On The Magic Pill Method: Odd Perfect Numbers and Odd Triperfect Numbers Do Not Exist.

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Perfect numbers have been studied for a long time since the times of the ancient Greeks. The question of whether odd perfect numbers exist must have been pondered over during the times of Euclid but there is no written evidence of this. In fact the earliest time when someone wrote down about the parity of perfect numbers was around the second century A.D. This person was a Greek mathematician called Nicomachus and was living in Gerasa.

In fact this person did not just ponder about the parity of perfect numbers by thinking about whether odd perfect numbers exist but he in stead went a step further and conjectured that odd perfect numbers do not exist. He did this by saying that all perfect numbers are even which is to imply that odd perfect numbers do not exist. This is the first written evidence of the odd perfect number conjecture. Therefore this conjecture is 1,900 years old.

D'ooge, Robbins and Karpinski translated Nicomachus's old book "Introductio de Arithmetica." into English in the early 20th century. The part of the translated version of the book where Nicomachus states his odd perfect number conjecture is pages 209-210. I will quote the relevant paragragh directly from the translated book. The very last sentence of the paragraph is where Nicomachus states his conjecture. The paragraph goes as follows:

"It comes about that even as fair and excellent things are few and easily enumerated, while ugly and evil ones are widespread, so also the superabundant and deficient numbers are found in great multitude and irregularly placed - for the method of their discovery is irregular - but the perfect numbers are easily enumerated and arranged with suitable order; for only one is found among the units, 6, only one other among the tens, 28, and a third in the rank of the hundreds, 496 alone, and a fourth within the limits of the thousands, that is, below ten thousand, 8,128. And it is their accompanying characteristics to end alternatively in 6 or 8, and always to be even."(D'ooge, Robbins and Karpinski,1926)

With this direct quote of Nicomachus himself, it is clear therefore that this odd perfect number conjecture is one of the oldest open conjectures in mathematics. Nicomachus however, did not provide any proof for his claims just like Fermat did not provide any proof regarding Fermat's last theorem. Nicomachus left it for future mathematicians to prove him right or wrong and no one has been able to do so ever since.

Since the time of Nicomachus, mathematicians have made some significant contributions to the odd perfect number conjecture. Euler proved that if an odd perfect number exists, then it has the form $p^k s^2$, where p is prime, $gcd(p, s) = 1$, and p and k are of the form $4n + 1$.

James Joseph Sylvester proved:

i) No odd perfect number is divisible by 105.

ii) An odd perfect number must have at least four distinct prime divisors.

iii) If an odd perfect number is not divisible by 3, then it must have at least 8 distinct prime divisors. (Beasley, 2013)

There has been a lot of advances made by many mathematicians since Euler and Sylvester and I cannot list all of those contributions here. However, I will mention only one recent contribution made by Ochem and Rao where they showed that an odd perfect number, if it exists, must be greater than 10^{1500}. Their research implies that the possibility of odd perfect numbers existing is slim to none.

PART I: Proving That Odd Perfect Numbers Do Not Exist

New Theorem 1: Odd perfect numbers do not exist

An odd perfect number N is a number whose total sum of divisors is equal to twice the number itself. Another way of writing it is $\sigma(N) = 2N$. The symbol σ is the sigma function and it counts the total number of divisors of any number N. Euler proved that if an odd perfect number exists then it has the form $p^k s^2$, where p is prime, $gcd(p, s) = 1$ and $p \equiv k \equiv 1(mod4)$. The Euler form of odd perfect numbers can also be expressed in another way (Ward, 2020). This second representation of the structure of odd perfect number is the same as the first one, the only difference being that s^2 is represented as a product of its prime factors. The first Euler structure is commonly used by many mathematicians in their work. However in this paper I will use the second, less commonly used, Euler structure of odd perfect numbers. This means that s^2 will have to be written as a product of its prime factors. That means that the odd perfect number N will be written as a product of its prime factors. I will first use a specific example before generalising it to a general case.

Analysing Euler's structure of odd perfect numbers

If odd perfect numbers exist they are of the form $p_1^{4k+1}p_2^{2b_1}p_3^{2b_2}...p_n^{2b_{n-1}}$. Where $p_1, p_2...p_n$ are all prime numbers. Note that the folowing two forms of odd perfect numbers are equal. The left side and the right side of the equation below are equal. It is for you to choose which version you want to use.
$p_1^{4k+1}p_2^{2b_1}p_3^{2b_2}...p_n^{2b_{n-1}}=p^k s^2$

We will however use this form $p_1^{4k+1}p_2^{2b_1}p_3^{2b_2}...p_n^{2b_{n-1}}$ of odd perfect numbers from now on. Looking at this structure, we notice that all prime numbers in the structure except one are raised to the power of even numbers. This means that only one prime number is raised to the power of an odd number. The total product of the prime numbers raised to the power of even numbers can be rewritten as an odd perfect square s^2.

Special case: Is there an odd perfect number of the form $p^5t^4s^2f^2$?

In this special case let us try to determine whether there is an odd perfect number of the form $p^5t^4s^2f^2$ where p,t,s and f are all prime numbers.

Let us assume that there is an odd perfect number N of the form $p^5t^4s^2f^2$ therefore we can write:

$p^5t^4s^2f^2$=N

The sum of divisors of N is equal to 2N therefore we can write:

$\sigma(N) = 2N \implies \sigma(p^5t^4s^2f^2) = 2N$

$\sigma(p^5t^4s^2f^2) = 2N$ can also be written as:

$\sigma(p^5t^4s^2f^2) = 2(p^5t^4s^2f^2)$

Dividing both sides by 2 we get:

$$\frac{\sigma(p^5t^4s^2f^2)}{2} = p^5t^4s^2f^2 \tag{1}$$

Dividing both sides by p we get:

$$\frac{\sigma(p^5t^4s^2f^2)}{2p} = \frac{\cancelto{p^4}{p^5}\ t^4s^2f^2}{\cancelto{1}{p}} \tag{2}$$

$$\frac{\sigma(p^5t^4s^2f^2)}{2p} = p^4t^4s^2f^2 \tag{3}$$

Notice that $p^4t^4s^2f^2$ is a perfect square and can also be written as y^2 where y is an odd integer. Therefore:

$$p^4t^4s^2f^2 = y^2 \tag{4}$$

Notice that equation 3 above can therefore be rewritten as:

$$\frac{\sigma(p^5t^4s^2f^2)}{2p} = y^2 \tag{5}$$

Equation 5 is the most important equation in this paper and it is the equation that will be used to prove that odd perfect numbers do not exist. Notice that equation 5 and 6 are the same equation, it is just that they are expressed differently.

$$\frac{(p^5+p^4+p^3+p^2+p+1)(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)}{2p} = y^2 \tag{6}$$

Notice that $(p^5+p^4+p^3+p^2+p+1)$ is divisible by 2 because it is an even number. We know this because it is an addition of 6 odd numbers. We know that when you add odd numbers even number of times you will always get an even number. Any even number is divisible by 2 hence $(p^5+p^4+p^3+p^2+p+1)$ is divisible by 2. When we divide $(p^5+p^4+p^3+p^2+p+1)$ by 2 we get the following expression:

$$\frac{\cancelto{g}{(p^5+p^4+p^3+p^2+p+1)}\quad(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)}{\cancelto{1}{2}p} = y^2 \tag{7}$$

Dividing $(p^5+p^4+p^3+p^2+p+1)$ by 2 we get a number g. We do not know much about about the properties of this number g but we do not need to know anything about it in this proof. We also do not know whether g is a perfect square number or not but again this is not going to matter in our analysis. After dividing $(p^5+p^4+p^3+p^2+p+1)$ by 2 we get the following equation:

$$\frac{g(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)}{p} = y^2 \qquad (8)$$

My main goal is to prove that the left side of the above equation (8) is not equal to the right side of that equation. My strategy for proving this is to prove that the left side of the equation is a not a perfect square while the right side is a perfect square. This would therefore mean that the left side of the equation is not equal to the right side and will therefore mean that there is no odd perfect number of the form $p^5t^4s^2f^2$. So how will I prove that $\frac{g(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)}{p}$ is not a perfect square? I will first do this by showing that there are only four possible cases we need to consider when we are diving $g(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)$ by p. The first case is when g is divisible by p. If g is not divisible by p then we go to cases 2(i), 2(ii) and 2(iii). Before we use these cases we need to separate $(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)$ into two using brackets as shown below:
$[(t^4+t^3+t^2+t+1)][(s^2+s+1)(f^2+f+1)]$

Case 2(i) is when $[(t^4+t^3+t^2+t+1)]$ is divisible by p. Case 2(ii) is when $[(s^2+s+1)(f^2+f+1)]$ is divisible by p. The fourth and the last case, case 2(iii) is when neither $[(t^4+t^3+t^2+t+1)]$ nor $[(s^2+s+1)(f^2+f+1)]$ is divisible by p.
I contend that these cases are exhaustive and that there is no other case that arises when one is trying to divide $g(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)$ by p. In fact the number of cases

should have been three because we can merge cases 2(i) and 2(ii) into a single case. However I want cases 2(i) and 2(ii) to remain as two separate cases for a particular reason which I will explain later. Therefore, if we can all agree that these four cases are exhaustive in showing all the possible cases we get when we divide $g(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)$ by p then I would like to proceed to these four cases and show that in each of these cases, the left side of the equation and the right side of the equation are not equal.

