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An existence and uniqueness of solution for p–Laplacian Kirchhoff type equation with singular term

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Abstract: This work is devoted to study the existence of positive solution for a class of p–Laplacian Kirchhoff type equation with singular nonlinearity:

$$\begin{cases} L_p(u) = f(x)|u|^{-\gamma} - \lambda|u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{P}_1)$$

where Ω is a smooth bounded domain in $\mathbb{R}^n(n \geq 3), \lambda > 0$ is a real parameter. Here $\gamma \in (0,1)$ is a constant, $a, b \geq 0$ such that $a+b > 0$ are parameters, the weight function $f : \Omega \to \mathbb{R}$ is positive and belonging to the Lebesgue space $L^\alpha(\Omega)$ with $\alpha := \frac{p^*}{p^*+\gamma-1}$, and $p^* := \frac{np}{n-p}$ is the Sobolev critical exponent in the Euclidian embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$. The operator is defined as

$$L_p(u) := -\left(a\int_\Omega |\nabla u|^p dx + b\right)^{p-1} \Delta_p u + \ell(x)|u|^{p-2}u,$$

and the operator Δ_p is the p–Laplacian for $1 < p < n$. Our approach relies on the variational methods and some analysis' techniques.

Key words: Kirchhoff type equation, p–Laplacian Kirchhoff type equation, Critical exponent of Sobolev

1. Introduction and Motivation

In this paper, we study the existence and uniqueness of positive solution to the following nonlinear elliptic p–Laplacian Kirchhoff equation:

$$\begin{cases} L_p(u) = f(x)|u|^{-\gamma} - \lambda|u|^{p^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{P}_1)$$

where, throughout this work, $\Omega \subset \mathbb{R}^n(n \geq 3)$ is a smooth bounded domain, $\lambda > 0$ is a real parameter. Here $\gamma \in (0,1)$ is a constant, $a, b \geq 0$ such that $a+b > 0$ are parameters, the weight function $f : \Omega \to \mathbb{R}$ is positive and belonging to the Lebesgue space $L^\alpha(\Omega)$ with $\alpha := \frac{p^*}{p^*+\gamma-1}$, and $p^* := \frac{np}{n-p}$ is the Sobolev critical exponent

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in the Euclidian embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$. The operator is defined as

$$L_p(u) := -\left(a\int_\Omega |\nabla u|^p dx + b\right)^{p-1} \Delta_p u + \ell(x)|u|^{p-2}u.$$

Such that $\ell \in L^{\frac{n}{p}}(\Omega)\cap L^\infty(\Omega)$ and the operator Δ_p is the $p-$Laplacian which be defined as for all $u \in W^{1,p}(\Omega)$:

$$\Delta_p u := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \sum_{1\le i\le n} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right).$$

The equation ($\mathcal{P}_1$) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a\int_\Omega |\nabla u|^p dx + b\right)\Delta_p u = g(x,u) \quad \text{in } \Omega, \tag{$\mathcal{P}_2$}$$

with $\Omega \subset \mathbb{R}^n (n \ge 3)$ is a smooth bounded domain, which was proposed by Kirchhoff in 1883 in [11] as an extension of the classical D'Alembert's wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = g(x,u), \tag{$\mathcal{P}_3$}$$

for free vibrations of elastic strings. The parameters in above equation have physical significant meanings as follows: L is the length of the string, E is the area of the cross section, ρ is the Young modulus of the material, ρ_0 is the mass density and is the initial tension.

In 2006, the authors in [8] considered two problems of the Kirchhoff type:

$$\begin{cases} -\left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1}\Delta_p u = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_4$}$$

and

$$\begin{cases} -\left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1}\Delta_p u = f(x,u) + \lambda|u|^{s-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{P}_5$}$$

Where $\Omega \subset \mathbb{R}^n (n \ge 3)$ is a smooth bounded domain for $1 < p < n, s \ge p^* := \frac{np}{n-p}$ and M and f are two continuous functions. Using Pass-Montain Theorem, they obtained the existence of positive solutions of problems ($\mathcal{P}_4$) and ($\mathcal{P}_5$).

