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Short Note on the Riemann Hypothesis

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Abstract Robin criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer. If the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that Robin inequality does not hold and we prove that $n^{\left(1-\frac{0.6253}{\log q_m}\right)} < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and q_m is the largest prime divisor of n. In addition, we show that q_m will not have an upper bound by some positive value for these counterexamples and therefore, the value of q_m tends to infinity as n goes to infinity.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

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1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [7]. Let $N_m = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times q_m$ denotes a primorial number of order m such that q_m is the m^{th} prime number [5]. As usual $\sigma(n)$ is the sum-of-divisors function of n [1]:

$$\sum_{d|n} d$$

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where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [7]. Moreover, if the Riemann hypothesis is false, then there are infinitely many natural numbers* $n > 5040$ *such that* $\mathsf{Robins}(n)$ *does not hold [7].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are not divisible by 2 [1]. We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [1].

Theorem 1.2 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *that are square free [1].*

Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [1]. Based on the theorem 1.1, we know this result:

Theorem 1.3 *If the Riemann hypothesis is false, then there are infinitely many natural numbers* $n > 5040$ *which are an Hardy-Ramanujan integer and* $\mathsf{Robins}(n)$ *does not hold [1].*

We prove if the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold and $n^{\left(1-\frac{0.6253}{\log q_m}\right)} < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and q_m is the largest prime divisor of n. Furthermore, we show that q_m will not have an upper bound by some positive value for these counterexamples and thus, the value of q_m tends to infinity as n goes to infinity.

2 Known Results

These are known results:

Theorem 2.1 *[1]. For* $n > 1$*:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}.$$

Theorem 2.2 *[2].*

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}.$$

Theorem 2.3 *[3]. Let $n > e^{e^{23.762143}}$ and let all its prime divisors be $q_1 < \cdots < q_m$, then*

$$\left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n.$$

Theorem 2.4 $\mathsf{Robins}(n)$ *holds for all natural numbers* $10^{10^{13.11485}} \geq n > 5040$ *[6].*

Theorem 2.5 *[9]. For $q_m \geq 20000$, we have*

$$\log q_m < \log\log N_m + \frac{0.1253}{\log q_m}.$$

Theorem 2.6 *[8]. For $x \geq 286$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log(x)}).$$

Theorem 2.7 *[4]. For $x > -1$:*

$$\frac{x}{x+1} \leq \log(1+x).$$

3 A Central Theorem

The following is a key theorem. It gives an upper bound on $f(n)$ that holds for all natural numbers n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all natural numbers n. Further the bound only uses the primes that divide n and not how many times they divide n.

Theorem 3.1 *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof Putting together the theorems 2.1 and 2.2 yields the proof:

$$f(n) < \prod_{i=1}^{m} \left(\frac{q_i}{q_i - 1}\right) = \prod_{i=1}^{m} \left(\frac{q_i + 1}{q_i} \times \frac{1}{1 - \frac{1}{q_i^2}}\right) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

4 A Particular Case

We can easily prove that $\mathsf{Robins}(n)$ is true for certain kind of numbers.

Theorem 4.1 $\mathsf{Robins}(n)$ *holds for $n > 5040$ when $q \leq 5$, where q is the largest prime divisor of n.*

Proof Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m \leq 5$, then we need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log\log n$$

according to the theorem 2.1. For $q_1 < \cdots < q_m \leq 5$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log\log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log\log(5040) < e^{\gamma} \times \log\log n$$

and therefore, the proof is complete when $q_1 < \cdots < q_m \leq 5$.

5 Robin on Divisibility

The next theorem implies that $\mathsf{Robins}(n)$ holds for a wide range of natural numbers $n > 5040$.

Theorem 5.1 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *when a prime* $q \leq 1771559$ *complies with* $q \nmid n$.

Proof Note that $f(n) < \frac{n}{\varphi(n)} = \prod_{q \mid n} \frac{q}{q-1}$ from the theorem 2.1, where $\varphi(x)$ is the Euler's totient function. We have that $f(n) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log(n)$ for any number $n > 10^{10^{13.11485}}$. Suppose that n is not divisible by a prime q for q less than or equal to some prime bound Q and $n > N = 10^{10^{13.11485}}$. Then,

$$\begin{aligned} f(n) &< \frac{n}{\varphi(n)} \\ &= \frac{n \times q}{\varphi(n \times q)} \times \frac{q-1}{q} \\ &< \frac{1771561}{1771560} \times \frac{q-1}{q} \times e^{\gamma} \times \log\log(n \times q) \end{aligned}$$

and

$$\begin{aligned}
\frac{f(n)}{e^{\gamma} \times \log\log(n)} &< \frac{1771561}{1771560} \times \frac{q-1}{q} \times \frac{\log\log(n \times q)}{\log\log(n)} \\
&\leq \frac{1771561}{1771560} \times \frac{Q-1}{Q} \times \frac{\log\log(n \times Q)}{\log\log(n)} \\
&= \frac{1771561}{1771560} \times \frac{Q-1}{Q} \times \frac{\log\log(n) + \log(1 + \frac{\log(Q)}{\log(n)})}{\log\log(n)} \\
&= \frac{1771561}{1771560} \times \frac{Q-1}{Q} \times \left(1 + \frac{\log(1 + \frac{\log(Q)}{\log(n)})}{\log\log(n)}\right)
\end{aligned}$$

So

$$\frac{f(n)}{e^{\gamma} \times \log\log(n)} < \frac{1771561}{1771560} \times \frac{Q-1}{Q} \times \left(1 + \frac{\log(1 + \frac{\log(Q)}{\log(n)})}{\log\log(n)}\right)$$

for $n > N = 10^{10^{13.11485}}$. The right hand side is less than 1 for $Q \leq 1771559$. Moreover, note that the inequality $10^{10^{13.11485}} > e^{e^{23.762143}}$ is satisfied. Therefore, $\mathsf{Robins}(n)$ holds as a consequence of the theorems 2.3 and 2.4.