Case 1: g is divisible by p.

Corollary 1: Polynomials of any degree, where every term has a coefficient of 1 is not a perfect square.

Examples of these types of polynomials described above include: $(t^4+t^3+t^2+t+1)$,$(p^5+p^4+p^3+p^2+p+1)$and (s^2+s+1). Since all of these polynomials have coefficients of 1. therefore none of these polynomials is a perfect square. This is the same as saying that the square-root of these polynomials cannot be an integer. This corollary follows from a well known theorem of squaring a summation. This theorem of squaring a summation will be briefly highlighted later in this paper

In this section we will assume that g is divisible by p and we will try to look at how that will affect our equation. We will see that whether g is divisible by p or not will not change the fact that our equation is not a perfect square.

If g is divisible by p, we will get the following equation:

$$\frac{\cancel{(g)}^{\,h}\ (t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)}{\cancel{p}^{\,1}} = y^2 \tag{9}$$

The final equation will look like this:

$$h(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1) = y^2 \tag{10}$$

Now we know that

$$h(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)$$

is not a perfect square because

$$((t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)) \tag{11}$$

is not a perfect square. This is true even if h is a perfect square. This is true because of theorem 1 below.

Therefore:

$$h(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1) \neq y^2 \tag{12}$$

Theorem 1: This theorem is one of the laws of exponents. This theorem is simply stated as:

$$a^2b^2c^2d^2e^2f^2 = (abcdef)^2 \tag{13}$$

From this theorem we can extract a corollary that states that the right hand side will not be a perfect square if any of the terms on the left hand side is not a perfect square. This means that if the first term on the left hand-side was for example, a^3 instead of a^2 then the equation on the right-hand side will not be a perfect square.In notation form:

$$a^3b^2c^2d^2e^2f^2 \neq (abcdef)^2$$

This means that for the equation on the right-hand side to be a perfect square then all the terms on the left-hand side must also be perfect squares. This is our corollary 2 and I will state it as such:

Corollary 2: If we have an equation of this nature:

$a^2b^2c^2d^2e^2f^2 = (abcdef)^2$, **for the equation on the right-hand side to be a perfect square then all the terms on the left-hand side must also be perfect squares.**

Let us label equation (14) below as q:

$$((t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)) = q \tag{14}$$

From now on, we will refer to $(t^4+t^3+t^2+t+1)$, $\quad (s^2+s+1) \quad and \quad (f^2+f+1)$ as the displayed factors of q. We will refer to them as displayed factors of q because those are the factors of q that we can see with our eyes without making any further calculations. It is important to notice that these three factors are not necessarily the only factors of q and q could potentially have many more factors but these are the factors that we can easily identify. However, we will not calculate all factors of q because that is not necessary and we need to only work with the displayed factors of q to achieve our goal.

Case 2: g is not divisible by p and q is divisible by p.

Separate into two (using brackets) the displayed factors of q. Since q in this case has 3 displayed factors, when we separate these displayed factors into two, one bracket will have one displayed factor within it while the other bracket will have two displayed factors within it. The separation will look like this:

$$[(t^4+t^3+t^2+t+1)] \quad [(s^2+s+1)(f^2+f+1)] = q \tag{15}$$

$$[(f^2+f+1)] \quad [(t^4+t^3+t^2+t+1)(s^2+s+1)] = q \tag{16}$$

$$[(s^2+s+1)] \quad [(t^4+t^3+t^2+t+1)(f^2+f+1)] = q \tag{17}$$

Now, we don't have to use all the three equations 15, 16, and 17 to solve our problem, we only need to pick one. This is because all these three equations are similar because the order of multiplication of the factors does not matter because of the commutative property of multiplication. For example, in equation 15, (s^2+s+1) and (f^2+f+1) are multiplied first and then $(t^4+t^3+t^2+t+1)$ is multiplied later while in equation 16, $(t^4+t^3+t^2+t+1)$ and (s^2+s+1) are multiplied first and then (f^2+f+1) is multiplied later. So it is only the order of multiplication that is different but the final equation is the same and all the 3 equations have the same total product of q. Therefore in order to avoid repetition, we only need to choose one of these pairs of displayed factors of q.

I want to emphasize that when you are separating (using brackets) the displayed factors of q into two, it does not matter how you do it. For example, if q had the following 6 displayed factors $(s^2+s+1)(f^2+f+1)(t^2+t+1)(v^2+v+1)(h^2+h+1)(k^2+k+1) = q$, we can separate them into two in any way we like including but not limited to the following examples:

A) $[(s^2+s+1)][(f^2+f+1)(t^2+t+1)(v^2+v+1)(h^2+h+1)(k^2+k+1)] = q$,

B) $[(s^2+s+1)(f^2+f+1)][(t^2+t+1)(v^2+v+1)(h^2+h+1)(k^2+k+1)] = q$,

C) $[(s^2+s+1)(f^2+f+1)(t^2+t+1)][(v^2+v+1)(h^2+h+1)(k^2+k+1)] = q$,

In the first example above, one bracket contains only one displayed factor while the other bracket contains 5 displayed factors. In the third example, both the first and the second brackets contain three displayed factors each. So it really doesn't matter how you do it as long as you separate (using brackets) the dispayed factors of q into two.

Please take note that cases 2A, 2B and 2C below are similar and you need to do only one of these cases. Here I am going to do all three of them just to show that they give us the same results. But in reality only one case is sufficient to prove the theorem.

Case 2 A: $[(t^4 + t^3 + t^2 + t + 1)] \quad [(s^2 + s + 1)(f^2 + f + 1)]$

If any of these two factors above is divisible by p then q will not be a perfect square.

Case 2 A (i): If $(t^4 + t^3 + t^2 + t + 1)$ is divisible by p then we get:

$$\frac{g\cancelto{b}{(t^4 + t^3 + t^2 + t + 1)} \quad (s^2 + s + 1)(f^2 + f + 1)}{\cancelto{1}{p}} = y^2 \tag{18}$$

$$gb(s^2 + s + 1)(f^2 + f + 1) = y^2 \tag{19}$$

$gb(s^2 + s + 1)(f^2 + f + 1)$ is not a perfect square because $(s^2 + s + 1)(f^2 + f + 1)$ is not a perfect square. This is true even if g and b are perfect squares.

Therefore:

$$gb(s^2 + s + 1)(f^2 + f + 1) \neq y^2 \tag{20}$$

Two other alternative statements that are equally correct are as follows:

i) $gb(s^2+s+1)(f^2+f+1)$ is not a perfect square because (f^2+f+1) is not a perfect square. This is true even if g and b are perfect squares.
Therefore:

$$gb(s^2+s+1)(f^2+f+1) \neq y^2 \tag{21}$$

Or

ii) $gb(s^2+s+1)(f^2+f+1)$ is not a perfect square because (s^2+s+1) is not a perfect square. This is true even if g and b are perfect squares.
Therefore:

$$gb(s^2+s+1)(f^2+f+1) \neq y^2 \tag{22}$$

NB: We know (f^2+f+1) is not a perfect square, (s^2+s+1) is not a perfect square and $(s^2+s+1)(f^2+f+1)$ is also not a perfect square. So it doesn't matter which one of the three polynomials you pick because none of them is a perfect square. Therefore you can pick whichever polynomial you like.