In 2009, the authors in [7] have considered the existence and multiplicity of solutions to a class of p-Kirchhoff type equation:

$$\begin{cases} -\left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1}\Delta_p u = f(x,u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{P}_6$}$$

They obtained the existence and multiplicity result of non trivial solution of the problem ($\mathcal{P}_6$).

Recently, the first author, S. Benmansour and Kh. Tahri in [21] showed the existence, nonexistence and multiplicity results for the following $p-$Laplacian Kirchhoff equation:

$$\begin{cases} -\left(a\int_\Omega |\nabla u|^p dx + b\right)\Delta_p u = \mu|u|^{p^*-2}u + \lambda|u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_7$}$$

where Ω is a bounded domain in $\mathbb{R}^n(n>3), \lambda, \mu>0$ are real parameters and $a, b \geq 0 : a+b>0$ are positive constants, $\Delta_p u$ is the $p-$Laplacian operator for $1<p<n$.
They established the following results:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n(n>3)$, assume $a, b \geq 0 : a+b>0$ and $p^*=4$ then the following assertions are true:*
(i). Assume that $a>0, b>0, 0<\mu<aK(n,p)^2$ and $0<\lambda<b\lambda_1$, then the equation ($\mathcal{P}_7$) has no positive nontrivial solution.
(ii). Assume that $a \geq 0, b>0, 0<\mu<aK(n,p)^2$ and $\lambda>b\lambda_1$, then the equation ($\mathcal{P}_7$) has a positive nontrivial solution.
(iii). Assume that $a \geq 0, b>0, 0<\mu<aK(n,p)^2$, then for any $k \in \mathbb{Z}^+$, there exists $\Lambda_k>0$ such that the equation ($\mathcal{P}_7$) has at least k pairs of nontrivial solutions for $\lambda>\Lambda_k>0$.

Some intersting studies for Kirchhoff type problems in a bounded domain of $\mathbb{R}^n(n \geq 3)$ by critical points theory and variational methods can be found in [1], [2], [3], [4], [5], [6], [10], [12], [13], [14], [15], [16], [17], [18], [20], [22], [23].
The following theorem is our main result.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n(n \geq 3)$ be a smooth bounded domain and assuming that $(H_i)_{1 \leq i \leq 2}$ are hold. The problem ($\mathcal{P}_1$) possesses a positive solution. Moreover, this solution is a global minimizer solution.*

This paper proceeds as follows. In the next section, we prove the energy functional J_λ satifies some geometric conditions. In section 3, by using critical point theory, we get the main result of this paper.

2. Variational Setting

Let $X=W_0^{1,p}(\Omega)$ be the usual Sobolev space, equipped with the norm

$$\|u\|=(\int_\Omega |\nabla u|^p\, dx)^{\frac{1}{p}}$$

and $\|u\|_p=(\int_\Omega |u|^p\, dx)^{\frac{1}{p}}$ denotes the norm in $L^p(\Omega)$.
A function $u \in X$ is said to be a weak solution of problem ($\mathcal{P}_1$) if $u>0$ in Ω and there holds

$$\begin{aligned} 0 =&(a+b\int_\Omega |\nabla u|^p\, dx)^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\, dx+\int_\Omega \ell(x)|u|^{p-2}u\varphi\, dx \\ &+\lambda\int_\Omega u^q\varphi\, dx-\int_\Omega f(x)u^{-\gamma}\varphi\, dx. \end{aligned} \tag{VF}$$

for all $\varphi \in X$.
We shall look for (weak) solutions of ($\mathcal{P}_1$) by finding critical points of the energy functional $J_\lambda : X \to \mathbb{R}$ given by

$$\begin{aligned} J_\lambda(u) =&\frac{1}{bp^2}\left[\left(a+b\int_\Omega |\nabla u|^p\, dx\right)^p-a^p\right]+\frac{1}{p}\int_\Omega \ell(x)|u|^p\, dx \\ &+\frac{\lambda}{1+q}\int_\Omega |u|^{1+q}\, dx-\frac{1}{1-\gamma}\int_\Omega f(x)|u|^{1-\gamma}\, dx, \end{aligned}$$

for all $u \in X$. By analyzing the associated minimization problems for the functional J_λ, one can study weak solutions for ($\mathcal{P}_1$). As we know, the functional J_λ fails to be Fréchet differentiable because of the singular term, then we cannot apply the critical point theory to obtain the existence of solutions directly.
Consider λ_1 the first eigenvalue of the problem:

