



A Nonlinear Programming Model to Solve Matrix Games with Pay-offs of Single-valued Neutrosophic Numbers

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Abstract. Single-valued neutrosophic number (SVNN) is an appropriate extension of the ordinary fuzzy number. The key feature of the SVNN is that it can capture indeterminacy in the imprecise information. In real-life problems, there are many situations where players of a matrix game can not assess their payoffs in terms of ordinary fuzzy or intuitionistic fuzzy numbers. The SVNN is used as an excellent tool to handle such situations. This paper explores matrix games with SVNN payoffs and investigates a non-linear programming approach to solve such a game. First, two auxiliary non-linear multi-objective programming problems have been formulated. Then, each of the two multi-objective programming problems is converted into two non-linear bi-objective programming problems. Finally, the lexicographic method is used to solve the reduced bi-objective programming problems. It is worth mentioning that the values of the game for both the players are obtained in SVNN forms, which is desirable. The applicability of the proposed approach is illustrated with a market share problem and results are compared and analyzed with an existing method.

Keywords: Matrix game; single-valued neutrosophic number; multi-objective optimization; lexicographic method.

1. Introduction

Matrix game theory [17] gives a mathematical framework to conceive strategies that help to overcome real-life conflicting situations. There are several types of mathematical games [22, 24, 34], which have been broadly studied and successfully utilized in many areas. Many of the real-life situations are uncertain due to the imprecision of data, asymmetric information, and conflict of interest between opponents in the same field of business. So, it is difficult to evaluate payoffs precisely. The players only approximate the payoffs with some imprecise

degrees. The most vital and argued issue among the researchers is to settle how to handle the uncertainty. Crisp data can not express most of these complicated structures correctly.

Fuzzy set (FS) was the first to successfully encounter the uncertainty which is not due to the randomness of an event. FS represents each element with a degree of membership (DOM) which lies between 0 and 1. In the literature, matrix games with fuzzy pay-offs have been broadly studied and analyzed by numerous researchers. Campos [6] used a linear programming approach to solve fuzzy matrix games. Bector et al. [3] used fuzzy linear programming duality to solve matrix games with fuzzy goals and fuzzy pay-offs. Li [15] evolved several methods to solve matrix games with payoffs as triangular fuzzy numbers. Verma and Kumar [30] proposed the Mehar method for fuzzy matrix games. Seikh et al. [19] implemented an α -cut based approach to solve fuzzy matrix games. Very recently, Seikh et al. [21] developed a methodology to solve matrix games with hesitant fuzzy payoffs. Some recent references on fuzzy matrix games are [12, 23, 25, 36].

FS considers only the DOM of the elements in a universe, which is a single value, and cannot provide any additional information regarding the incomplete concept of the elements. Atanassov [2] generalized the idea of FS to intuitionistic fuzzy set (IFS) where the degree of non-membership (DONM) $\nu_X(x) \in [0, 1]$ is also attached with DOM $\mu_X(x) \in [0, 1]$ for each element $x \in X$ in a universe where $0 \leq \mu_X(x) + \nu_X(x) \leq 1$. IFS outlines uncertainty more correctly and descriptively than FS, as IFS considers both complete and incomplete imprecise data. Nan et al. [16] proposed solution methodologies to study matrix games with payoffs of triangular intuitionistic fuzzy numbers (TIFNs) [18]. Seikh et al. [20] accomplished an approach to solve matrix games with intuitionistic fuzzy payoffs. Xing and Qiu [35] used the accuracy function method to solve a matrix game where the payoffs are considered as TIFN.

However, in reality, the available information always contains some imprecise data which consists of conflicting, unpredictable, and indeterminate information. The FS can not express the DONM and the IFS does not control the indeterminacy of information. Neutrosophic sets (NSs) [27] considers the degree of indeterminacy (DOI) $\omega_X(x) \in [0, 1]$ together with the DOM and DONM. Therefore, NS can capture more realistic data than that of FS and IFS. NSs are represented by DOM (μ), DOI (ω), and DONM (ν) which are independent and $0 \leq \mu, \omega, \nu \leq 1$ provided $0 \leq \mu + \omega + \nu \leq 3$. NS generalizes the classical set when $\omega = 0$, μ, ν either 0 or 1 and $\mu + \omega + \nu = 1$; the FS when $\omega = 0$, $0 \leq \mu, \omega, \nu \leq 1$ and $\mu + \omega + \nu = 1$; the IFS when $0 \leq \mu, \omega, \nu \leq 1$ and $0 < \mu + \omega + \nu < 1$. This concludes the fact that NS is a generalization of a classical set, FS and IFS.