<u>Case 2 A (ii):</u> If $[(s^2+s+1)(f^2+f+1)]$ is divisible by p then:

$$\frac{g(t^4+t^3+t^2+t+1)\quad \cancelto{c}{[(s^2+s+1)(f^2+f+1)]}}{\cancelto{1}{p}} = y^2 \tag{23}$$

$$g(t^4+t^3+t^2+t+1)c = y^2 \tag{24}$$

$g(t^4+t^3+t^2+t+1)c$ is not a perfect square because $(t^4+t^3+t^2+t+1)$ is not a perfect square. This is true even if g and c are perfect squares.

Therefore:

$$g(t^4+t^3+t^2+t+1)c \neq y^2 \tag{25}$$

<u>Case 2 B:</u> $(f^2+f+1)\quad [(t^4+t^3+t^2+t+1)(s^2+s+1)]$

If any of these two factors above is divisible by p then q will not be a perfect square.

<u>Case 2 B (i):</u> If (f^2+f+1) is divisible by p then:

$$\frac{\cancelto{d}{g(f^2+f+1)}\;[(t^4+t^3+t^2+t+1)(s^2+s+1)]}{\cancelto{1}{p}} = y^2 \tag{26}$$

$$gd[(t^4+t^3+t^2+t+1)(s^2+s+1)] = y^2 \tag{27}$$

$gd[(t^4+t^3+t^2+t+1)(s^2+s+1)]$ is not a perfect square because $[(t^4+t^3+t^2+t+1)(s^2+s+1)]$ is not a perfect square. This is true even if g and d are perfect squares.

Therefore:

$$gd[(t^4+t^3+t^2+t+1)(s^2+s+1)] \neq y^2 \tag{28}$$

Case 2 B (ii): If $[(t^4+t^3+t^2+t+1)(s^2+s+1)]$ is divisible by p then:

$$\frac{g(f^2+f+1)\;\cancelto{j}{[(t^4+t^3+t^2+t+1)(s^2+s+1)]}}{\cancelto{1}{p}} = y^2 \tag{29}$$

$$g(f^2+f+1)j = y^2 \tag{30}$$

$g(f^2+f+1)j$ is not a perfect square because (f^2+f+1) is not a perfect square. This is true even if g and j are perfect squares.

Therefore:

$$g(f^2 + f + 1)j \neq y^2 \tag{31}$$

<u>Case 2 C:</u> $(s^2 + s + 1) \quad [(t^4 + t^3 + t^2 + t + 1)(f^2 + f + 1)]$

If any of these two factors above is divisible by p then q will not be a perfect square.

<u>Case 2 C (i):</u> If $(s^2 + s + 1)$ is divisible by p then:

$$\frac{g\cancelto{k}{(s^2 + s + 1)}\,[(t^4 + t^3 + t^2 + t + 1)(f^2 + f + 1)]}{\cancelto{1}{p}} = y^2 \tag{32}$$

$$gk[(t^4 + t^3 + t^2 + t + 1)(f^2 + f + 1)] = y^2 \tag{33}$$

$gk[(t^4 + t^3 + t^2 + t + 1)(f^2 + f + 1)]$ is not a perfect square because $[(t^4 + t^3 + t^2 + t + 1)(f^2 + f + 1)]$ is not a perfect square. This is true even if g and k are perfect squares.

Therefore:

$$gk[(t^4 + t^3 + t^2 + t + 1)(f^2 + f + 1)] \neq y^2 \tag{34}$$

Case 2 C (ii): If $[(t^4+t^3+t^2+t+1)(f^2+f+1)]$ is divisible by p then:

$$\frac{g(s^2+s+1)\quad \cancelto{m}{[(t^4+t^3+t^2+t+1)(f^2+f+1)]}}{\cancelto{1}{p}} = y^2 \tag{35}$$

$$g(s^2+s+1)m = y^2 \tag{36}$$

$g(s^2+s+1)m$ is not a perfect square because (s^2+s+1) is not a perfect square. This is true even if g and m are perfect squares.
Therefore:

$$g(s^2+s+1)m \neq y^2 \tag{37}$$

Therefore, we have proved that if any factor of q is divisible by p then the resultant integer will not be a perfect square. Essentially, if one of the factors of q is divisible by p, even if the resultant integer is a perfect square, this perfect square must be multiplied by another non perfect square factor of q. The final integer we get after this multiplication cannot be a perfect square because we are multiplying a number that may be a perfect square by a factor of q which is not a perfect square. Recall corollary 2 says that when we multiply a number x that is a perfect square by a number y that is not a perfect square then we get another number z that is not a perfect square.

Case 2 (iii) P is not divisible by a factor of q

If the displayed factors of q are not divisible by p then q is not divisible by p. Therefore (q/p) is an irreducible fraction. Also remember that in this case we are assuming that g is not divisible by p. Therefore, gq is not divisible by p and hence (gq/p) is also an irreducible fraction. The irreducible fraction (gq/p) is not a perfect square because this irreducible fraction (gq/p) is not an integer and all perfect squares are integers. This is the same as saying that the square-root of an irreducible fraction cannot be an integer. Therefore (gq/p) is not a perfect square even if g is a perfect square.

Therefore

$$\frac{gq}{p} \neq y^2 \tag{38}$$

Therefore, there is no odd prime number of the form $p^5t^4s^2f^2$. Q.E.D.

There is an odd perfect number of a specific structure (if it exists) that cannot be solved using this method.

The only odd perfect number that cannot be solved using this method is an odd perfect number with only 2 unique prime factorisations. An example of such an odd perfect number would look like this: $(p^5+p^4+p^3+p^2+p+1)(s^2+s+1)$. This is proven below. However, we know that an odd perfect number with only 2 unique prime factors does not exist because (Pace Nielsen 2007) proved that an odd perfect number must have at least 9 unique prime factorisations.

An Important Exception:

proof: This method does not give a solution for an odd perfect number that has only two unique prime factorisations.

Let us assume that our odd perfect number has only two unique prime factorisations as follows: p^5s^2.

$$\frac{(p^5+p^4+p^3+p^2+p+1)(s^2+s+1)}{2p} = y^2 \tag{39}$$

Obviously, we know that $(p^5+p^4+p^3+p^2+p+1)$ is divisible by 2. So we get:

$$\frac{\cancelto{g}{(p^5+p^4+p^3+p^2+p+1)}(s^2+s+1)}{\cancelto{1}{2}p} = y^2 \tag{40}$$

$$\frac{g(s^2+s+1)}{p} = y^2 \tag{41}$$

Hence, if we assume that (s^2+s+1) is divisible by p we get:

$$\frac{g(\cancel{s^2+s+1})^{n}}{\cancel{p}^{1}} = y^2 \tag{42}$$

$$gn = y^2 \tag{43}$$

Now we know almost nothing about g and n except the fact that they are odd integers. But perhaps that is all we can know about their properties. We do not know for example, whether they are perfect squares or not. We cannot rule out the possibility that both numbers are perfect squares. If both numbers are perfect squares then gn is also a perfect square. If gn is a perfect square then $[(p^5+p^4+p^3+p^2+p+1)(s^2+s+1)]$ is an odd perfect number. However, we know that an odd perfect number must have atleast 9 unique prime factorisations (Pace Nielsen, 2007) and hence there is no odd perfect number with only two unique prime factorisations. Therefore, we are lucky that the only structure of an odd perfect number where this method fails to give us a solution, it also turns out that such a structure does not exist. Therefore the solution is complete. And we have proven that an odd perfect number of the form $p^5t^4s^2f^2$ does not exist. Q.E.D

The General Solution

What I have proven so far is that a specific case of an odd perfect number does not exist. What I need to do is to prove that all cases of odd perfect numbers do not exist. I will prove that in this section. I will use algebra to do that.

Some of the general facts that we know about all odd perfect numbers that you need to keep at the back of your mind are:

1) The structure of all odd perfect numbers as given by Euler has only one prime that is raised to the power of an odd number.

2) Following from statement number 1, we know that the rest of the prime numbers in the Euler structure of odd perfect numbers are prime numbers raised to the power of an even number.