6 A Main Insight

The next theorem is a main insight.

Theorem 6.1 *Let* $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ *for some natural number* $n > 5040$ *such that* n' *is the square free kernel of the natural number* n. *Then* $\mathsf{Robins}(n)$ *holds.*

Proof Let n' be the square free kernel of the natural number n, that is the product of the distinct primes $q_1, \ldots, q_m$. By assumption we have that

$$\frac{\pi^2}{6} \times \log\log n' \leq \log\log n.$$

For all square free $n' \leq 5040$, $\mathsf{Robins}(n')$ holds if and only if $n' \notin \{2,3,5,6,10,30\}$ [1]. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ when $n' \in \{2,3,5,6,10,15,30\}$ due to the theorem 4.1. When $n' > 5040$, we know that $\mathsf{Robins}(n')$ holds and so

$$f(n') < e^{\gamma} \times \log\log n'$$

because of the theorem 1.2. By the previous theorem 3.1:

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Suppose by way of contradiction that $\mathsf{Robins}(n)$ fails. Then

$$f(n) \geq e^{\gamma} \times \log\log n.$$

We claim that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i} > e^{\gamma} \times \log\log n.$$

Since otherwise we would have a contradiction. This shows that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i} > \frac{\pi^2}{6} \times e^{\gamma} \times \log\log n'.$$

Thus

$$\prod_{i=1}^{m} \frac{q_i+1}{q_i} > e^{\gamma} \times \log\log n',$$

and

$$\prod_{i=1}^{m} \frac{q_i+1}{q_i} > f(n'),$$

This is a contradiction since $f(n')$ is equal to

$$\frac{(q_1+1) \times \cdots \times (q_m+1)}{q_1 \times \cdots \times q_m}$$

according to the formula $f(x)$ for the square free numbers [1].

7 Proof of Main Theorem

Theorem 7.1 *If the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold and $n^{\left(1-\frac{0.6253}{\log q_m}\right)} < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and q_m is the largest prime divisor of n. In addition, q_m will not have an upper bound by some positive value for these counterexamples and therefore, the value of q_m tends to infinity as n goes to infinity.*

Proof Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of some natural number $n > 5040$ as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. The primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since the natural number $n > 5040$ will be an Hardy-Ramanujan integer. We assume that $\mathsf{Robins}(n)$ does not hold. Indeed, we know there are infinitely many Hardy-Ramanujan integers such as $n > 5040$ when the Riemann hypothesis is false according to the theorem 1.3. From the theorem 5.1, we know that necessarily $q_m \geq 1771559$. So,

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)})$$

because of the theorems 2.1 and 2.6. Hence,

$$\log\log n < \log q_m + \frac{0.5}{\log(q_m)}.$$

From the theorem 2.5, we have that

$$\log\log n < \log\log N_m + \frac{0.1253}{\log q_m} + \frac{0.5}{\log(q_m)}.$$

That is the same as

$$\log\log n - \log\log N_m < \frac{0.6253}{\log q_m}.$$

Then,

$$\begin{aligned}
\log\log n - \log\log N_m &= \log\left(\log N_m + \log(\frac{n}{N_m})\right) - \log\log N_m \\
&= \log\left(\log N_m \times (1 + \frac{\log(\frac{n}{N_m})}{\log N_m})\right) - \log\log N_m \\
&= \log\log N_m + \log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}) - \log\log N_m \\
&= \log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}).
\end{aligned}$$

In addition, we know that

$$\log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}) \geq \frac{\log(\frac{n}{N_m})}{\log n}$$

using the theorem 2.7 since $\frac{\log(\frac{n}{N_m})}{\log N_m} > -1$. Certainly, we will have that

$$\log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}) \geq \frac{\frac{\log(\frac{n}{N_m})}{\log N_m}}{\frac{\log(\frac{n}{N_m})}{\log N_m} + 1} = \frac{\log(\frac{n}{N_m})}{\log(\frac{n}{N_m}) + \log N_m} = \frac{\log(\frac{n}{N_m})}{\log n}.$$

In this way, we have that

$$\frac{\log(\frac{n}{N_m})}{\log n} < \frac{0.6253}{\log q_m}$$

which is equivalent to

$$\log(\frac{n}{N_m}) < \log(n^{\frac{0.6253}{\log q_m}})$$

and thus

$$\frac{n}{N_m} < n^{\frac{0.6253}{\log q_m}}.$$

Finally, we obtain that

$$n^{\left(1 - \frac{0.6253}{\log q_m}\right)} < N_m.$$

Moreover, we know that q_m will not have an upper bound by some positive value for these counterexamples because of the theorem 6.1. Certainly, if there is a possible upper bound for q_m, then it cannot exist infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold as a consequence of the theorem 6.1.

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