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# On Independence Nuetrosophic Random Variables

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**Abstract**. In this article we study independence neutrosophic random variables and conditioned expectation, we prove that conditional variance is equal to neutrosophic conditional variance. These notions are important for neutrosophic probability due to neutrosophic random variables, neutrosophic central limit theorem and neutrosophic laws of large numbers can be studied.

**Keywords:** Neutrosophic random vector; neutrosophic expectation; independence neutrosophic random variables; Neutrosophic Logic.

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## 1. **Introduction**

F. Smarandache introduced the notion of neutrosophic probability measure as a function $\mathcal{NP} : Y \to [0,1]^3$ where $Y$ is a neutrosophic sample space, and introduced the probability mapping to take the form $\mathcal{NP}(B) = (ch(B), ch(neutB), ch(antiB)) = (\alpha, \beta, \gamma)$ where $0 \leq \alpha, \beta, \gamma \leq 1$ and $0 \leq \alpha + \beta + \gamma \leq 3$ [33]. Moreover, many researchers have investigated many neutrosophic probability distributions like Poisson, exponential, binomial, normal, uniform, Weibull,...etc. (See [32], [2], [18], [26]). Furthermore, researchers have introduced the notion of neutrosophic queueing theory in [35], [36] this is one branch of neutrosophic stochastic modelling. Besides, researchers have also investigated neutrosophic time series prediction and modelling in many cases like neutrosophic moving averages, neutrosophic logarithmic models, neutrosophic linear models and so on. [3], [4], [12].

On the other hand, neutrosophic logic is an extension of intuitionistic fuzzy logic by adding indeterminacy component $(I)$ where $I^2 = I, ..., I^n = I, 0.I = 0; n \in \mathbb{N}$ and $I^{-1}$ is undefined (see [20], [32]). Neutrosophic logic has a huge brand of applications in many fields including decision making [29], [19], [25], machine learning [6], [27], intelligent disease diagnosis [30], [11], communication services [8], pattern recognition [28], social network analysis and e-learning systems [21], physics [34], sequences spaces [14] and so on. Neutrosophic logic has solved many decision-making problems efficiently like evaluating green credit rating, personnel selection, etc. [22], [23], [24], [1]. For more notions related to neutrosophic theory, we refer the reader to [9, 10, 14–17].

The study of neutrosophic random variables has become one of the fundamental pillars in neutrosophic theory and probability. Recent results of great importance can be seen in [37] and [13]. Taking into account mentioned above, in this article we study independent neutrosophic random variables and conditioned expectation.

## 2. Preliminaries

In this section, we show some well-known definitions and properties of neutrosophic logic and neutrosophic probability which are useful for the development of this paper.

**Definition 2.1.** (see [31]) Let $X$ be a non-empty fixed set. A neutrosophic set $A$ is an object having the form $\{x, (\mu A(x), \delta A(x), \gamma A(x)) : x \in X\}$, where $\mu A(x), \delta A(x)$ and $\gamma A(x)$ represent the degree of membership, the degree of indeterminacy , and the degree of non-membership respectively of each element $x \in X$to the set $A$.

**Definition 2.2.** (see [5]) Let $K$ be a field, the neutrosophic filed generated by $K$ and $I$ is denoted by $\langle K \cup I \rangle$ under the operations of $K$, where $I$ is the neutrosophic element with the property $I^2 = I$.

**Definition 2.3.** (see [32]) Classical neutrosophic number has the form $a + bI$ where $a, b$ are real or complex numbers and $I$ is the indeterminacy such that $0.I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers $n$.

**Definition 2.4.** (see [33]) The neutrosophic probability of event $A$ occurrence is $NP(A) = (ch(A), ch(neutA), ch(antiA)) = (T, I, F)$ where $T, I, F$ are standard or non-standard subsets of the non-standard unitary interval $]^{-}0, 1^{+}[$.

Recently, Bisher and Hatip [37] introduced and studied the notions of neutrosophic random variables by using the concepts presented by [33], these notions were defined as follows:

**Definition 2.5.** Consider the real valued crisp random variable $X$ which is defined as follows:

$$X : \Omega \to \mathbb{R}$$

where $\Omega$ is the events space. Now, they defined a neutrosophic random variable $X_N$ as follows:

$$X_N : \Omega \to \mathbb{R}(I)$$

and

$$X_N = X + I$$

where $I$ is indeterminacy.