Let us assume that there is an odd perfect number N of the general form ${p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}}$ where $p_1, p_2, p_3 ... p_{\mathrm{n}}$ are all prime numbers. therefore we can write:

$${p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}} = N$$

The sum of divisors of N is equal to 2N therefore we can write:

$$\sigma(N) = 2N \implies \sigma({p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}}) = 2N$$

$\sigma({p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}}) = 2N$ can also be written as:

$$\sigma({p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}}) = 2({p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}})$$

Dividing both sides by 2 we get:

$$\frac{\sigma({p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}})}{2} = {p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}} \tag{44}$$

Dividing both sides by p_1 we get:

$$\frac{\sigma({p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_\mathrm{n}}^{2b_\mathrm{n\text{-}1}})}{2p_1} = \frac{\cancelto{{p_1}^{2x}}{{p_1}^{4m+1}}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_\mathrm{n}}^{2b_\mathrm{n\text{-}1}}}{\cancelto{1}{p_1}} \tag{45}$$

$$\frac{\sigma({p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_\mathrm{n}}^{2b_\mathrm{n\text{-}1}})}{2p_1} = {p_1}^{2x}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_\mathrm{n}}^{2b_\mathrm{n\text{-}1}} \tag{46}$$

Notice that ${p_1}^{2x}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_\mathrm{n}}^{2b_\mathrm{n\text{-}1}}$ is a perfect square and can also be written as y^2 where y is an odd integer. Therefore:

$${p_1}^{2x}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_\mathrm{n}}^{2b_\mathrm{n\text{-}1}} = y^2 \tag{47}$$

Notice that equation (46) above can therefore be rewritten as:

$$\frac{\sigma(p_1{}^{4m+1}p_2{}^{2b_1}p_3{}^{2b_2}\cdots p_\mathrm{n}{}^{2b_\mathrm{n-1}})}{2p_1} = y^2 \tag{48}$$

General proof:

Important points to understand.

1) In the Euler structure of odd perfect numbers, let the product of all prime numbers raised to the power of an even number be equal to q.

For example, $t^4s^6f^8 = q$ While q in the above example has only three terms, in reality q can have any number of terms and the proof will still hold. For example q can have 20,000 terms and the proof will still hold. The important thing to understand is that the terms in q have similar characteristics in the sense that all of them are prime numbers raised to the power of an even number and that makes q a perfect square.

2) The prime number that is raised to the power of an odd number is of the form 4n+1. The odd power itself is also of the form 4n+1.

Suppose that odd perfect numbers exist and they have the following Euler form, written as a product of primes:

$$
{p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n\text{-}1}}} = N \tag{49}
$$

Then using the multiplicative property of the sum of divisors function we get:

$$
\sigma({p_1}^{4m+1}{p_2}^{2b_1}{p_3}^{2b_2}\cdots{p_{\mathrm{n}}}^{2b_{\mathrm{n\text{-}1}}}) = 2N \tag{50}
$$

$$
\sigma({p_1}^{4m+1})\sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_{\mathrm{n}}}^{2b_{\mathrm{n\text{-}1}}}) = 2N \tag{51}
$$

Notice that in the equation above we have assumed that each unique prime factor is raised to a unique power. But this is not always the case and it is possible for two unique prime factors of an odd perfect number to be raised to the same power. Either way, whether some unique prime factors are raised to the same power or whether each unique prime factor is raised to a different power does not matter because the proof remains valid regardless of whether the unique prime factors share the same powers or not.

For example, the equation below shows an odd perfect number with all the unique prime factors of q raised to the same power of $2b_1$. The equation below is a special case of the equation above but both equations give us the same result. If we use this magic pill method on both equations, we still get the same result, that is, odd perfect numbers do not exist.

$$\sigma({p_1}^{4m+1})\sigma({p_2}^{2b_1})\sigma({p_3}^{2b_1})\cdots\sigma({p_\text{n}}^{2b_1}) \tag{52}$$

So it doesn't matter whether the powers of the primes are shared or are unique because the proof still remains valid. However, it is important to note that the primes themselves must be unique. However, for the purpose of writing this proof we will go as general as possible and therefore we will use the equation (51) which assumes that each unique prime factor also has a unique power.
Also note that:

$$\sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_\text{n}}^{2b_\text{n-1}}) = q \tag{53}$$

So the general equation will look like this:

$$\frac{\sigma({p_1}^{4m+1})\sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_\text{n}}^{2b_\text{n-1}})}{2p_1} = y^2 \tag{54}$$

Since $\sigma({p_1}^{4m+1})$ is an even number, we divide it by 2 to get:

$$\frac{\cancelto{h}{\sigma({p_1}^{4m+1})}\,\sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_\text{n}}^{2b_\text{n-1}})}{\cancelto{1}{2}p_1} = y^2 \tag{55}$$

$$\frac{h \quad \sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_{\text{n}}}^{2b_{\text{n-1}}})}{p_1} = y^2 \tag{56}$$

My main goal is to prove that the left side of the above equation (56) is not equal to the right side of that equation. I will do that by proving that the number on the left side of the equation is not a perfect square while the number on the right side is a perfect square which means that the number on the left side is not equal to the number on the right side of the equation.

To prove that the number on the left side of the equation is not a perfect square, I will show that there are only four cases that arise when we divide $h \quad \sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_{\text{n}}}^{2b_{\text{n-1}}})$ by p_1. These four cases are exhaustive and they are listed below as follows:

i) Case 1: When h is divisible by p_1.

ii) Case 2(i): When $[\sigma({p_2}^{2b_1})]$ is divisible by p_1

iii) Case 2(ii): When $[\sigma({p_3}^{2b_2})\cdots\sigma({p_{\text{n}}}^{2b_{\text{n-1}}})]$ is divisible by p_1

iv) Case 2(iii): When neither $[\sigma({p_2}^{2b_1})]$ nor $[\sigma({p_3}^{2b_2})\cdots\sigma({p_{\text{n}}}^{2b_{\text{n-1}}})]$ is divisible by p_1

.

Case 1: If h is divisible by p_1 then we get:

$$\frac{\cancelto{b}{h} \quad \sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_\text{n}}^{2b_\text{n-1}})}{\cancelto{1}{p_1}} = y^2 \tag{57}$$

$$b\sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_\text{n}}^{2b_\text{n-1}}) = y^2 \tag{58}$$

Now we know that $b\sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_\text{n}}^{2b_\text{n-1}})$ is not a perfect square because $\sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_\text{n}}^{2b_\text{n-1}})$ is not a perfect square. This is true even if b is a perfect square. Therefore:

$$b\sigma({p_2}^{2b_1})\sigma({p_3}^{2b_2})\cdots\sigma({p_\text{n}}^{2b_\text{n-1}}) \neq y^2 \tag{59}$$

Case 2: If h is not divisible by $\mathbf{p_1}$ then we get:

Arrange the displayed factors of q into a pair of displayed factors as shown below:

$$[\sigma({p_2}^{2b_1})][\sigma({p_3}^{2b_2})\cdots\sigma({p_n}^{2b_{n-1}})] \tag{60}$$

The general equation looks like this:

$$\frac{h[\sigma({p_2}^{2b_1})][\sigma({p_3}^{2b_2})\cdots\sigma({p_n}^{2b_{n-1}})]}{p_1} \tag{61}$$

<u>**Case 2 (i): If $[\sigma({p_2}^{2b_1})]$ is divisible by $\mathbf{p_1}$ we get:**</u>

$$\frac{h\cancelto{j}{[\sigma({p_2}^{2b_1})]}\;[\sigma({p_3}^{2b_2})\cdots\sigma({p_n}^{2b_{n-1}})]}{\cancelto{1}{p_1}} = y^2 \tag{62}$$

$$hj[\sigma({p_3}^{2b_2})\cdots\sigma({p_n}^{2b_{n-1}})] = y^2 \tag{63}$$

Therefore we know that $hj[\sigma({p_3}^{2b_2})\cdots\sigma({p_n}^{2b_{n-1}})]$ is not a perfect square because $[\sigma({p_3}^{2b_2})\cdots\sigma({p_n}^{2b_{n-1}})]$ is not a perfect square. This is true even if both h and j are perfect squares.

Therefore:

$$hj[\sigma({p_3}^{2b_2})\cdots\sigma({p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}})] \neq y^2 \tag{64}$$

<u>Case 2 (ii): If $[\sigma({p_3}^{2b_2})\cdots\sigma({p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}})]$ is divisible by p_1 we get:</u>

$$\frac{h[\sigma({p_2}^{2b_1})]\cancelto{k}{[\sigma({p_3}^{2b_2})\cdots\sigma({p_{\mathrm{n}}}^{2b_{\mathrm{n-1}}})]}}{\cancelto{1}{p_1}} = y^2 \tag{65}$$

$$h[\sigma({p_2}^{2b_1})]k = y^2 \tag{66}$$

We know that $h[\sigma({p_2}^{2b_1})]k$ is not a perfect square because $[\sigma({p_2}^{2b_1})]$ is not a perfect square. This is true even if both h and k are perfect squares.