Wang et al. [31] conceptualized a single-valued neutrosophic set (SVNS), a special form of NS for realistic applications. SVNNs are a distinctive case of SVNSs. SVNN is an extension of a fuzzy number (FN) and an intuitionistic fuzzy number (IFN). SVNS outlined the variables

which are completely relevant for human prediction because of the imperfection of information that human observes from the outer world. For example, the proposition “The newly launched car Ψ would be the best seller”, has not any precise answer for the human brain as far as yes or no, since indeterminacy is the segment of ignorance of the value of a proposition between truth and lie. For that reason, the three components of the neutrosophic set are very much suitable to exhibit the indeterminacy and inconsistency in the information.

To express the uncertainty of a conflicting circumstance, SVNNs are better to utilize instead of FS/IFS. The following example shows the implication and the relevance of the use of SVNN in real-life problems.

Suppose a smartphone manufacturing company ‘*Alpha*’ is going to launch a new item ‘ Ω ’ in the new year 2021. The company *Alpha* wants to estimate the number of selling a unit of new products before starting its production. The selling of the new item depends on various uncertain parameters such as new attracting features, the capacity of supply, the selling price, advertisement of the product, etc. But, the company wants to know whether the guaranteed selling unit would be 1 billion in the year or less’. The existence of this uncertain guaranteed selling unit always contains some knowledge of ‘neutral’ (indeterminate/unknown) thought besides ‘truth/membership’ and ‘falsehood/non-membership’ components that lie in FS/IFS. This situation can not be revealed by FNs or IFNs and SVNN comes into consideration. Some experts (say α , β , γ) are consulted for their opinions and they express their views in SVNNs. Let the expert α give its opinion about the guaranteed selling unit as $\langle 0.8, 0.3, 0.4 \rangle$. This implies that the company has 80% chance to meet the goal positively and is unable to reach the guaranteed selling unit by 40%. In this case, the expert α has a neutral thought that *Alpha* has an indeterminacy to meet the goal by 30%.

Hence, the application of SVNS theory has been growing quickly in many research areas [4, 29, 32, 37, 38]. Selvachandran et al. [26] designed a modified TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) with maximizing deviation method based on the SVNS model. Garg and Nancy [9] discussed some new distance measures under the SVNS environment and developed an algorithm for single-valued neutrosophic decision making based on TOPSIS method. Garai et al. [8] investigated the weighted possibility mean for SVNNs and developed a rigorous ranking methodology to solve multi-attribute decision making. Haque et al. [11] presented a new exponential operational law for trapezoidal neutrosophic numbers and studied pollution-related MCGDM problems in megacities. Chakraborty et al. [7] explored the classification of trapezoidal bipolar neutrosophic numbers and implemented the de-bipolarization in cloud service-based MCGDM problem. Ahmed et al. [1] developed a new approach to solve linear programming problems in bipolar single-valued neutrosophic environments.

2. Motivation

With the use of SVNNS, the decision-maker can set greater flexibility and better dependability in the strategy-making process. Many uncertain situations are better to present by SVNNS than FS/IFS. Moreover, in matrix games due to lack of information in the available data, the DOI plays a vital role while assessing the payoff values. So, there are many uncertain situations where players can assess payoffs of the matrix game problems in SVNNS forms. This useful requirement motivates us to investigate the matrix game with SVNNS payoffs.

Here we develop a new methodology to solve matrix game problems with SVNNS payoffs. First, two multi-objective non-linear programming problems are constructed to get the optimal value and strategies for the players. Then multi-objective programming problems are transferred into bi-objective non-linear programming problems by considering the same importance of the objective functions. Then the Lexicographic method [14] is used to solve the reduced bi-objective programming problems. A market share problem is illustrated to show the validity of the proposed methodology. The obtained results are discussed and compared and the physical significance of the obtained results is explored. The key contributions of this paper are augmented as under.