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

According to the work developed by Peral in [19] it has been shown that:

$$\lambda_1 := \inf_{u \in W_0^{1,p}(\Omega)-\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}.$$

the first eigenvalue is isolated and simple and its corresponding first eigenfunction named ϕ_1 is positive.
Let ν_1 be the first eigenvalue of the following eigenvalue problem:

$$\begin{cases} L_p(u) = \nu |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Where

$$L_p(u) := -\left(a \int_\Omega |\nabla u|^p dx + b\right)^{p-1} \Delta_p u + \ell(x)|u|^{p-2}u.$$

According to the work developed by the authors of [9] it has been shown that:

$$\nu_1 := \inf_{u \in W_0^{1,p}(\Omega)-\{0\}} \frac{\int_\Omega u L_p(u) dx}{\int_\Omega |u|^p dx}.$$

The first eigenvalue is simple and strictly positive and its corresponding first eigenfunction named Ψ_1 is simple and strictly positive also, and the operator L_p possesses unbounded eigenvalues sequence:

$$\nu_1 < \nu_2 \leq \nu_3 \leq \ldots \leq \nu_n \to +\infty \text{ as } n \to +\infty.$$

Then we have the following proposition as a caracterisation of the values of the sequence $(\nu_k)_{k\geq 1}$.

Proposition 2.1. *If ν is an eigenvalue of the operator L_p, then there exist ν_k and u_k such that*

$$\nu := \nu_k \left(a \int_\Omega |\nabla u_k|^p \, dx + b\right)^{p-1}.$$

Throughout this paper, we make the following assumptions:

(H_1) $0 < \gamma < 1, 0 < q \leq p^* - 1$.

(H_2) $f \in L^{\frac{p^*}{p^*+\gamma-1}}(\Omega)$ avec $f(x) \geq 0$ pour tout $x \in \Omega$.

3. Some useful Lemmas

In this section, we will recall and prove some lemmas which are crucial in the proof ofthe main theorem.

Lemma 3.1. *The energy functional* J_λ *has a minimum* α *in* X *with* $\alpha < 0$.

Proof. Since $0 < \gamma < 1$, $\lambda \geq 0$, by the Hölder inequality, we have

$$\int_\Omega f(x)|u|^{1-\gamma}\,dx \leq (\int_\Omega |f(x)|^{\frac{p^*}{p^*-1+\gamma}}\,dx)^{\frac{p^*-1+\gamma}{p^*}}(\int_\Omega |u|^{(1-\gamma)(\frac{p^*}{1-\gamma})}\,dx)^{\frac{1-\gamma}{p^*}}.$$
$$\leq \|f\|_{\frac{p^*}{p^*-1+\gamma}}\|u\|^{1-\gamma}_{\frac{p^*}{1-\gamma}}.$$

Furthermore, by the Sobolev embedding theorem, we obtain that

$$\frac{1}{1-\gamma}\|u\|^{1-\gamma}_{\frac{p^*}{1-\gamma}} \leq C\|u\|^{1-\gamma}.$$

Hense

$$J_\lambda(u) = -\frac{a^p}{bp^2} + \frac{1}{bp^2}(a+b\|u\|^p)^p + \frac{1}{p}\int_\Omega \ell(x)|u|^p\,dx$$
$$+\frac{\lambda}{1+q}\int_\Omega |u|^{1+q}\,dx - \frac{1}{1-\gamma}\int_\Omega f(x)|u|^{1-\gamma}\,dx.$$

Hense

$$J_\lambda(u) \geq \frac{1}{bp^2}(a+b\|u\|^p)^p - C\|f\|_{\frac{p^*}{p^*-1+\gamma}}\|u\|^{1-\gamma}. \tag{1}$$

where $C > 0$ is a constant.
This implies that J_λ is coercive and bounded from below on X.
Then

$$\alpha = \inf_{u\in X} J_\lambda$$

is well defined.
Moreover, since $0 < \gamma < 1$ and $f(x) > 0$ for almost every $x \in \Omega$, we have $J_\lambda(t\delta) < 0$ for all $\delta \neq 0$ and small $t > 0$.
Thus, we obtain

$$\alpha = \inf_{u\in X} J_\lambda < 0.$$

The proof is complete. □

The validity of the next lemma will be crucial in the sequel.