**Theorem 2.6.** *Consider the neutrosophic random variable $X_N = X + I$ where cumulative distribution function of $X$ is $F_X(x) = P(X \leq x)$. Then, the following statements hold:*

(1) $F_{X_N}(x) = F_X(x - I)$,
(2) $f_{X_N}(x) = f_X(x - I)$.

*Where $F_{X_N}$ and $f_{X_N}$ are cumulative distribution function and probability density function of $X_N$, respectively.*

**Theorem 2.7.** *Consider the neutrosophic random variable $X_N = X + I$, expected value can be found as follows:*

$$E(X_N) = E(X) + I.$$

**Proposition 2.8** (Properties of expected value of a neutrosophic random variable)**.** *Let $X_N$ and $Y_N$ be neutrosophic random variables, then the following properties holds:*

(1) $E(aX_N + b + cI) = aE(X_N) + b + cI; a, b, c \in \mathbb{R}$,
(2) *If $X_N$ and $Y_N$ are neutrosophic random variables, then* $E(X_N \pm E_(Y_N) = E(X_N) \pm E(Y_N)$,
(3) $E[(a + bI)X_N] = aE(X_N) + bIE(X_N); a, b \in \mathbb{R}$,
(4) $|E(X_N)| \leq E|X_N|$.

**Theorem 2.9.** *Consider the neutrosophic random variable $X_N = X + I$, variance of $X_N$ is equal to variance of $X$, i.e. $V(X_N) = V(X)$.*

Now, Granados [13] studied the notions of neutrosophic random vector and joint neutrosophic random variable, these notions were defined as follows:

**Definition 2.10.** A neutrosophic random vector of two dimension is a vector $(X_N, Y_N)$ in which each coordinate is a neutrosophic random variable. Analogously, we can define a neutrosophic random vector multidimensional as follows $(X_{N_1}, X_{N_2}, ..., X_{N_n})$ in which $X_{N_1}, X_{N_2}, ..., X_{N_n}$ are neutrosophic random variables for each $n = 1, 2, ....$

**Definition 2.11.** Let $(X_N, Y_N)$ be a neutrosophic random vector, we define probability function of a neutrosophic continuous random vector $(X_N, Y_N)$. Then, joint probability neutrosophic function of a discrete random vector $(X_N, Y_N)$ $f_N(x, y) : \mathbb{R}^2 \to [0, \infty)$ in which is non-negative and integrable, and for any $(x, y) \in \mathbb{R}^2$ , it is defined as follows

$$P(X_N \leq x, Y_N \leq y) = P(X \leq x - I, Y \leq y - I) = \int_{-\infty}^{y-I} \int_{-\infty}^{x-I} f_{(X_N,Y_N)}(u, v)dvdu$$

Similarly, probability function of a neutrosophic discrete random vector $(X_N, Y_N)$ is defined similar by using sum.

**Definition 2.12.** Let $(X_N, Y_N)$ be a neutrosophic random vetor, we define neutrosophic joint distribution function which will be denoted by $F_{(X_N,Y_N)}(x,y) = P(X_N \leq x, Y_N \leq y) = P(X \leq x - I, Y \leq y - I)$.

**Definition 2.13.** Let $f_{(X_N,Y_N)}(x,y)$ be a joint probability neutrosophic function of a continuous random variable $(X_N, Y_N)$. We define neutrosophic marginal function of $X_N$ as follows:

$$f_{X_N}(x) = \int_{-\infty}^{+\infty} f_{(X_N,Y_N)}(x,y)dy$$

and we define neutrosophic marginal function of $Y_N$ as follows:

$$f_{Y_N}(y) = \int_{-\infty}^{+\infty} f_{(X_N,Y_N)}(x,y)dx$$

Similarly, joint probability neutrosophic function of a discrete random variable is defined similar by using sum.

**Definition 2.14.** Expected value of a neutrosophic random vector $(X_N, Y_N)$ in which expected value of $X_N$ and $Y_N$ exist, we define $E(X_N, Y_N) = (E(X_N), E(Y_N))$.

Next, we will show some new notions on neutrosophic random variables which have not bee studied so far and are needed.