- (1) This is the first attempt to solve matrix game with SVNNS payoffs.
- (2) To get the optimal strategies and optimal values of the players, two different non-linear multi-objective programming problems are constructed. These multi-objective programming problems are transferred to bi-objective problems by considering the same importance of the two objective functions. Then Lexicographic method is used to solve the bi-objective problems.
- (3) Owen [17] proposed the concept of optimal strategies of the players and value of the game for the crisp matrix game problem. Here, we extend the definition for SVNNS matrix game.
- (4) The optimal values of the game for both the players are obtained in the SVNNS forms, which is desirable.
- (5) A real-life market share problem is illustrated to check the applicability and validity of the proposed approach.

The remainder of this paper is sorted out as follows. Some preliminaries on NS are recalled in Section 3. Section 4 conceptualized the idea of SVNNS matrix games. Section 5 is dedicated to the solution process. In Section 6, a market share problem is illustrated to validate our proposed methodology. Section 7 concludes the paper.

3. Preliminaries

Neutrosophic sets and their basic operations are recalled in this section.

Definition 3.1. [28] $\tilde{M} = \{ \langle \xi, (\mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi)) \rangle \mid \xi \in \Omega, \mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi) \in]^{-0}, 1^+[\}$ is said to be a neutrosophic set (NS) over a universe Ω , where $\mu_{\tilde{M}} : \Omega \rightarrow]^{-0}, 1^+[$; $\omega_{\tilde{M}} : \Omega \rightarrow]^{-0}, 1^+[$ and $\nu_{\tilde{M}} : \Omega \rightarrow]^{-0}, 1^+[$ are called respectively the membership function, indeterminacy membership function and non-membership function and $^{-0} \leq \mu_{\tilde{M}}(\xi) + \omega_{\tilde{M}}(\xi) + \nu_{\tilde{M}}(\xi) \leq 3^+$.

Definition 3.2. [31] $\tilde{M} = \{ \langle \xi, (\mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi)) \rangle \mid \xi \in \Omega, \mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi) \in [0, 1] \}$ is said to be a single-valued neutrosophic set (SVNS) over a universe Ω , where $\mu_{\tilde{M}} : \Omega \rightarrow [0, 1]$, $\omega_{\tilde{M}} : \Omega \rightarrow [0, 1]$ and $\nu_{\tilde{M}} : \Omega \rightarrow [0, 1]$ are called respectively the membership function, indeterminacy membership function and non-membership function and $0 \leq \mu_{\tilde{M}}(\xi) + \omega_{\tilde{M}}(\xi) + \nu_{\tilde{M}}(\xi) \leq 3$.

For convenience, $\langle \mu_{\tilde{M}}(\xi), \omega_{\tilde{M}}(\xi), \nu_{\tilde{M}}(\xi) \rangle$ is called a single-valued neutrosophic number (SVNN) which is usually represented by $\tilde{\alpha} = \langle \mu, \omega, \nu \rangle$.

Definition 3.3. [31] Let \tilde{M} and \tilde{M}' be two SVNSs in the set Ω . $\tilde{M} \subseteq \tilde{M}'$ if and only if $\mu_{\tilde{M}}(\xi) \leq \mu_{\tilde{M}'}(\xi)$, $\omega_{\tilde{M}}(\xi) \geq \omega_{\tilde{M}'}(\xi)$ and $\nu_{\tilde{M}}(\xi) \geq \nu_{\tilde{M}'}(\xi)$ for any $\xi \in \Omega$. Again $\tilde{M} \cong \tilde{M}'$ if and only if $\mu_{\tilde{M}}(\xi) = \mu_{\tilde{M}'}(\xi)$, $\omega_{\tilde{M}}(\xi) = \omega_{\tilde{M}'}(\xi)$ and $\nu_{\tilde{M}}(\xi) = \nu_{\tilde{M}'}(\xi)$ for any $\xi \in \Omega$.

Definition 3.4. [31] Let \tilde{M} and \tilde{M}' be two SVNSs in the set Ω . The intersection of \tilde{M} and \tilde{M}' is a SVNS \tilde{C} , written as $\tilde{C} = \tilde{M} \cap \tilde{M}'$, whose membership function, indeterminacy membership function and non-membership functions are related to those of \tilde{M} and \tilde{M}' by $\mu_{\tilde{C}}(\xi) = \min\{\mu_{\tilde{M}}(\xi), \mu_{\tilde{M}'}(\xi)\}$, $\omega_{\tilde{C}}(\xi) = \max\{\omega_{\tilde{M}}(\xi), \omega_{\tilde{M}'}(\xi)\}$ and $\nu_{\tilde{C}}(\xi) = \max\{\nu_{\tilde{M}}(\xi), \nu_{\tilde{M}'}(\xi)\}$ for any $\xi \in \Omega$.