Lemma 3.2. *Assume that conditions (H1) and (H2) hold. Then* J_λ *attains the global minimizer in* X, *that is, there exists* $u_* \in X$ *such that*

$$J_\lambda(u_*) = \alpha < 0.$$

Proof. From Lemma 1, there exists a minimizing sequence $\{u_n\} \subset X$ such that

$$\lim_{n\to\infty} J_\lambda(u_n) = \alpha < 0.$$

Since $J_\lambda(u_n) = J_\lambda(|u_n|)$, we may assume that $u_n \geq 0$ for almost every $x \in \Omega$.
By (1), the sequence $\{u_n\}$ is bounded in X.
Since X is reflexive, we may extract a subsequence that for simplicity we call again $\{u_n\}$, there exists $u_* \geq 0$ such that

$$\begin{cases} u_n \rightharpoonup u_* & \text{weakly in } X, \\ u_n \to u_* & \text{strongly } L^s(\Omega),\ 1 \leq s \leq p^*, \\ u_n(x) \to u_*(x) & \text{p.p. in } \Omega, \end{cases} \tag{1}$$

as $n \to \infty$. As usual, letting $w_n = u_n - u_*$, we need to prove that $\|w_n\| \to 0$ as $n \to \infty$. By Vitali's theorem, we find

$$\lim_{n\to\infty} \int_\Omega f(x)|u_n|^{1-\gamma}\,dx = \int_\Omega f(x)|u_*|^{1-\gamma}\,dx \tag{2}$$

and

$$\lim_{n\to\infty} \int_\Omega \ell(x)|u_n|^p\,dx = \int_\Omega \ell(x)|u_*|^p\,dx. \tag{3}$$

Moreover, by the weak convergence of $\{u_n\}$ in X and Brézis-Lieb's Lemma, one obtains

$$\|u_n\|^p = \|w_n\|^p + \|u_*\|^p + o(1) \tag{4}$$

$$(\|u_n\|^p)^p = (\|w_n\|^p)^p + (\|u_*\|^p)^p + A\left(\|w_n\|, \|u_*\|\right) + o(1) \tag{5}$$

where $o(1)$ is an infinitesimal as $n \to \infty$.
Hence, in the case that $0 < q < p^* - 1$, from (4)-(7), we deduce that

$$\begin{aligned}
\alpha =& \lim_{n\to\infty} J_\lambda(u_n) \\
=& \lim_{n\to\infty} (-\frac{a^p}{bp^2} + \frac{1}{bp^2}(a + b\|u_n\|^p)^p + \frac{1}{p}\int_\Omega \ell(x)|u_n|^p\,dx \\
&+ \frac{\lambda}{1+q}\int_\Omega |u_n|^{1+q}\,dx - \frac{1}{1-\gamma}\int_\Omega f(x)|u_n|^{1-\gamma}\,dx) \\
=& \lim_{n\to\infty} (-\frac{a^p}{bp^2} + \frac{1}{bp^2}(a + b\left(\|w_n\|^p + \|u_*\|^p\right))^p + \frac{1}{p}\int_\Omega \ell(x)|u_n|^p\,dx \\
&+ \frac{\lambda}{1+q}\int_\Omega |u_n|^{1+q}\,dx - \frac{1}{1-\gamma}\int_\Omega f(x)|u_n|^{1-\gamma}\,dx) \\
=&(-\frac{a^p}{bp^2} + \frac{1}{bp^2}(a + b\|u_*\|^p)^p + \frac{1}{p}\int_\Omega \ell(x)|u_*|^p\,dx + \frac{\lambda}{1+q}\int_\Omega |u_*|^{1+q}\,dx \\
&- \frac{1}{1-\gamma}\int_\Omega f(x)|u_*|^{1-\gamma}\,dx) + \lim_{n\to\infty}(-\frac{a^p}{bp^2} + \frac{1}{bp^2}(a + b\|w_n\|^p)^p) \\
=&J_\lambda(u_*) + \lim_{n\to\infty}(-\frac{a^p}{bp^2} + \frac{1}{bp^2}(a + b\|w_n\|^p)^p) \\
\geq& J_\lambda(u_*) \geq \inf_{u_n \in X} J_\lambda(u_n) = \alpha
\end{aligned}$$