Let $(X_N, Y_N)$ be neutrosophic vector and $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a function, then $\phi(X_N, Y_N)$ is a neutrosophic random variable and its expectation is defined as follows:

$$E[\phi(X_N, Y_N)] = \int_{-\infty}^{+\infty} (x - I)dF_{\phi(X_N,Y_N)}(x),$$

as well as one dimensional case, we are required to find out the distribution $\phi(X_N, Y_n)$ by which can be difficult in some cases. Next, we establish an alternative way in which we can calculate expectation of $\phi(X_N, Y_N)$ without known its distribution, but we must know $(X_N, Y_N)$ distribution. Let $(X_N, Y_N)$ be neutrosophic vector and $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $\phi(X_N, Y_N)$ has finite expected, then:

$$E[\phi(X_N, Y_N)] = \int_{\mathbb{R}^2} \phi(x - I, y - I)dF_{\phi(X_N,Y_N)}(x,y),$$

This can be proved easily due to this is a Riemmann-Stieltjes integral in two dimensional.

In case of $X_N$ and $Y_N$ be independence (see section 3), this increment is

$$F_X(x_i - I)F_Y(y_j - I) - F_X(x_i - I)F_Y(y_{j-1} - I) - F_X(x_{i-1} - I)F_Y(y_j - I)$$
$$+ F_X(x_i - I)F_Y(y_j - I) = \Delta F_X(x_i - I)\Delta F_Y(y_j - I),$$

i.e, bidimensional integral can be separated in two integral and can be written as follows

$$E[\phi(X_N, Y_N)] = \int_{\mathbb{R}^2} \phi(x-I, y-I) dF_{X_N}(x) dF_{Y_N}(y).$$

When $(X_N, Y_N)$ is a discrete vector, we have

$$E[\phi(X_N, Y_N)] = \sum_{x-I, y-I} \phi(x-I, y-I) P(X = x-I, Y = y-I),$$

in which the sum is applied over all possible valued $(x-I, y-I)$ on $(X_N, Y_N)$.

**Theorem 2.15.** *Let $X_N$ and $Y_N$ be two neutrosophic random variable with finite expectation, then*

$$E(X_N + Y_N) = E(X_N) + E(Y_N).$$

*Proof.* Let $\phi(x-I, y-I) = x-I+y-I$, $\phi_1(x-I, y-I) = x-I$ and $\phi_2(x-I, y-I) = y-I$. Then,

$$\begin{aligned} E(X_N + Y_N) &= E(\phi(X_N, Y_N)) \\ &= \int_{\mathbb{R}^2} (x-I+y-I) dF_{X_N,Y_N}(x, y) \\ &= \int_{\mathbb{R}^2} (x-I) dF_{X_N,Y_N}(x, y) + \int_{\mathbb{R}^2} (y-I) dF_{X_N,Y_N}(x, y) \\ &= E(\phi_1(X_N, Y_N)) + E(\phi_2(X_N, Y_N)) \\ &= E(X_N) + E(Y_N) \end{aligned}$$

□

**Theorem 2.16.** *Let $X_N$ and $Y_N$ be two independence neutrosophic random variable and $g$ and $h$ be two functions such that $g(X_N)$ and $h(Y_N)$ have finite expected, then*

$$E[g(X_N)h(Y_N)] = E[g(X_N)]E[h(Y_N)].$$

*Proof.*

$$\begin{aligned} E[g(X_N)h(Y_N)] &= \int_{\mathbb{R}^2} g(x-I)h(y-I) dF_{X_N,Y_N}(x, y) \\ &= \int_{\mathbb{R}^2} g(x-I)h(y-I) dF_{X_N}(x) dF_{Y_N}(y) \\ &= \int_{\mathbb{R}} g(x-I) dF_{X_N}(x) \int_{\mathbb{R}} g(y-I) dF_{Y_N}(y) \\ &= E[g(X_N)]E[h(Y_N)] \end{aligned}$$

□

## 3. Independence neutrosophic random variables

Let $X_N$ and $Y_N$ be two neutrosophic random variables. $X_N$ and $Y_N$ are independence if the events $(X \leq x - I)$ and $(Y \leq y - I)$ are independence for any real-value $x$ and $y$, i.e., if the following equality is satisfied

$$P[(X \leq x - I) \cap (Y \leq y - I)] = P(X \leq x - I)P(Y \leq y - I). \tag{1}$$

The left side of the equality (1) can be written as $P(X \leq x-I, Y \leq y-I)$ or $F_{X,Y}(x-I, y-I)$, and it is said to be the joint distribution function of $X_N$ and $Y_N$ evaluate in the point $(x, y)$. Therefore, note that (1) can be expressed as follows

$$F_{X,Y}(x - I, y - I) = F_X(x - I)F_Y(y - I),\ for\ x, y \in \mathbb{R}. \tag{2}$$

In this way, in order to determine whether two neutrosophic random variables are independent, it is necessary to know both the joint probabilities $P(X \leq x-I, Y \leq y-I)$ as individual probabilities $P(X \leq x-I)$ and $P(Y \leq y-I)$, and verify the identity (2) for each real numbers $x$ and $y$. Hence, it is enough that there exists a pair $(x - I, y - I)$ for which the equality (2) does not hold to be able to conclude that $X$ and $Y$ are not independent. Granados [13] studied on random vectors and explained how to obtain the individual distributions from the joint distribution of two random variables.

