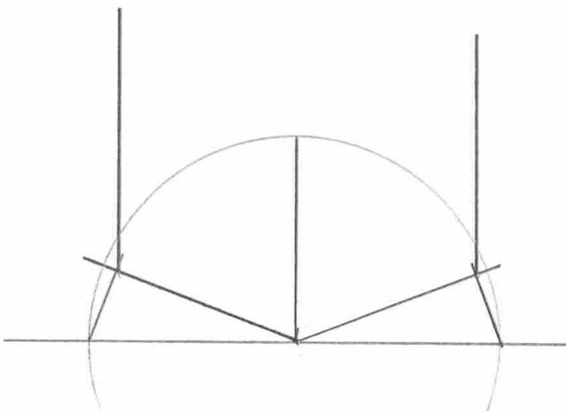


Figure 26: Construction showing foreshortening in Douglas projection

The angle between the bases in the conventional view is (180 - 30 - 30) = 120°.
In the Douglas views, it is (180 - 20.91 - 20.91) = 138.18°. This is 0.68° different to the angle created by dividing a circle into φ (or φ², which produces the same diagram), or 137.5°. This "golden division" may play a part in the aesthetic appeal of the Douglas projection.

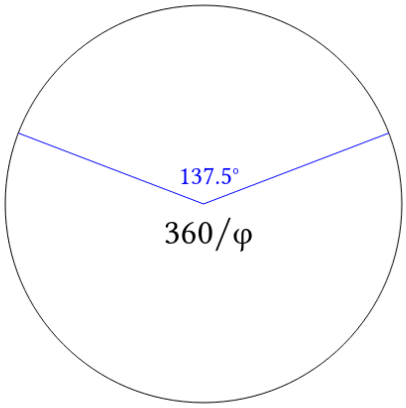


Figure 27: Dividing a circle in the golden ratio

8. Curiosities

1. The Canaanite Goddess, Tanit, [4] wife/consort of Baal, was represented by a simple diagram of a circle, bar, and triangle. The actual representation was open to artistic interpretation, but we can make one version by placing two Douglas Triangles back to back. The circle diameter is half the base, or the short side of one triangle, and the arms are equal to the base.

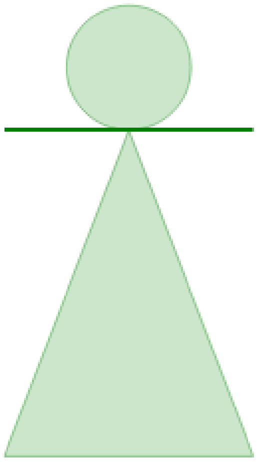


Figure 28: Symbol for Tanit constructed with Douglas triangles

2. Bernoulli's Lemniscate [5] has the general formula

$$(x^2+y^2) = a^2(x^2-y^2)$$

and produces a curve that looks like this:

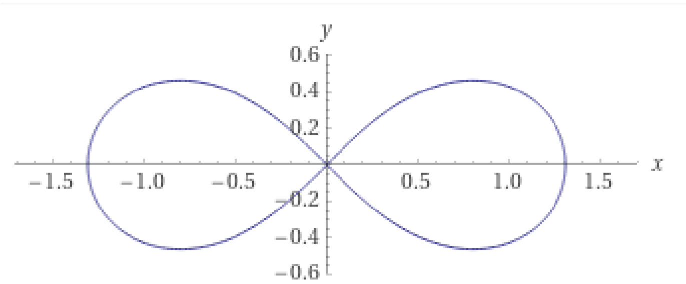


Figure 29: Typical graph of Bernoulli's Lemniscate

It should be reasonably obvious why an equation based around squares might appeal to the architects of Giza, especially as it produces such an elegant curve.

We can replace the a^2 with $(\varphi^2/2)^2$, as this gives a maximum x value of $\varphi^2/2$, or half the long side of the Douglas triangle.

$$(x^2+y^2)=\left(\frac{\varphi^2}{2}\right)^2(x^2-y^2)$$

Using just the positive values for x, we can plot this curve over the double Douglas Triangle, after arranging it so that the triangle centre is at the (0;0) origin.

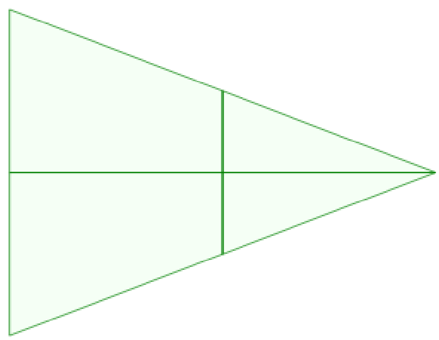


Figure 30: Double Douglas triangles, length divided in half.