Definition 3.5. [5] Let $\tilde{\alpha} = \langle \mu, \omega, \nu \rangle$, $\tilde{\alpha}_1 = \langle \mu_1, \omega_1, \nu_1 \rangle$ and $\tilde{\alpha}_2 = \langle \mu_2, \omega_2, \nu_2 \rangle$ be three SVNNs, and $\lambda > 0$, then their algebraic operations are defined as follows:

1. $\tilde{\alpha}_1 + \tilde{\alpha}_2 = \langle \mu_1 + \mu_2 - \mu_1\mu_2, \omega_1\omega_2, \nu_1\nu_2 \rangle$;
2. $\tilde{\alpha}_1 - \tilde{\alpha}_2 = \langle 1 - (1 - \mu_1)(1 - \mu_2)^{-1}, \omega_1\omega_2^{-1}, \nu_1\nu_2^{-1} \rangle$;
3. $\tilde{\alpha}_1 \times \tilde{\alpha}_2 = \langle \mu_1\mu_2, \omega_1 + \omega_2 - \omega_1\omega_2, \nu_1 + \nu_2 - \nu_1\nu_2 \rangle$;
4. $\lambda\tilde{\alpha} = \langle 1 - (1 - \mu)^\lambda, \omega^\lambda, \nu^\lambda \rangle$;
5. $\tilde{\alpha}^\lambda = \langle \mu^\lambda, 1 - (1 - \omega)^\lambda, 1 - (1 - \nu)^\lambda \rangle$.

4. Matrix Games with Pay-offs represented by SVNN

Consider \mathfrak{R}_+^n as the non-negative orthant of n -dimensional Euclidean space. Let the pure strategies ϵ_h and ζ_k are chosen by Player-I and II with probabilities u_h and v_k respectively, for $h \in \Delta_1$ and $k \in \Delta_2$ where $\Delta_1 = \{1, 2, \dots, p\}$ and $\Delta_2 = \{1, 2, \dots, q\}$. If $\sum_{h=1}^p u_h = 1$ and

$\sum_{k=1}^q v_k = 1$ for $(\mathbf{u}, \mathbf{v}) \in \mathfrak{R}_+^p \times \mathfrak{R}_+^q$ where $\mathbf{u} = (u_1, u_2, \dots, u_p)$ and $\mathbf{v} = (v_1, v_2, \dots, v_q)$, then \mathbf{u}

and \mathbf{v} are called the mixed strategies for Player-I and II, respectively. Let the sets of all mixed strategies for Player-I and Player-II are denoted by U and V respectively, where

$$U = \left\{ \mathbf{u} = (u_1, u_2, \dots, u_p) \in \mathfrak{R}_+^p : \sum_{h=1}^p u_h = 1 \right\},$$

$$V = \left\{ \mathbf{v} = (v_1, v_2, \dots, v_q) \in \mathfrak{R}_+^q : \sum_{k=1}^q v_k = 1 \right\}.$$

Let us consider the matrix $\tilde{N} = (\tilde{n}_{hk})_{p \times q}$, where $\tilde{n}_{hk} = \langle \mu_{hk}, \omega_{hk}, \nu_{hk} \rangle$ is a SVNN, which represents the payoff for the Player-I. Then, the matrix game with SVNN payoffs is represented by $\{U, V, \tilde{N}\}$. From this, the two person matrix game $\{U, V, \tilde{N}\}$ with payoffs of SVNNs is supposed to call as a SVNN matrix game \tilde{N} .