**Example 3.1.** Let $(X_N, Y_N)$ be a neutrosophic random vector with density function $f(x - I, y - I) = 4(x - I)(y - I)$ for $I \leq x, y \leq 1 - I$.

The marginal density function of $X_N$ is calculated as follows for $I \leq x \leq 1 - I$

$$f_X(x - I) = \int_I^{1-I} 4(x - I)(y - I)dy = 2(x - I).$$

Analogously, we can prove that $f_{Y_N}(y) = 2(y - I)$ for $I \leq x \leq 1 - I$. Therefore, $X_N$ and $Y_N$ are independence, due to $F_{X,Y}(x - I, y - I) = F_X(x - I)F_Y(y - I)$.

**Proposition 3.2.** *Let $X_N$ and $Y_N$ be two independence neutrosophic random variables , and let $g$ and $h$ be two functions of $\mathbb{R} \to \mathbb{R}$. Then, the neutrosophic random variables $g(X_N)$ and $h(Y_N)$ are independence neutrosophic random variables.*

**Proof**: Let $\mathcal{A} = (-\infty, x - I]$ and $\mathcal{B} = (-\infty, y - I]$, then

$$\begin{aligned} P(g(X_N) \leq x, h(Y_N) \leq y) &= P(g(X_N) \in \mathcal{A}, h(Y_N) \in \mathcal{B}) \\ &= P(X_N \in g^{-1}(\mathcal{A}), Y_N \in h^{-1}(\mathcal{B})) \\ &= P(X_N \in g^{-1}(\mathcal{A}))P(Y_N \in h^{-1}(\mathcal{B})) \\ &= P(g(X_N) \in \mathcal{A})P(h(Y_N) \in \mathcal{B}) \\ &= P(g(X_N) \leq x)P(h(Y_N) \leq y) \end{aligned}$$

**Theorem 3.3.** *Let $X_N$ and $Y_N$ two discrete neutrosophic random variables, then*

(1) *Using $P(X \leq x - I, Y \leq y - I) = \sum_{u \leq x-I} \sum_{v \leq y-I} P(X = u, Y = v)$, we have*

$$\begin{aligned} P(X = x - I, Y = y - I) = & P(X \leq x - I, Y \leq y - I) \\ & - P(X \leq x - I - 1, Y \leq y - I) \\ & - P(X \leq x - I, Y \leq y - I - 1) \\ & + P(X \leq x - I - 1, Y \leq y - I - 1). \end{aligned}$$

(2) *Using (1), independence condition $P(X \leq x - I, Y \leq y - I) = P(X \leq x - I)P(Y \leq y - I)$ is equivalent to $P(X = x - I, Y = y - I) = P(X = x - I)P(Y = y - I)$.*

**Proof**:

(1) This result can be obtained by using the following equality

$$\begin{aligned} (X = x - I, Y = y - I) = & (X \leq x - I, Y \leq y - I) - (X \leq x - I - 1, Y \leq y - I) \\ & - (X \leq x - I, Y \leq y - I - 1) \\ & + (X \leq x - I - 1, Y \leq y - I - 1). \end{aligned}$$

(2) Using (1) and by hypothesis of independence, we have

$$\begin{aligned} P(X = x - I, Y = y - I) = & P(X \leq x - I)P(X \leq y - I) - \\ & P(X \leq x - I - 1)P(Y \leq y - I) \\ & - P(X \leq x - I)P(Y \leq y - I) \\ & + P(X \leq x - I - 1)P(Y \leq y - I - 1) \\ & = [P(X \leq x - I) - P(X \leq x - I - 1)] \\ & \times [P(Y \leq y - I) - P(Y \leq y - I - 1)] \\ & = P(X = x - I)P(Y = y - I). \end{aligned}$$

**Theorem 3.4.** *Let $X_N$ and $Y_N$ be two independence neutrosophic random variables which have finite expectation. Then,*

$$E(X_N Y_N) = E(X_N)E(Y_N). \tag{3}$$

**Proof**: We prove the case when $X_N$ and $Y_N$ are discrete neutrosophic random variables, the case for continuous neutrosophic random variables are proved similarly.