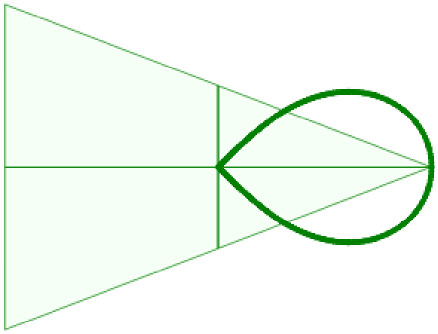


Figure 31: Double Douglas triangles with Bernoulli's Lemniscate

Then we thicken some lines and drop others.

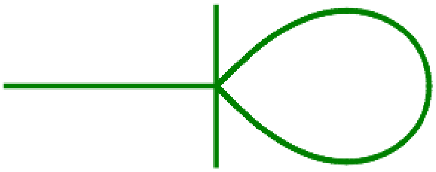


Figure 32: Ankh

9. Conclusion

The Douglas triangle, and the way it leads to the collection of favourite irrationals that are used repeatedly at Giza, appears to have appealed to the designers so much that they used it for the base sizes of the two largest pyramids. This elegantly links 440 and 411 via π and φ, arguably their two favourite numbers. Also, $100\sqrt{3}\varphi$ ($100 \times$ the hypotenuse) rounds to 280, Khufu's height.

10. Acknowledgements

Thanks as always to my "guides," whoever or whatever they are, for their constant prompting and ideas.

Thanks to the team behind the Libertinus fonts [2].

11. Proofs

Proof 1

Before we calculate the hypotenuse, we need a little helper result. Remember $1 + \varphi = \varphi^2$.

$$\begin{aligned}\varphi^4 = (\varphi^2)^2 = (1+\varphi)^2 = (1+\varphi)(1+\varphi) &= 1+2\varphi+\varphi^2 \\ &= 1+2\varphi+\varphi+1 \\ &= 3\varphi+2\end{aligned}$$

Now applying Pythagoras to find the hypotenuse:

$$\begin{aligned}\text{hypotenuse} &= \sqrt{1^2+(\varphi^2)^2} = \sqrt{1+\varphi^4} \\ &= \sqrt{1+3\varphi+2} = \sqrt{3\varphi+3} \\ &= \sqrt{3(\varphi+1)} = \sqrt{3\varphi^2} = \sqrt{3}\,\varphi\end{aligned}$$

Proof 2

$$\begin{aligned}\text{Area} &= \frac{1}{2}\,base\times height \\ &= \frac{1}{2}\times\varphi^2 \\ &= \frac{\varphi^2}{2}\end{aligned}$$

Proof 3

$$\begin{aligned}\text{Area} &= \frac{1}{2}\,base\times height \\ &= \frac{1}{2\varphi}\times\varphi \\ &= \frac{1}{2}\end{aligned}$$

Proof 4

$$\begin{aligned}\text{ratio} &= \frac{3\varphi^2}{\frac{\varphi^2}{2}} = \frac{2\times3\varphi^2}{\varphi^2} \\ &= 6\end{aligned}$$

Proof 5

$$\begin{aligned}\text{ratio} &= \frac{3}{\frac{1}{2}} = \frac{2\times3}{1} \\ &= 6\end{aligned}$$

Proof 6

$$\text{ratio} = \frac{\varphi^4}{\frac{\varphi^2}{2}} = \frac{2\,\varphi^4}{\varphi^2}$$
$$= 2\,\varphi^2$$
$$= 5.236\ldots$$

Proof 7

$$\text{ratio} = \frac{\varphi^2}{\frac{1}{2}}$$
$$= 2\,\varphi^2$$
$$= 5.236\ldots$$

Proof 8

$$\frac{\text{Large area}}{\text{small area}} = \frac{\left(\sqrt{5}\right)^2 \pi}{\left(\frac{\sqrt{3}\,\varphi}{2}\right)^2 \pi}$$
$$= \frac{5}{\left(\frac{3\,\varphi^2}{4}\right)}$$
$$= \frac{20}{3\,\varphi^2}$$
$$= 2.546440075$$
$$\approx 2\sqrt{\varphi}$$

$2\sqrt{\varphi}$ is 2.544039299.