Therefore:

$$h[\sigma({p_2}^{2b_1})]k \neq y^2 \tag{67}$$

Case 2 (iii): If q is not divisible by p_1 we get:

If the displayed factors of q are not divisible by p_1 then q is not divisible by p_1. Therefore (q/p_1) is an irreducible fraction. Also remember that in this case we are assuming that h is not divisible by p_1. Therefore, hq is not divisible by p_1 and hence (hq/p_1) is also an irreducible fraction. The irreducible fraction (hq/p_1) is not a perfect square because this irreducible fraction (hq/p_1) is not an integer and all perfect squares are integers. This is the same as saying that the square-root of an irreducible fraction cannot be an integer. Therefore (hq/p_1) is not a perfect square even if g is a perfect square.
Therefore:

$$\frac{hq}{p_1} \neq y^2 \tag{68}$$

Therefore, an odd perfect number does not exist. Q.E.D.

An Alternative General Solution

This method is similar to the previous general proof the only difference is that instead of using $\sigma(s^2)$ we use (s^2+s+1). In this general solution, we will follow all the initial steps that we followed in the previous general solution so I will not repeat those initial steps here.

Let us assume that our general equation looks like this:

$$p_1{}^{4b+1}p_2{}^{2c}p_3{}^{2d}\cdots p_{\mathrm{n}}{}^{2y} = N \tag{69}$$

$$\frac{\sigma(p_1{}^{4b+1}p_2{}^{2c}p_3{}^{2d}\cdots p_{\mathrm{n}}{}^{2y})}{2p_1} = h^2 \tag{70}$$

Then using the multiplicative property of the sum of divisors function we get:

$$\frac{\sigma(p_1{}^{4b+1})\sigma(p_2{}^{2c})\sigma(p_3{}^{2d})\cdots\sigma(p_{\mathrm{n}}{}^{2y})}{2p_1} = h^2 \tag{71}$$

We have the following general equation of an odd perfect number:

$$[(p_1^{4b+1} + p_1^{(4b+1)-1} + ... + p_1^1 + 1)\times$$
$$(p_2^{2c} + p_2^{(2c)-1} + ... + p_2^1 + 1)(p_3^{2d} + p_3^{(2d)-1} + ... + p_3^1 + 1)...(p_n^{2y} + p_n^{(2y)-1} + ... + p_n^1 + 1)] \div 2p_1 = h^2$$

We will label the following polynomial as q for brevity:

$$(p_2^{2c} + p_2^{(2c)-1} + ... + p_2^1 + 1)(p_3^{2d} + p_3^{(2d)-1} + ... + p_3^1 + 1)...(p_n^{2y} + p_n^{(2y)-1} + ... + p_n^1 + 1) = q. \tag{72}$$

Since we know that $(p_1^{4b+1} + p_1^{(4b+1)-1} + ... + p_1^1 + 1)$ is divisible by 2 because it is an even number, we will divide it by 2 straight away to get g. Therefore we will have an equation like this:

$$\frac{\cancelto{g}{(p_1^{4b+1} + p_1^{(4b+1)-1} + ... + p_1^1 + 1)}q}{\cancelto{1}{2}p_1} = h^2 \tag{73}$$

$$\frac{gq}{p_1} = h^2 \tag{74}$$

In the two equations above, we have replaced the long polynomial product in equation (72) with q for the purpose of brevity.

In the equation below we have replaced q with the long polynomial product. So equation (74) and (75) are the same, the only difference is that one is longer while the other one is shorter.

$$\frac{g(p_2^{2c}+p_2^{(2c)-1}+...+p_2^1+1)(p_3^{2d}+p_3^{(2d)-1}+...+p_3^1+1)...(p_n^{2y}+p_n^{(2y)-1}+...+p_n^1+1)}{p_1}=h^2 \tag{75}$$

Case 1: If g is divisible by p_1 then we get:

$$\frac{\cancelto{w}{g}q}{\cancelto{1}{p_1}}=h^2 \tag{76}$$

$$wq=h^2 \tag{77}$$

wq is not a perfect square because q is not a perfect square. This is true even if w is a perfect square.

Therefore:

$$wq \neq h^2 \tag{78}$$

Case 2: If g is not divisible by p_1 we get:

Separate the displayed factors of q using brackets into two as shown below:

$$[(p_2^{2c}+p_2^{(2c)-1}+...+p_2^1+1)][(p_3^{2d}+p_3^{(2d)-1}+...+p_3^1+1)...(p_n^{2y}+p_n^{(2y)-1}+...+p_n^1+1)]$$

The general equation looks like this:

$$\frac{g[(p_2^{2c}+p_2^{(2c)-1}+...+p_2^1+1)][(p_3^{2d}+p_3^{(2d)-1}+...+p_3^1+1)...(p_n^{2y}+p_n^{(2y)-1}+...+p_n^1+1)]}{p_1}=h^2 \tag{79}$$

Case 2 (i): If $[(p_2^{2c}+p_2^{(2c)-1}+...+p_2^1+1)]$ is divisible by p_1 we get:

$$\frac{g\cancelto{h}{[(p_2^{2c}+p_2^{(2c)-1}+...+p_2^1+1)]}[(p_3^{2d}+p_3^{(2d)-1}+...+p_3^1+1)...(p_n^{2y}+p_n^{(2y)-1}+...+p_n^1+1)]}{\cancelto{1}{p_1}}=h^2 \tag{80}$$

$$gh[(p_3^{2d}+p_3^{(2d)-1}+...+p_3^1+1)...(p_n^{2y}+p_n^{(2y)-1}+...+p_n^1+1)]=h^2 \tag{81}$$

$gh[(p_3^{2d}+p_3^{(2d)-1}+...+p_3^1+1)...(p_n^{2y}+p_n^{(2y)-1}+...+p_n^1+1)]$ is not a perfect square because $[(p_3^{2d}+p_3^{(2d)-1}+...+p_3^1+1)...(p_n^{2y}+p_n^{(2y)-1}+...+p_n^1+1)]$ is not a perfect square. This is true even if g and h are perfect squares.

Therefore:

$$gh[(p_3^{2d} + p_3^{(2d)-1} + ... + p_3^1 + 1)...(p_n^{2y} + p_n^{(2y)-1} + ... + p_n^1 + 1)] \neq h^2 \tag{82}$$

Case 2 (ii): If $[(p_3^{2d} + p_3^{(2d)-1} + ... + p_3^1 + 1)...(p_n^{2y} + p_n^{(2y)-1} + ... + p_n^1 + 1)]$ **is divisible by** p_1 **we get:**

$$\frac{g\cancelto{m}{[(p_3^{2d} + p_3^{(2d)-1} + ... + p_3^1 + 1)...(p_n^{2y} + p_n^{(2y)-1} + ... + p_n^1 + 1)]}[(p_2^{2c} + p_2^{(2c)-1} + ... + p_2^1 + 1)]}{\cancelto{1}{p_1}} = h^2 \tag{83}$$

$$gm[(p_2^{2c} + p_2^{(2c)-1} + ... + p_2^1 + 1)] = h^2 \tag{84}$$

$gm[(p_2^{2c} + p_2^{(2c)-1} + ... + p_2^1 + 1)]$ is not a perfect square because $[(p_2^{2c} + p_2^{(2c)-1} + ... + p_2^1 + 1)]$ is not a perfect square. This is true even if g and m are perfect squares.

Therefore:

$$gm[(p_2^{2c} + p_2^{(2c)-1} + ... + p_2^1 + 1)] \neq h^2 \tag{85}$$

Case 2 (iii): If q is not divisible by p_x we get:

If q is not divisible by p_x then q/p_x is an irreducible fraction. This means that q/p_x is not a perfect square because all perfect squares are integers and q/p_x is not an integer. This also means that gq/p_x is also an irreducible fraction because we know that g is not divisible by p_x. This means that gq/p_x cannot be a perfect square because it is an irreducible fraction and irreducible fractions cannot be perfect squares. That is to say that the square-root of an irreducible fraction can never be an integer.