Let

$$\begin{aligned} E(X_N Y_N) &= \sum_{x,y}(x-I)(y-I)P(X_N = x, Y_N = y) \\ &= \sum_{x}\sum_{y}(x-I)(y-I)P(X_N = x, Y_N = y) \\ &= (\sum_{x}(x-I)P(X_N = x))(\sum_{y}(y-I)P(Y_N = y)) \\ &= E(X_N)E(Y_N). \end{aligned}$$

**Definition 3.5.** Let $X_{N_1}, X_{N_2}, ..., X_{N_n}$ be a collection of neutrosophic random variables with joint function distribution $F_X(x_1-I, x_2-I, ..., x_n-I)$, and consider marginals functions distribution $F_{X_{N_1}}(x_1), F_{X_{N_2}}(x_2), ..., F_{X_{N_n}}(x_n)$, respectively. Then, we say that $X_{N_1}, X_{N_2}, ..., X_{N_n}$ are independence if for any real numbers $x_1 - I, x_2 - I, ..., x_n - I$ the following equality holds

$$F_X(x_1 - I, x_2 - I, ..., x_n - I) = F_{X_{N_1}}(x_1)F_{X_{N_2}}(x_2)...F_{X_{N_n}}(x_n).$$

Analogously, we can define it in terms of neutrosophic density function $f(x_1-I, x_2-I, ..., x_n-I)$ if the following equality holds

$$f(x_1 - I, x_2 - I, ..., x_n - I) = f_{X_{N_1}}(x_1)f_{X_{N_2}}(x_2)...f_{X_{N_n}}(x_n).$$

**Example 3.6.** Let $X_N$ and $Y_N$ be two continuous neutrosophic random variables with joint density function

$$f_{(X,Y)}(x-I, y-I) = \begin{cases} e^{-x-y+2I} & if\ x, y > I, \\ \\ 0 & otherwise. \end{cases}$$

Nuetrosophic marginals probability functions are defined as follows

$$f_X(x-I) = \begin{cases} e^{-x+I} & if\ x > I, \\ \\ 0 & otherwise. \end{cases}$$

and

$$f_Y(y-I) = \begin{cases} e^{-y+I} & if\ y > I, \\ \\ 0 & otherwise. \end{cases}$$

Therefore, $f_{(X,Y)}(x-I, y-I) = f_X(x-I)f_Y(y-I)$ for any real numbers $x - I$ and $y - I$, and hence we conclude $X_N$ and $Y_N$ are independence.

**Example 3.7.** Let $X_N$ and $Y_N$ be two discrete neutrosophic random variables with joint density function

$$f_{(X,Y)}(x-I,y-I)=\begin{cases}\frac{1}{4} & if\ x,y\in\{I,1+I\},\\ \\ 0 & otherwise.\end{cases}$$

Nuetrosophic marginals probability functions are defined as follows

$$f_X(x-I)=\begin{cases}\frac{1}{2} & if\ x\in\{I,1+I\},\\ \\ 0 & otherwise.\end{cases}$$

and

$$f_Y(y-I)=\begin{cases}\frac{1}{2} & if\ y\in\{I,1+I\},\\ \\ 0 & otherwise.\end{cases}$$

Therefore, $f_{(X,Y)}(x-I,y-I)=f_X(x-I)f_Y(y-I)$ for any real numbers $x-I$ and $y-I$, and hence we conclude $X_N$ and $Y_N$ are independence.

**Remark 3.8.** It can be said that an infinite set of neutrosophic random variables is independence if any finite subset is independence.

This statement can be useful due to for future work the concepts of neutrosophic central limit theorem and neutrosophic laws of large numbers can be studied.

## 4. **Conditional expectation**

In this section, we introduce the concept of conditional expectation of a neutrosophic random variable with respect to a $\sigma$-algebra, and some of its elemental properties are studied. We will consider that has a base probability space $(\Omega,\mathcal{F},P)$, and $\mathcal{G}$ is a sub-algebra of $\mathcal{F}$. We have defined expectation of a neutrosophic random variable as a Riemann-Stieltjes integral as follows

$$E(X_N)=\int_{-\infty}^{+\infty}(x-I)dF_{X_N}(x).$$

however, to make the notation simpler in this section, it is sometimes convenient to adopt the notation of measure theory and denote the expectation of a netrusophic random variable $X_N$ as follows