Proof 9

$$\frac{\text{Large circle area}}{\text{middle square area}} = \frac{\left(\sqrt{5}\right)^2 \pi}{\varphi^4}$$
$$= \frac{5\,\pi}{\varphi^4}$$
$$= 2.291760955$$
$$\approx \sqrt{3}\,\rho$$

$\sqrt{3}\rho$ is 2.294478808, so correct to 2 decimals.

Proof 10

$$\frac{\text{Square on hypotenuse}}{\text{Circle on short side}} = \frac{\left(\sqrt{3}\,\varphi\right)^2}{0.5^2\,\pi}$$
$$= \frac{3\,\varphi^2}{0.25\,\pi}$$
$$= \frac{3 \times 2.618}{0.25 \times 3.1416}$$
$$= 10$$

Proof 11

$$\frac{\text{Square on long side}}{\text{Circle on short side}} = \frac{\left(\varphi^2\right)^2}{0.5^2\,\pi}$$
$$= \frac{\left(\varphi^2\right)^2}{0.25\,\pi}$$
$$= \frac{2.618^2}{0.25 \times 3.1416}$$
$$= 3\frac{1}{3}\,\varphi^2$$

Proof 12

The square has side φ, so the diagonal is $\sqrt{2}\,\varphi$
The height will create a right-angled triangle bisecting the base, so the short side is $\frac{\sqrt{2}\,\varphi}{2}$

Using Pythagoras, $height = \sqrt{\left(\sqrt{3}\,\varphi\right)^2 - \left(\frac{\sqrt{2}\,\varphi}{2}\right)^2}$

$$= \sqrt{3\,\varphi^2 - \frac{2\,\varphi^2}{4}}$$
$$= \sqrt{\frac{12\,\varphi^2 - 2\,\varphi^2}{4}}$$
$$= \sqrt{\frac{10\,\varphi^2}{4}}$$
$$= \sqrt{2.5}\,\varphi$$

Proof 13

$$\frac{\text{Area of red triangle}}{\text{Area of Douglas triangle}} = \frac{\frac{\left(\sqrt{2}\,\varphi\right)}{2}\times\sqrt{2.5}\,\varphi}{\frac{\varphi^2}{2}}$$

$$= \frac{\frac{\left(\sqrt{5}\,\varphi^2\right)}{2}}{\frac{\varphi^2}{2}}$$

$$= \sqrt{5}$$

Proof 14

$$\text{The diagonal} = \sqrt{2}\,\varphi^2 = \sqrt{2\,\varphi^4} = \sqrt{2\,\varphi^2\,\varphi^2}$$

$$= \sqrt{\frac{5\times2\,\varphi^2\,\varphi^2}{5}}$$

$$\text{Now } \frac{\varphi^2}{5} \approx \frac{\pi}{6} \approx \mathcal{G}, \text{so}$$

$$\text{The diagonal} \approx \sqrt{\frac{5\times2\,\varphi^2\,\pi}{6}} \text{ or } \sqrt{5\times2\,\varphi^2\,\mathcal{G}}$$

$$\approx \sqrt{\frac{5\,\varphi^2\,\pi}{3}} \text{ or } \sqrt{10\,\varphi^2\,\mathcal{G}}$$

126. Bibliography

[1] I. Douglas, ‘Zep Tepi Mathematics 101’, Zenodo, Jan. 2021. doi: 10.5281/zenodo.4445312.

[2] Douglas, Ian, ‘The Beautiful Cubit System’. Jun. 30, 2019. [Online]. Available: https://doi.org/10.5281/zenodo.3263864

[3] Douglas, Ian, ‘Diskerfery and the Alignment of the Four Main Giza Pyramids’. [Online]. Available: https://doi.org/10.5281/zenodo.3263926

[4] ‘tanit - Google Search’. https://www.google.com/search?q=tanit&tbm=isch&hl=en-GB&chips=q:tanit,online_chips:symbol:McH6rOBtFB4%3D,online_chips:carthaginian:gc8iBPzaGfc%3D (accessed Dec. 28, 2021).

[5] ‘Lemniscate of Bernoulli’, *Wikipedia*. Oct. 05, 2021. Accessed: Dec. 19, 2021. [Online]. Available: https://en.wikipedia.org/w/index.php?title=Lemniscate_of_Bernoulli&oldid=1048378469

[6] C. Maclennan, *LIbertinus font*. 2020. Accessed: Jan. 01, 2021. [Online]. Available: https://github.com/alerque/libertinus