which implies

$$J_\lambda(u_*) = \alpha.$$

In the case that $q = p^* - 1$, it follows from (5)–(8) that

$$\begin{aligned}\alpha =& J_\lambda(u_*) + \lim_{n\to\infty}(-\frac{a^p}{bp^2} + \frac{1}{bp^2}(a + b\|w_n\|^p)^p + \frac{\lambda}{p^*}\|w_n\|_{p^*}^{p^*})\\ \geq& J_\lambda(u_*) \geq \alpha\end{aligned}$$

which yields

$$J_\lambda(u_*) = \alpha.$$

Thus

$$\inf_{u_n\in X} J_\lambda(u_n) = J_\lambda(u_*)$$

and this completes the proof of Lemma 2. The proof is complete.

□

We are now in a position to prove Theorem 2.

4. Proof of Theorem 2

We only need to prove that u_* is a weak solution of ($\mathcal{P}_1$) and $u_* > 0$ in Ω. Firstly, we show that u_* is a weak solution of ($\mathcal{P}_1$), From Lemma 1, we see that

$$\min_{u_n\in X} J_\lambda(u_n) = J_\lambda(u_*), \qquad \forall\varphi \in X$$

thus

$$J'_\lambda(u_* + t\varphi)|_{t=0} = 0.$$

This implies that

$$\begin{aligned}0 =& (a + b\int_\Omega |\nabla u|^p\, dx)^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\, dx + \int_\Omega \ell(x)|u|^{p-2}u\varphi\, dx\\ &+ \lambda\int_\Omega u^q\varphi\, dx - \int_\Omega f(x)u^{-\gamma}\varphi\, dx.\end{aligned} \tag{6}$$

for all $\varphi \in X$, Thus, u_* is a weak solution of ($\mathcal{P}_1$).
Secondly, we prove that $u_* > 0$ for almost every $x \in \Omega$.
Since $J_\lambda(u_*) = \alpha < 0$, we obtain $u_* \geq 0$ and $u_* \neq 0$.
Then, $\forall\phi \in X$ and $\phi \geq 0$ and $t > 0$, we have

$$\begin{aligned}0 \leq& \frac{J_\lambda(u_* + t\phi) - J_\lambda(u_*)}{t}\\ =& -\frac{a^p}{bp^2} + \frac{1}{bp^2}(a + b\int_\Omega \frac{|\nabla(u_* + t\phi)|^p - |\nabla u_*|}{t}\, dx)^p\\ &+ \frac{1}{p}\int_\Omega \ell(x)\frac{|u_* + t\phi|^p - |u_*|^p}{t}\, dx\\ &+ \frac{\lambda}{1+q}\int_\Omega \frac{(u_* + t\phi)^{1+q} - u_*^{1+q}}{t}\, dx\\ &- \frac{1}{1-\gamma}\int_\Omega f(x)\frac{(u_* + t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t}\, dx.\end{aligned} \tag{7}$$

Using the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{t\to 0^+}\frac{1}{p}\int_\Omega \ell(x)\frac{(u_*+t\phi)^p-u_*^p}{t}\,dx=\frac{1}{p}\int_\Omega \ell(x)u_*^{p-1}\phi\,dx. \tag{8}$$

and

$$\lim_{t\to 0^+}\frac{1}{1+q}\int_\Omega \frac{(u_*+t\phi)^{1+q}-u_*^{1+q}}{t}\,dx=\int_\Omega u_*^q\phi\,dx. \tag{9}$$