$$E(X_N)=\int_\Omega X_N dP.$$

We shall recall that if we know distribution of neutrosophic vector $(X_N,Y_N)$ and we take valued $y-I$ such that $f_Y(y-I)=f_{Y_N}(y)\neq 0$, conditional expectation of $X_N$ known $Y=y-I$

is the function

$$y - I \mapsto E(X_N|Y_N = y) = \int_{-\infty}^{+\infty} (x - I) dF_{X_N|Y_N}(x|y), \tag{4}$$

when $f_{Y_N}(y) \neq 0$. (4) is equivalent to write

$$E(X_N|Y_N = y) = \int_{-\infty}^{+\infty} (x - I) f_{X_N|Y_N}(x|y) dx.$$

If we make a change in order of integration, we can see that

$$E(X_N) = \int_{-\infty}^{+\infty} E(X_N|Y_N = y) f_{Y_N}(y) dy,$$

if we apply this expression by using total probability theorem in terms of expectation. In the case when $(X_N, Y_N)$ is a discrete neutrosophic vector, we have

$$\begin{aligned} E(X_N|Y_N = y) &= \sum_x (x - I) f_{X_N|Y_N}(x|y) \\ &= \sum_x (x - I) P(X_N = x|Y_N = y), \end{aligned}$$

considering $f_{Y_N}(y) \neq 0$ and sum is absolutely convergent. Again, applying a change in order of sum, we have

$$E(X_N) = \sum_y E(X_N|Y_N = y) P(Y_N = y).$$

In any cases, we can also see, when $Y_N$ and $X_N$ are independence, we have

$$E(X_N|Y_N = y) = E(X_N).$$

**Example 4.1.** We will find expectation of $E(X_N|Y_N = y)$ for each $y \in (I, 1 + I)$ when $X_N$ and $Y_N$ have the following joint neutrosophic density function.

$$f_{X|Y}(x - I, y - I) = \begin{cases} 12(x - I)^2 & if\ I < x < y < 1 + I, \\ \\ 0 & otherwise. \end{cases}$$

For $I < y < 1 + I$,

$$
\begin{aligned}
E(X_N|Y_N=y) &= \int_0^{y-I}(x-I)\frac{12(x-I)^2}{4(y-I)^3}dx \\
&= \frac{3}{(y-I)^3}\int_0^{y-I}(x-I)^3dx \\
&= \frac{3}{4}\frac{(y-2I)^4-I}{(y-I)^3}
\end{aligned}
$$

Analogously to (4), if $Q$ is an event with positive probability and $X_N$ is a integrable neutrosophic random variable, conditional expectation of $X_N$ known $Q$ is

$$E(X_N|Q) = \int_{-\infty}^{+\infty}(x-I)dF_{X_N|Q}(x),$$

where $F_{X_N|Q}(x) = P(X_N \leq x|A) = P(X \leq x-I|A) = \dfrac{P(X \leq x-I, Q)}{P(Q)}$. Next, we will show a more generally definition which generalized concepts showed so far.

**Definition 4.2.** Let $X_N$ be a neutrosophic random variable with finite expectation, and let $\mathcal{G}$ be a sub-algebra of $\mathcal{F}$. Conditional expectation of $X_N$ known $\mathcal{G}$, it is a neutrosophic random variable which will be denoted by $E(X_N|\mathcal{G})$ which satisfies the following conditions:

(1) It is $\mathcal{G}$-measurable,

(2) It has finite expectation,

(3) For any event $G \in \mathcal{G}$, $\int_G E(X_N|\mathcal{G})dP = \int_G X_N dP$.

**Remark 4.3.** Just as it happens in a random variable, part of the difficulty in understanding this general definition is that an explicit formula is not provided for this neutrosophic random variable but only the properties it satisfies. The objective of this section is to find the meaning of this neutrosophic random variable, interpret its meaning, and explain its relationship with the concept of elementary conditional expectation.

**Remark 4.4.** When $\mathcal{G} = \sigma(Y_N)$ for any neutrosophic random variable $Y_N$, conditional expected will be written by $E(X_N|Y_N)$ instead of $E(X_N|\sigma(Y_N))$.

**Remark 4.5.** Let $Q$ be any event, then conditional expected $E(1_Q|\mathcal{G})$ will be denoted by $P(Q|\mathcal{G})$.

Now, we will show some properties on $E(X_N|Y_N)$ when $Y_N$ is a discrete neutrosophic random variable.