Therefore:

$$\frac{gq}{p_x} \neq h^2 \tag{86}$$

Chapter 2: Expounding on four key concepts underpinning the magic pill method.

There are four simple but major concepts underpinning this method and I will give more information about them in this section for fellow amateur mathematicians who may not be familiar with them. These ideas are rather obvious to most professional mathematicians. These are probably high school level concepts but still I am going to write about them because they are an important part of my proof. For people who are already familiar with these concepts you can move on to the next chapter. These four key concepts are:

1) A polynomial of any degree with all terms having coefficient of 1 is not a perfect square.

2) If a is not divisible by c and b is not divisible by c then ab is not divisible by c provided c is a prime number and a and b are integers.

3) If we have an equation of this nature:

$a^2b^2c^2d^2e^2f^2 = (abcdef)^2$, for the equation on the right-hand side to be a perfect square then all the terms on the left-hand side must also be perfect squares. Otherwise if even one term on the left hand side is not a perfect square then the number on the right hand side will definitely not be a perfect square.

4) An irreducible fraction cannot be a perfect square. This is the same as saying that the square-root of an irreducible fraction cannot be an integer.

Concept 1

Here using a well known method of square of summation, I will show that a polynomial of any degree with all coefficients being 1 is not a perfect square? I will give the following examples just to be clear about what I am saying.

$$x^2 + x + 1 \neq n^2 \tag{87}$$

$$x^3 + x^2 + x + 1 \neq n^2 \tag{88}$$

$$x^4 + x^3 + x^2 + x + 1 \neq n^2 \tag{89}$$

$$x^5 + x^4 + x^3 + x^2 + x + 1 \neq n^2 \tag{90}$$

Where n in the above equations is an integer and x is a prime number.

Let us square the first two polynomials and see what we get:

$$(x^2 + x + 1)^2 = x^4 + 2x^3 + 3x^2 + 2x + 1 \tag{91}$$

$$(x^3 + x^2 + x + 1)^2 = x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1 \tag{92}$$

We can immediately notice that only the first and the last terms have a coefficient of 1. All the other terms have coefficients greater than 1. Therefore, we can immediately conjecture that most of the coefficients of polynomials that are prefect squares have have values greater than 1. The key question to ask now is whether all polynomials that are perfect squares must have atleast one coefficient greater than 1. If the answer is yes then it means that there are no polynomials that are perfect squares where all its coefficients are 1.

To prove that all polynomials that are perfect squares must have atleast one coefficient greater than 1, I will use a well known theorem on squaring a summation.

Theorem : Squaring a summation

Find the sum of:

$$\left(\sum_{i=1}^{n} x_{\mathrm{i}}\right)^2 = (x_1 + x_2 + x_3 + \cdots + x_{\text{n-1}} + x_{\mathrm{n}})(x_1 + x_2 + x_3 + \cdots + x_{\text{n-1}} + x_{\mathrm{n}})$$

When you multiply the two brackets together, each term is squared as shown below:

$$= ({x_1}^2 + {x_2}^2 + {x_3}^2 + \cdots + {x_{\text{n-1}}}^2 + {x_{\text{n}}}^2) + \tag{93}$$

And we will add to this the cross product terms. This is where x_1 is multiplied by x_2, x_1 is multiplied by x_3 and so on. However, since there is a reverse of the same process therefore we have to multiply each row of cross product terms by 2. This process is shown below:

$$2\begin{cases} (x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_{\text{n-1}} + x_1x_{\text{n}}) \\ +(x_2x_3 + x_2x_4 + \cdots + x_2x_{\text{n-1}} + x_2x_{\text{n}}) \\ +(x_3x_4 + x_3x_5 + \cdot + x_3x_{\text{n-1}} + x_3x_{\text{n}}) \\ . \\ . \\ . \\ +(x_{\text{n-1}}x_{\text{n}}) \end{cases} \tag{94}$$

$$= \sum_{i=1}^{n} {x_{\text{i}}}^2 \quad + \quad 2\left[\left(\sum_{j=1}^{n} x_{\text{i}}\right)^2 \left(\sum_{k=j+1}^{n} x_{\text{i}}\right)^2 x_{\text{j}}x_{\text{k}}\right]$$

It is therefore clear from the above generalized squaring of a summation that when you square a polynomial with all its coefficient being 1, you will get a polynomial that has at least one of its terms having a coefficient greater than 1. Obviously that means that the square-root of a polynomial with all its coefficients being 1 cannot be an integer.

Concept 2

In concept 2, I want to prove the following theorem: If a is not divisible by c and b is not divisible by c then ab is not divisible by c provided c is a prime number and a and b are integers. I will do this by proving that this statement is the contrapositive statement to Euclid's lemma (which of course has already been proven for a long time to be true).
An example of a conditional statement would be "A implies B" which can be written as "$A \implies B$". The contrapositive of this conditional statement would be "Not B implies Not A" which can also be written as " $\sim B \implies \sim A$".(Shorser)

And we know that if the conditional statement is true then the contrapositive statement is also true. The conditional statement is the Euclid's lemma which states that "If a prime p divides the product ab of two integers a and b, then p must divide at least one of these integers a and b." (Bajnok, 2013). We know this statement to be true and there are several different ways of proving that it is true. The contrapositive of Euclid's lemma is "If p does not divide at least one of these integers a and b, then prime p does not divide the product ab of two integers a and b. You can clearly see that this contrapositive is the same statement as the original statement that I was trying to prove. Therefore I do not need to prove that the contrapositive is true since it is already proven by other mathematicians that the conditional statement (Euclid's lemma) is true. So we take it for granted that the contrapositive of Euclid's lemma is also true.

Concept 3:

If we have an equation of this nature:

$a^2b^2c^2d^2e^2f^2 = (abcdef)^2$, for the equation on the right-hand side to be a perfect square then all the terms on the left-hand side must also be perfect squares. This is a well known theorem and it is one of the laws of the exponents as I have indicated earlier in this paper. Here, I just want to provide 5 tangible examples of this law of exponents so that everyone can understand what this law of exponent means and its implications.

Examples of concept 3

1) $15 \times 16 = 240$

We know that 16 is a perfect square and we also know that 15 is not a perfect square. We also know that 240 is not a perfect square. Hence we have multiplied a perfect square with non perfect square number and the product that we have got is a non perfect square. Hence this example obeys this law of exponents.

2) $12 \times 15 \times 17 \times 25 = 76,500$

Here we see that three terms are non perfect squares and only one term (25) is a perfect square. The total product is also a non perfect square because the square-root of 76,500 is not an integer. Hence this second example also obeys this law of exponents.

3) $16 \times 25 \times 36 \times 41 = 590,400$

In this third example, we have three perfect squares and one non-perfect square (41). The total product is not a perfect square because we know that the square-root of 590,400 is not an integer. Hence this third example also obeys this law of exponents.

4) $12 \times 15 \times 17 \times 21 = 64,260$

In this example, all four terms are non perfect square numbers. The total product of these numbers is also a non perfect square number because 64,260 is a non perfect square number. This example obeys this law of exponents.

5) $16 \times 25 \times 36 \times 49 = 705,600$

In this final example, all four terms are perfect squares. The total product is also a perfect square because 705,600 is a perfect square and its square root is 840 which is an integer. Hence this example also obeys this law of exponents.

The key point to understand here is that if you are multiplying several integers together $a \times b \times c \times \cdots \times n$, the total of their product will only be a perfect square if all these integers are also perfect squares. However, if even one of these integer is a non perfect square number then the product of these numbers will not be a perfect square number. This is a very key concept in my proof of the nonexistence of odd perfect numbers.

Concept 4

It is very trivial to show that the square-root of an irreducible fraction cannot be an integer. An irreducible fraction is a fraction where the numerator and the denominator share no common factors which therefore means that this fraction cannot be reduced any further. This fraction cannot be written as an integer. For example 15/7 is an irreducible fraction and it cannot be expressed as an integer. If we divide 15 by 7 we get 2.142857... this is clearly not an integer. Since there is no irreducible fraction that can be expressed an integer then there is no irreducible fraction which is a perfect square because all perfect squares are integers.

Chapter 3: Limitations of the magic pill method.