For any $x\in\Omega$, we denote

$$g(t)=f(x)\frac{[u_*(x)+t\phi(x)]^{1-\gamma}-u_*^{1-\gamma}(x)}{(1-\gamma)t}$$

Then

$$g^{'}(t)=f(x)\frac{u_*^{1-\gamma}(x)-[\gamma t\phi(x)+u_*(x)][u_*(x)+t\phi(x)]^{-\gamma}}{t^2(1-\gamma)}\leq 0$$

which implies that $g(t)$ is non increasing for $t>0$.
Moreover, we have

$$\lim_{t\to 0^+}g(t)=([u_*(x)+t\phi(x)]^{1-\gamma})^{'}|_{t=0}=f(x)u_*^{-\gamma}(x)\phi(x).$$

for every $x\in\Omega$, which may be $+\infty$ when $u_*(x)=0$ and $\phi(x)>0$.
Consequently, by the Monotone Convergence Theorem, we obtain

$$\lim_{t\to 0^+}\frac{1}{1-\gamma}\int_\Omega f(x)\frac{(u_*+t\phi)^{1-\gamma}-u_*^{1-\gamma}}{t}\,dx=\int_\Omega f(x)u_*^{-\gamma}\,dx.$$

which may equal to $+\infty$.
Combining this with (8 and 9), let $t\to 0$, it follows from (7) that

$$\begin{aligned}0\leq&(a+b\int_\Omega|\nabla u_*|^p\,dx)^{p-1}\int_\Omega|\nabla u_*|^{p-2}\nabla u_*\nabla\phi\,dx\\&+\int_\Omega \ell(x)|u_*|^{p-2}u_*\phi\,dx+\lambda\int_\Omega u_*^q\phi\,dx-\int_\Omega f(x)u_*^{-\gamma}\phi\,dx.\end{aligned} \tag{10}$$

Then, we have

$$\begin{aligned}&\int_\Omega f(x)u_*^{-\gamma}\phi\,dx\\\leq&(a+b\|u_*\|^p)^{p-1}\int_\Omega|\nabla u_*|^{p-2}\nabla u_*\nabla\phi\,dx+\frac{1}{p}\int_\Omega \ell(x)u_*^{p-1}\phi\,dx+\lambda\int_\Omega u_*^q\phi\,dx\end{aligned} \tag{11}$$

for all $\phi\in X$ with $\phi>0$.

Let $\phi_1 \in X$ be the first eigenfunction of the operator $-\Delta_p$ with $\phi_1 > 0$ and $\|\phi_1\| = 1$. Particularly, taking $\phi = \phi_1$ in (6), one gets

$$\begin{aligned}
\int_\Omega f(x)u_*^{-\gamma}\phi_1\,dx \leq & (a+b\|u_*\|^p)^{p-1}\int_\Omega |\nabla u_*|^{p-2}\nabla u_*\nabla\phi_1\,dx + \frac{1}{p}\int_\Omega \ell(x)u_*^{p-2}u_*\phi_1\,dx \\
& + \lambda\int_\Omega u_*^q\phi_1\,dx \\
\leq & (a+b\|u_*\|^p)^{p-1}\int_\Omega |\nabla u_*|^{p-1}\nabla\phi_1\,dx + \frac{1}{p}\int_\Omega \ell(x)u_*^{p-1}\phi_1\,dx \\
& + \lambda\int_\Omega u_*^q e_1\,dx \\
\leq & (a+b\|u_*\|^p)^{p-1}[(\int_\Omega |\nabla u_*|^{(p-1)(\frac{p}{p-1})}\,dx)(\int_\Omega |\nabla\phi_1|^p\,dx)^{\frac{1}{p}}] \\
& + \frac{1}{p}(\int_\Omega (\ell(x)u_*^{p-1})^{\frac{p}{p-1}}\,dx)^{\frac{p-1}{p}}(\int_\Omega |\nabla\phi_1|^p\,dx)^{\frac{1}{p}} \\
& + \lambda(\int_\Omega |u_*|^{q(\frac{p}{p-1})}\,dx)^{\frac{p-1}{p}}(\int_\Omega |\nabla\phi_1|^p\,dx)^{\frac{1}{p}} \\
\leq & (a+b\|u_*\|^p)^{p-1}[(\int_\Omega |\nabla u_*|^{(p-1)(\frac{p}{p-1})}\,dx)(\int_\Omega |\nabla\phi_1|^p\,dx)^{\frac{1}{p}}] \\
& + \frac{1}{p}(\int_\Omega \ell(x)u_*^p\,dx)^{\frac{p-1}{p}}(\int_\Omega |\nabla\phi_1|^p\,dx)^{\frac{1}{p}} \\
& + \lambda(\int_\Omega |u_*|^{q(\frac{p}{p-1})}\,dx)^{\frac{p-1}{p}}(\int_\Omega |\nabla\phi_1|^p\,dx)^{\frac{1}{p}} \\
\leq & (a+b\|u_*\|^p)^{p-1}[(\int_\Omega |\nabla u_*|^{(p-1)(\frac{p}{p-1})}\,dx)(\int_\Omega |\nabla\phi_1|^p\,dx)^{\frac{1}{p}}] \\
& + \frac{1}{p}(\int_\Omega \ell(x)^{\frac{N}{p}}\,dx)^{\frac{p}{N}\frac{p-1}{p}}(\int_\Omega u_*^{p\frac{p^*}{p}})^{\frac{p^*}{p}\frac{p-1}{p}}(\int_\Omega |\nabla\phi_1|^p\,dx)^{\frac{1}{p}} \\
& + \lambda(\int_\Omega |u_*|^{q(\frac{p}{p-1})}\,dx)^{\frac{p-1}{p}}(\int_\Omega |\nabla\phi_1|^p\,dx)^{\frac{1}{p}} \\
\leq & (a+b\|u_*\|^p)^{p-1}(\|u_*\|^{p-1})(\|\phi_1\|) + \|\ell\|_{\frac{N}{p}}^{\frac{p-1}{p}}\|u_*\|_{p^*}^p\|\phi_1\| \\
& + \lambda\|u_*\|_{\frac{p}{p-1}}\|\phi_1\| \\
< & \infty
\end{aligned}$$