Let $X_N$ and $Y_N$ be two neutrosophic random variables. Now, consider that $X_N$ has finite expected and $Y_N$ is discrete with possible values $y_1 - I, y_2 - I, \ldots$ Conditional expectation of

$X_N$ known event $(Y_N = y_j) = (Y = y_j - I)$ is $E(X_N|Y_N = y_j)$, this value depends of the event $(Y_N = y_j)$, and we can consider that we have a function defined over $\Omega$ as follows: If $\omega$ is such that $Y(\omega) = y_j - I$, then

$$\omega \mapsto E(X_N|Y_N)(\omega) = E(X_N|Y_N = y_j).$$

We can see that $E(X_N|Y_N)$ takes at much different values as $Y_N$ does. Generally, function can be rewritten in terms of indicator function as follows

$$E(X_N|Y_N)(\omega) = \sum_{j=1}^{\infty} E(X_N|Y_N = y_j)1_{(Y_N=y_j)}(\omega).$$

In this way, we can define the function $E(X_N|Y_N) : \Omega \to \mathbb{R}$ which is denoted by

$$\omega \mapsto E(X_N|Y_N)(\omega) = E(X_N|Y_N = y_j) \ if \ Y(\omega) = y_j - I$$

is a neutrosophic random variable.

**Theorem 4.6.** *Let $X_N$ be a integrable neutrosophic random variable, and let $Y_N$ discrete with possible values $y_1 - I, y_2 - I, ...$ $E(X_N|Y_N) : \Omega \to \mathbb{R}$ is a neutrosophic random variable which satisfies the following conditions:*

(1) *It is $\sigma(Y_N)$-measurable,*
(2) *It has finite expected,*
(3) *For any event $G \in \sigma(Y_N)$, $\int_G E(X_N|Y_N)dP = \int_G X_N dP$.*

**Proof**:

(1) Through possibles values, the neutrosophic random variables $Y_N$ divides in different part $\Omega$, i.e $(Y_N = y_1), (Y_N = y_2), ...$ are disjunct events. $\sigma(Y_N) = \sigma\{(Y_N = y_1), (Y_N = y_2), ...\} \subset \mathcal{F}$. Since $E(X_N|Y_N)$ is constant in each element of partition, implies that $E(X_N|Y_N)$ is $\sigma(Y_N)$-measurable, and hence it is a neutrosophic random variable.
(2) Taking the event $G$ as $\Omega$ in the third property, we get that $X_N$ and $E(X_N|Y_N)$ have the same finite expectation.
(3) Since each element of $\sigma(Y_N)$ is union of disjunct elements of $(Y_N = y_j)$, by properties of integral it is enough to show that $\int_G E(X_N|Y_N)dP = \int_G X_N dP$ for these simple

events. Then, we have

$$\begin{aligned}\int_{(Y_N=y_j)} E(X_N|Y_N)(\omega)dP(\omega) &= E(X_N|Y_N=y_j)P(Y_N=y_j)\\ &= \int_\Omega X_N(\omega)dP(\omega|Y_N=y_j)P(Y_N=y_j)\\ &= \int_\Omega X_N(\omega)dP(\omega, Y_N=y_j)\\ &= \int_{(Y_N=y_j)} X_N(\omega)dP(\omega).\end{aligned}$$

**Remark 4.7.** We have to see the different between $E(X_N|Y_N=y_j)$ and $E(X_N|Y_N)$. First term is a possible numerical value, second term is a neutrosophic random variable. However, both expressions are called conditional expectation. We will see next a particular case of this neutrosophic random variable. Besides, we will show that the conditional expectation can be seen as a generalization of the basic concept of conditional probability, and it can also be considered as a generalization of the concept of expectation.

**Proposition 4.8.** *Let $X_N$ be a neutrosophic random variable with finite expectation, then $E(X_N|\{\emptyset,\Omega\}) = E(X_N) = E(X)+I$.*

**Proof**: This proofs follows since $E(X_N|\mathcal{G})$ is measurable respected to $\mathcal{G}$, and any measurable function respected to $\{\emptyset,\Omega\}$ is constant. Now, for any event $G\in\{\emptyset,\Omega\}$, $\int_G E(X_N|\{\emptyset,\Omega\})dP = \int_G X_N dP = \int_\Omega X_N dP = E(X_N) = E(X)+I$.