While the magic pill method is useful and can be used to solve some of these intractable divisibility problems in number theory, the method has some limitations and anybody who wants to use this method should be aware of these limitations. It is very easy to misuse this method if one does not understand its limitations. So I intend to explain the two conditions that must be met for this method to be used. If those two conditions are not met then this method cannot be used. I will also explain why this method in its current form would make it difficult to solve the problem of whether any odd quasi-perfect numbers exist. In my opinion, the structure of the quasi-perfect number is incompatible with this method and another method should be used to solve it.

Two compulsory conditions that must be met before this method is used.

Condition 1

The first condition that must be met before this method is used is the condition that the odd number N for which we are trying to calculate its sum of divisors must have at least two unique prime factors if N is a perfect square or it must have atleast 3 unique prime factors if N has a structure similar to that of an odd perfect number. This is because if N is a perfect square and if N has only one unique prime factor then we cannot separate $\sigma(N)$ (using brackets) into two. Likewise if N has the structure of an odd perfect number and if N has Less than three unique prime factors then we cannot use this method and I have already explained why this is the case in the previous part of this paper.

For example, if $N = 2^4$ its $\sigma(2^4) = (2^4 + 2^3 + 2^2 + 2 + 1)$ cannot be separated into two separate parts because it is just one item, we need two such items or more for us to use the magic pill method.

N is supposed to have two or more unique prime factors for example N can be of the form $N = 2^4 3^2$. Therefore $\sigma(N) = (2^4 + 2^3 + 2^2 + 2 + 1)(3^2 + 3 + 1)$. This is good because $\sigma(N)$ has two separate items that we can use in our calculations. This is an important condition that must be met.

Condition 2

The second condition that must be met is that either N or $\sigma(N)$ must be a perfect square.

The main goal of the magic pill method is to make the right side of the equation a perfect square and then go on to prove that the left side of the equation is not a perfect square. Since the right side of the equation must be a perfect square, then it means that either N itself must be a perfect square or sum of divisors of N which is $\sigma(N)$ must be a perfect square. If N or $\sigma(N)$ is a perfect square then we can make the right side of the equation to be a perfect square and therefore we can use this method.

Example 1: When N is a perfect square

$\sigma(N) = 2N$ where N is a perfect square that can also be written as t^2.
When you divide both sides by 2 you get:

$\frac{\sigma(N)}{2} = N$
Which can also be written as:

$$\frac{\sigma(N)}{2} = t^2 \tag{95}$$

In equation (95) above, the right hand side is a perfect square. So if the left hand side of the equation is equal to the right hand side of the equation then the left side must also be a perfect square. If you manage to prove that the left side is not a perfect square then you have proven that the left and the right side are not equal hence a contradiction because we expect them to be equal if N exists.

Example 2: When $\sigma(N)$ is a perfect square

$\sigma(N) = 2N$ where $\sigma(N)$ is a perfect square that can also be written as t^2. Here since $2N$ itself is a perfect square then it means that the right hand side of the equation is a perfect square. All we have to do now is just to prove that the left hand side of the equation is either a perfect square or its not a perfect square depending on what you are trying to solve.

The most important thing is to understand that the right hand side of the equation must either be a perfect square or must be divisible by a certain integer to make it a perfect square. In example 1 we had to divide $2N$ by 2 to make the right side a perfect square while in example 2, we did not have to divide the right side by any integer because the right side was already a perfect square.

Why it would be difficult to use this method to solve the odd quasi-perfect number conjecture.

If sum of divisors of N are of the form $AN \pm B$ where A,N and B are integers then it may be a little bit more difficult to get a solution to such a problem using the magic pill method. Therefore it is my view that if N is of the following form: $2N+1, \quad 2N-1, \quad 3N+7, \quad 3N-7, \quad 6N+5, \quad 6N-5$ e.t.c., then it may be a little bit challenging to solve such problems. I will give an example by using the odd quasi-perfect number conjecture as an example.

Quasi-perfect numbers are numbers whose sum of divisors is equal to $2N + 1$. That is $\sigma(N) = 2N + 1$.

Cattaneo already proved that odd quasi-perfect numbers if they exist, are perfect squares. Hagis and Cohen proved that quasi-perfect numbers if they exist, have at least 7 unique prime factors. Since N is a perfect number, it can also be rewritten as t^2.

$N = t^2$

$\sigma(N) = 2N + 1$

If we subtract 1 from both sides we get:

$\sigma(N) - 1 = 2N$

If we divide 2 on both sides we get:

$$\frac{\sigma(N) - 1}{2} = t^2 \tag{96}$$

The left side of the equation (96) above is very difficult to deal with. There are some complications and unknowns that arise from trying to analyse this equation. For example, we know that σ (odd perfect number) = odd number.

This means that σ (N) = odd number. (If N is an odd perfect number.)

We know:

odd number - 1 = even number.

Therefore $\sigma(N) - 1$= even number.

$\frac{\sigma(N)-1}{2} = \frac{\text{even number}}{2}$

We have no way of knowing the characteristics of $\frac{\text{even number}}{2}$.
Some unanswered questions are: 1) When you divide this even number by 2, do you get an even number or an odd number? 2) If you get an odd number, is that odd number a perfect square square or not? It does not look like we can answer these two questions using this method. Question number 2 seems to be especially difficult to answer and that is why I suspect problems like these cannot be solved using the magic pill method.

Basically what I am trying to say in simple terms is that the magic pill method is very good at solving questions of the form:

1) $\sigma(N) = 2N$

2) $\sigma(N) = 3N$

3) $\sigma(N) = 4N$ e.t.c

(as long as we know the characteristics of N)

I believe the magic pill method is not very good in solving equations of the form:

4) $\sigma(N) = 2N + 1$

5) $\sigma(N) = 3N + 5$

6) $\sigma(N) = 4N + 6$ e.t.c

Chapter 4: Proving that odd triperfect numbers do not exist

New Theorem 2: odd triperfect numbers do not exist.

A number N is triperfect if $\sigma(n) = 3N$. The existence of an odd triperfect number is an open question. According to Beck and Najar, a German mathematician called Kanold proved that an odd triperfect number, if it exists, must be a square and must have at least 9 distinct prime factors. (Kanold, 1957)(Beck and Najar, 1982). These two characteristics alone are enough for us to prove that odd triperfect numbers do not exist. In fact it will be much easier to prove that odd triperfect numbers do not exist than to prove that odd perfect numbers do not exist.

Since an odd triperfect number is a perfect square, we already know that the structure of an odd triperfect number looks like this: t^4s^2 or $t^4s^2f^2$ e.t.c. Basically, an odd triperfect number has no prime number raised to the power of an odd number in its structure. For example, we know that p^5, where p is a prime number, is not a factor of an odd triperfect number. Therefore because all prime factors of an odd triperfect number are raised to an even power, we can reduce all these products of primes raised to an even power into a single odd perfect square number. For example $t^4s^2f^2$ can be reduced to y^2, that is, $t^4s^2f^2 = y^2$. where t,s and f are prime numbers.

$$\sigma(t^4s^2f^2) = 3N \tag{97}$$

$$\sigma(t^4s^2f^2) = 3(t^4s^2f^2) \tag{98}$$

Dividing both sides by 3 we get:

$$\frac{\sigma(t^4s^2f^2)}{3} = (t^4s^2f^2) \tag{99}$$

Notice that $t^4s^2f^2$ is a perfect square and can also be rewritten as y^2 where y is an odd integer.

Therefore:

$$t^4s^2f^2 = y^2 \tag{100}$$

Therefore replacing $t^4s^2f^2$ with y^2 in equation 99 above we get:

$$\frac{\sigma(t^4s^2f^2)}{3} = y^2 \tag{101}$$

Using the multiplicative property of the sum of divisor function, we get:

$$\frac{\sigma(t^4)\sigma(s^2)\sigma(f^2)}{3} = y^2 \tag{102}$$

$$\frac{(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)}{3} = y^2 \tag{103}$$

By the way remember that

$$(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1) = q \tag{104}$$

It is easy to see that proving that the left hand side of equation (103) above equation is not a perfect square is a trivial affair. Nevertheless, we will still proceed to prove it. Notice that equation (103) is almost similar to the equation (6) for odd perfect numbers, the only difference is that p^5 is not part of this structure and the denominator is 2p instead of 3.