which implies that $u_* > 0$ for almost every $x \in \Omega$.
Moreover, according to Lemma 2, we have

$$J_\lambda(u_*) = \inf_{u\in X} J_\lambda(u).$$

Thus u_* is a global minimizer solution.

Assuming that φ and ψ are two distint weak solutions of problem (1), then we test equation (VF) by $(\varphi-\psi)$:

$$\int_\Omega L_p(\varphi)(\varphi-\psi)dx = \int_\Omega f(x)|\varphi|^{-\gamma}(\varphi-\psi)dx - \lambda\int_\Omega |\varphi|^{p^*-2}\varphi(\varphi-\psi)dx \tag{12}$$

and

$$\int_\Omega L_p(\psi)(\varphi-\psi)dx = \int_\Omega f(x)|\psi|^{-\gamma}(\varphi-\psi)dx - \lambda\int_\Omega |\psi|^{p^*-2}\psi(\varphi-\psi)dx \tag{13}$$

Then we have

$$\begin{aligned}\int_\Omega L_p(\varphi)(\varphi-\psi)dx = &(a\|\varphi\|^p+b)^{p-1}\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla(\varphi-\psi)dx\\ &+\int_\Omega \ell(x)|\varphi|^{p-2}\varphi(\varphi-\psi)dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega L_p(\psi)(\varphi-\psi)dx = &(a\|\psi\|^p+b)^{p-1}\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\nabla(\varphi-\psi)dx\\ &+\int_\Omega \ell(x)|\psi|^{p-2}\psi(\varphi-\psi)dx.\end{aligned}$$

From (12), one obtains

$$\begin{gathered}\int_\Omega L_p(\varphi)(\varphi-\psi)dx =\\ (a\|\varphi\|^p+b)^{p-1}\left[\|\varphi\|^p - \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla\psi dx\right]\\ +\int_\Omega \ell(x)|\varphi|^p dx - \int_\Omega \ell(x)|\varphi|^{p-2}\varphi\psi dx\end{gathered} \tag{14}$$

and also, from (13), one obtains

$$\begin{gathered}\int_\Omega L_p(\psi)(\varphi-\psi)dx =\\ (a\|\psi\|^p+b)^{p-1}\left[-\|\psi\|^p + \int_\Omega |\nabla\psi|^{p-2}\nabla\varphi\nabla\psi dx\right]\\ -\int_\Omega \ell(x)|\psi|^p dx + \int_\Omega \ell(x)|\psi|^{p-2}\psi\varphi dx.\end{gathered} \tag{15}$$