**Theorem 4.9.** *Let $X_N$ and $Y_N$ be two neutrosophic random variables with finite expectation and $b\in\mathbb{R}$. Then, the following statements hold:*

(1) *If $X\geq I$, then $E(X_N|\mathcal{G})\geq 0$,*
(2) *$E(bX_N+Y_N|\mathcal{G}) = bE(X_N|\mathcal{G})+E(Y_N|\mathcal{G})$,*
(3) *If $X_N\leq Y_N$, then $E(X_N|\mathcal{G})\leq E(Y_N|\mathcal{G})$,*
(4) *$E(E(X_N|\mathcal{G})) = E(X_N)$,*
(5) *If $X_N$ is $\mathcal{G}$-measurable, then $E(X_N|\mathcal{G}) = X_N$ a.s. In particular, $E(b|\mathcal{G}) = b$,*
(6) *If $\mathcal{G}_1\subset\mathcal{G}_2$, then*

$$E(E(X_N|\mathcal{G}_1)|\mathcal{G}_2) = E(E(X_N|\mathcal{G}_2)|\mathcal{G}_1) = E(X_N|\mathcal{G}_1),$$

(7) *$|E(X_N|\mathcal{G})|\leq E(|X_N||\mathcal{G})$,*
(8) *$E|E(X_N|\mathcal{G})|\leq E(|X_N|)$.*

**Proof**:

(1) Follows directly from definition and the fact that $X\geq I\equiv X_N\geq 0$.

(2) For all $G \in \mathcal{G}$, we have

$$\int_G E(bX_N + Y_N|\mathcal{G})dP = \int_G (bX_N + Y_N)dP$$
$$= b\int_G X_N dP + \int_G Y_N dP.$$

Now,

$$\int_G [bE(X_N|\mathcal{G}) + E(Y_N|\mathcal{G})]dP = b\int_G E(X_N|\mathcal{G})dP + \int_G E(Y_N|\mathcal{G})dP$$
$$= b\int_G X_N dP + \int_G Y_N dP.$$

(3) Follows directly from definition and the fact that $X_N \le Y_N$.

(4) Taking $G = \Omega$, we get the equality.

(5) Since $X_N$ is $\mathcal{G}$-measurable, three condition of the definition hold. Now, $\int_G E(X_N)dP = \int_G E(X_N)dP$. Therefore, $X_N = E(X_N|\mathcal{G})$ a.s.

(6) For all $G \in \mathcal{G}_1 \subset \mathcal{G}_2$, we have

$$\int_G E(E(X_N|\mathcal{G}_1)|\mathcal{G}_2)dP = \int_G E(X_N|\mathcal{G}_1)dP$$
$$= \int_G X_N dP.$$

Analogously,

$$\int_G E(E(X_N|\mathcal{G}_2)|\mathcal{G}_1)dP = \int_G E(X_N|\mathcal{G}_2)dP$$
$$= \int_G X_N dP.$$

(7) Consider that $\int_G |E(X_N|\mathcal{G})|dP = |\int_G E(X_N|\mathcal{G})dP|$. Then,

$$|\int_G E(X_N|\mathcal{G})dP| = |\int_G X_N dP|$$
$$\le \int_G |X_N|dP$$
$$= \int_G E(|X_N||\mathcal{G})dP.$$

(8) Proof follows from parts (4) and (7) of this theorem.

**Definition 4.10.** Let $X_N$ be a neutrosophic random variable with finite second moment, and $\mathcal{G}$ be a sub-algebra of $\mathcal{F}$. Conditional variance of $X_N$ known $\mathcal{G}$ which will be denoted by $Var(X_N|\mathcal{G})$, it is defined as a neutrosophic random variable as follows

$$Var(X_N|\mathcal{G}) = E[(X_N - E(X_N|\mathcal{G}))^2|\mathcal{G}].$$

We shall recall that neutrosophic conditional variance is not a number, it is a neutrosophic random variable. Therefore, the only way that we have the neutrosophic variance of a random variable from conditional variance is $Var(X_N) = Var(X) = Var(X_N|\{\emptyset, \Omega\})$.

On the other hand, we have that

$$\begin{aligned} Var(X_N|\mathcal{G}) &= Var(X + I|\mathcal{G}) \\ &= Var(X|\mathcal{G}) \\ &= E[(X - E(X|\mathcal{G}))^2|\mathcal{G}], \end{aligned}$$

this shows that conditional covariance is equal to neutrosophic conditional covariance.

## 5. Conclusion

In this article we study the notion of neutrosophic random variable taking into account the notions previously studied by [37] and [13]. These results are of great importance because convergence on neutrosophic random variables, neutrosophic central limit theorem and neutrosophic laws of large numbers can be studied. Secondly, this results can be applied in quality control, stochastic modeling, reliability theory, queueing theory, electrical engineering and so on.

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## Conflicts of Interest

The author declares no conflict of interest.

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