We will again arrange the displayed factors of q in pairs as follows. Pick only one of the three options below. I will pick the first one:

$$(t^4+t^3+t^2+t+1) \quad [(s^2+s+1)(f^2+f+1)] = q$$
$$(f^2+f+1) \quad [(t^4+t^3+t^2+t+1)(s^2+s+1)] = q$$
$$(s^2+s+1) \quad [(t^4+t^3+t^2+t+1)(f^2+f+1)] = q$$

My main goal is to prove that the left side of the above equation (93) is not equal to the right side of that equation. I will do that by proving that the number on the left side of the equation is not a perfect square while the number on the right side is a perfect square which means that the number on the left side is not equal to the number on the right side of the equation.

To prove that the number on the left side of the equation is not a perfect square, I will show that there are only three cases that arise when we divide $(t^4+t^3+t^2+t+1)(s^2+s+1)(f^2+f+1)$ by 3. These three cases are exhaustive and they are listed below as follows:

i) Case 1(i): When $(t^4+t^3+t^2+t+1)$ is divisible by 3.

ii) Case 1(ii): When $[(s^2+s+1)(f^2+f+1)]$ is divisible by 3.

iii) Case 1(iii): When neither $(t^4+t^3+t^2+t+1)$ nor $[(s^2+s+1)(f^2+f+1)]$ is divisible by 3.

Case 1 (i): If $(t^4+t^3+t^2+t+1)$ is divisible by 3 we get:

$$\frac{\overset{v}{\cancel{(t^4+t^3+t^2+t+1)}}(s^2+s+1)(f^2+f+1)}{\overset{1}{\cancel{3}}} = y^2 \tag{105}$$

$$v(s^2+s+1)(f^2+f+1) = y^2 \tag{106}$$

$v(s^2+s+1)(f^2+f+1)$ is not a perfect square because $(s^2+s+1)(f^2+f+1)$ is not a perfect square. This is true even if v is a perfect square.

Therefore:

$$v(s^2+s+1)(f^2+f+1) \neq y^2 \tag{107}$$

<u>Case 1 (ii): If $[(s^2+s+1)(f^2+f+1)]$ is divisible by 3 then we get:</u>

$$\frac{(t^4+t^3+t^2+t+1)\cancelto{a}{(s^2+s+1)(f^2+f+1)}}{\cancelto{1}{3}} = y^2 \tag{108}$$

$$(t^4+t^3+t^2+t+1)a = y^2 \tag{109}$$

$(t^4+t^3+t^2+t+1)a$ is not a perfect square because $(t^4+t^3+t^2+t+1)$ is not a perfect square This is true even if number a is a perfect square.

Therefore:

$$(t^4 + t^3 + t^2 + t + 1)a \neq y^2 \tag{110}$$

Case 1 (iii): If q is not divisible by 3 we get:.

If no displayed factor of q is divisible by 3 then q is not divisible by 3. Therefore (q/3) is an irreducible fraction. The irreducible fraction (q/3) is not a perfect square because this irreducible fraction (q/3) is not an integer and all perfect squares are integers. Therefore q/3 is not a perfect square.

Therefore:

$$\frac{q}{3} \neq y^2 \tag{111}$$

Therefore, there is no odd triperfect number of the form $t^4 s^2 f^2$. Q.E.D

The general solution:

The odd triperfect number, if it exists, has the following structure:

$$ {p_1}^{2b_1} {p_2}^{2b_2} \cdots {p_{\mathrm{n}}}^{2b_{\mathrm{n}}} = N \tag{112} $$

$$ \sigma({p_1}^{2b_1} {p_2}^{2b_2} \cdots {p_{\mathrm{n}}}^{2b_{\mathrm{n}}}) = 3N \tag{113} $$

Using the multiplicative property of the sum of divisor function we get:

$$ \sigma({p_1}^{2b_1})\sigma({p_2}^{2b_2}) \cdots \sigma({p_{\mathrm{n}}}^{2b_{\mathrm{n}}}) = 3N \tag{114} $$

Also let $\sigma({p_1}^{2b_1})\sigma({p_2}^{2b_2}) \cdots \sigma({p_{\mathrm{n}}}^{2b_{\mathrm{n}}}) = q$

Arrange the above displayed factors of q into a pair such that one member of the pair consists of at least one displayed factor of q and the other member of the pair consists of the rest of the displayed factors of q.

After doing that, you should get something like this:

$[\sigma(p_1{}^{2b_1})] \quad [\sigma(p_2{}^{2b_2}) \cdots \sigma(p_{\mathrm{n}}{}^{2b_{\mathrm{n}}})] = q$

Dividing both sides by 3 we get:

$$\frac{[\sigma(p_1{}^{2b_1})] \quad [\sigma(p_2{}^{2b_2}) \cdots \sigma(p_{\mathrm{n}}{}^{2b_{\mathrm{n}}})]}{3} = N \tag{115}$$

Since N is a perfect square, we can replace it with y^2 where y is a positive integer.

My main goal is to prove that the left side of the above equation (115) is not equal to the right side of that equation. I will do that by proving that the number on the left side of the equation is not a perfect square while the number on the right side is a perfect square which means that the number on the left side is not equal to the number on the right side of the equation.

To prove that the number on the left side of the equation is not a perfect square, I will show that there are only three cases that arise when we divide $[\sigma(p_1{}^{2b_1})] \quad [\sigma(p_2{}^{2b_2}) \cdots \sigma(p_{\mathrm{n}}{}^{2b_{\mathrm{n}}})]$ by 3. These three cases are exhaustive and they are listed below as follows:

i) Case 1(i): When $[\sigma(p_1{}^{2b_1})]$ is divisible by 3.

ii) Case 1(ii): When $[\sigma({p_2}^{2b_2})\cdots\sigma({p_\mathrm{n}}^{2b_\mathrm{n}})]$ is divisible by 3.

iii) Case 1(iii): When neither $[\sigma({p_1}^{2b_1})]$ nor $[\sigma({p_2}^{2b_2})\cdots\sigma({p_\mathrm{n}}^{2b_\mathrm{n}})]$ is divisible by 3.

Case 1: If $\sigma({p_1}^{2b_1})$ is divisible by 3 then we get:

$$\frac{\overset{h}{\cancel{[\sigma({p_1}^{2b_1})]}}\,[\sigma({p_2}^{2b_2})\cdots\sigma({p_\mathrm{n}}^{2b_\mathrm{n}})]}{\overset{1}{\cancel{3}}} = y^2 \tag{116}$$

$$h[\sigma({p_2}^{2b_2})\cdots\sigma({p_\mathrm{n}}^{2b_\mathrm{n}})] = y^2 \tag{117}$$

We know that $h[\sigma({p_2}^{2b_2})\cdots\sigma({p_\mathrm{n}}^{2b_\mathrm{n}})]$ is not a perfect square because $[\sigma({p_2}^{2b_2})\cdots\sigma({p_\mathrm{n}}^{2b_\mathrm{n}})]$ is not a perfect square. This is true even if h is a perfect square.

Therefore:

$$h[\sigma({p_2}^{2b_2})\cdots\sigma({p_n}^{2b_n})] \neq y^2 \tag{118}$$

Case 2: If $[\sigma({p_2}^{2b_2})\cdots\sigma({p_n}^{2b_n})]$ is divisible by 3 we get:

$$\frac{[\sigma({p_1}^{2b_1})]\quad\overbrace{[\sigma({p_2}^{2b_2})\cdots\sigma({p_n}^{2b_n})]}^{g}}{\cancelto{1}{3}} = y^2 \tag{119}$$

$$[\sigma({p_1}^{2b_1})]g = y^2 \tag{120}$$

We know that $[\sigma({p_1}^{2b_1})]g$ is not a perfect square because $[\sigma({p_1}^{2b_1})]$ is not a perfect square. This is true even if g is a perfect square.

Therefore:

$$[\sigma({p_1}^{2b_1})]g \neq y^2 \tag{121}$$

Case 3 : If q is not divisible by 3 we get:.

If no displayed factor of q is divisible by 3 then q is not divisible by 3. Therefore (q/3) is an irreducible fraction. The irreducible fraction (q/3) is not a perfect square because this irreducible fraction (q/3) is not an integer and all perfect squares are integers. Therefore q/3 is not a perfect square.
Therefore:

$$\frac{q}{3} \neq y^2 \tag{122}$$

Therefore it is clear that odd triperfect numbers do not exist because we have looked at all possible scenarios and found that they do not exist. Q.E.D

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