Combining (14) and (15), we have

$$\int_\Omega L_p(\varphi)(\varphi-\psi)dx - \int_\Omega L_p(\psi)(\varphi-\psi)dx =$$

$$\|\varphi\|^p\left(a\|\varphi\|^p+b\right)^{p-1} + \|\psi\|^p\left(a\|\psi\|^p+b\right)^{p-1}$$

$$-\left(a\|\varphi\|^p+b\right)^{p-1}\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla\psi dx - \left(a\|\psi\|^p+b\right)^{p-1}\int_\Omega |\nabla\psi|^{p-2}\nabla\varphi\nabla\psi dx$$

$$+\int_\Omega \ell(x)\left(|\varphi|^p+|\psi|^p\right)dx - \int_\Omega \ell(x)\left(|\varphi|^{p-2}+|\psi|^{p-2}\right)\varphi\psi dx.$$

With the same computation for the right term, we have

$$\int_\Omega L_p(\varphi)(\varphi-\psi)dx - \int_\Omega L_p(\psi)(\varphi-\psi)dx =$$

$$\int_\Omega f(x)\left(|\varphi|^{-\gamma}-|\psi|^{-\gamma}\right)(\varphi-\psi)dx \tag{16}$$

$$-\lambda\int_\Omega \left(|\varphi|^{p^*-2}\varphi - |\psi|^{p^*-2}\psi\right)(\varphi-\psi)dx.$$

Put

$$Q_1(\varphi,\psi) := \|\varphi\|^p\left(a\|\varphi\|^p+b\right)^{p-1} - \left(a\|\varphi\|^p+b\right)^{p-1}\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla\psi dx,$$

$$Q_2(\varphi,\psi) := \|\psi\|^p\left(a\|\psi\|^p+b\right)^{p-1} - \left(a\|\psi\|^p+b\right)^{p-1}\int_\Omega |\nabla\psi|^{p-2}\nabla\varphi\nabla\psi dx,$$

and

$$Q_3(\varphi,\psi) := \int_\Omega \ell(x)\left(|\varphi|^p+|\psi|^p\right)dx - \int_\Omega \ell(x)\left(|\varphi|^{p-2}+|\psi|^{p-2}\right)\varphi\psi dx.$$

Using Holder inequality, we have

$$Q_1(\varphi,\psi) \geq \|\varphi\|^{p-1}\left(a\|\varphi\|^p+b\right)^{p-1}\left(\|\varphi\|-\|\psi\|\right),$$

$$Q_2(\varphi,\psi) \geq \|\psi\|^{p-1}\left(a\|\psi\|^p+b\right)^{p-1}\left(\|\psi\|-\|\varphi\|\right),$$

and

$$Q_3(\varphi,\psi) \geq \left(\|\varphi\|_p-\|\psi\|_p\right)\|\ell\|_\infty\left[\|\varphi\|_p^{p-1}-\|\psi\|_p^{p-1}\right].$$

We divided in three cases:

1. Case if $\|\varphi\|-\|\psi\|>0$, then

$$Q_1(\varphi,\psi)-Q_2(\varphi,\psi)+Q_3(\varphi,\psi)>0.$$

2. Case if $\|\varphi\|-\|\psi\|<0$, then

$$Q_1(\varphi,\psi)-Q_2(\varphi,\psi)+Q_3(\varphi,\psi)<0.$$

3. Case if $\|\varphi\|-\|\psi\|=0$, then

$$Q_1(\varphi,\psi)-Q_2(\varphi,\psi)+Q_3(\varphi,\psi)\geq 0.$$

Since $\gamma \in (0,1)$ and p > 0, it is well known the following inequalities:

$$\forall x, y > 0 : \begin{cases} (x^p - y^p)(x-y) \geq 0, \\ (x^{-\gamma} - y^{-\gamma})(y-x) \geq 0. \end{cases}$$

Thus

$$\int_\Omega L_p(\varphi)(\varphi - \psi)dx - \int_\Omega L_p(\psi)(\varphi - \psi)dx \geq 0.$$

Consequently, we obtain a contradiction with the equation (16).
Then

$$\varphi = \psi.$$

This completes the proof of the theorem 1.2.

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