For the choice of mixed strategy $(\mathbf{u}, \mathbf{v}) \in U \times V$ by Player-I and II, the expected payoff $\tilde{E}(\mathbf{u}, \mathbf{v})$ for Player-I will be calculated as

$$\begin{aligned} \tilde{E}(\mathbf{u}, \mathbf{v}) &= \mathbf{u}^T \tilde{A} \mathbf{v} = \sum_{h=1}^p \sum_{k=1}^q \tilde{n}_{hk} u_h v_k \\ &= \sum_{h=1}^p \sum_{k=1}^q \langle \mu_{hk}, \omega_{hk}, \nu_{hk} \rangle u_h v_k \\ &= \left\langle 1 - \prod_{k=1}^q \prod_{h=1}^p (1 - \mu_{hk})^{u_h v_k}, \prod_{k=1}^q \prod_{h=1}^p \omega_{hk}^{u_h v_k}, \prod_{k=1}^q \prod_{h=1}^p \nu_{hk}^{u_h v_k} \right\rangle, \end{aligned}$$

which is still a SVNN.

Irrespective of the use of best strategies of the players, the maximum guaranteed gain (or the minimum possible loss) is the value of the game for Player-I (or Player-II). According to the maximin and minimax principles for Player-I and Player-II respectively, if for some $(\mathbf{u}^0, \mathbf{v}^0) \in U \times V$, such that

$$\mathbf{u}^{0T} \tilde{N} \mathbf{v}^0 = \max_{\mathbf{u} \in U} \min_{\mathbf{v} \in V} \{ \mathbf{u}^T \tilde{N} \mathbf{v} \} = \min_{\mathbf{v} \in V} \max_{\mathbf{u} \in U} \{ \mathbf{u}^T \tilde{N} \mathbf{v} \},$$

then in the sense of Definition 3.3, \mathbf{u}^0 and \mathbf{v}^0 are called optimal strategies for Player-I and Player-II, respectively and $\mathbf{u}^{0T} \tilde{N} \mathbf{v}^0$ is the value of SVNN matrix game \tilde{N} .

Bector et al. [3] introduced the concept of a reasonable solution of the fuzzy matrix game. Here we extend the definition of reasonable solution and solution for the SVNN matrix game in the following definitions.

Definition 4.1. Let $\tilde{\theta}$ and $\tilde{\phi}$ be two SVNNs. Assume that there exist $\mathbf{u}^* \in U$ and $\mathbf{v}^* \in V$ such that $\mathbf{u}^* \tilde{N} \mathbf{v} \subset \tilde{\theta}$ and $\mathbf{u} \tilde{N} \mathbf{v}^* \supset \tilde{\phi}$ hold for any $\mathbf{u} \in U$ and $\mathbf{v} \in V$, then $(\mathbf{u}^*, \mathbf{v}^*, \tilde{\theta}, \tilde{\phi})$ is called a reasonable solution of the SVNN matrix game \tilde{N} .

In this case, $\tilde{\theta}$ and $\tilde{\phi}$ are called reasonable values and $\mathbf{u}^* \in U$ and $\mathbf{v}^* \in V$ are called reasonable strategies for Player-I and Player-II, respectively. The reasonable solution, which

is defined in the above definition, does not represent the solution of the SVNN matrix game \tilde{N} . In the following definition, the concept of solution of SVNN matrix game \tilde{N} is explored.

Definition 4.2. Assume that Θ and Φ are the sets of reasonable values for Player-I and II respectively and $\tilde{\theta}^* \in \Theta$ and $\tilde{\phi}^* \in \Phi$. If there do not exist $\tilde{\theta}' \in \Theta$ ($\tilde{\theta}' \neq \tilde{\theta}^*$) and $\tilde{\phi}' \in \Phi$ ($\tilde{\phi}' \neq \tilde{\phi}^*$) such that $\tilde{\theta}^* \subset \tilde{\theta}'$ and $\tilde{\phi}^* \supset \tilde{\phi}'$, then $(\tilde{\mathbf{u}}^*, \tilde{\mathbf{v}}^*, \tilde{\theta}^*, \tilde{\phi}^*)$ is said to be a solution of the SVNN matrix game \tilde{N} .

In this case, \mathbf{u}^* and \mathbf{v}^* are respectively called the maximin strategy for Player-I and minimax strategy for Player-II. $\tilde{\theta}^*$ is the *gain floor* for Player-I and $\tilde{\phi}^*$ is the *loss ceiling* for Player-II.

5. Mathematical Model and Solution Approach for SVNN Matrix game

Suppose $E(\mathbf{u})_k$ is the expected payoff for Player-I when Player-I uses the mixed strategy $\mathbf{u} \in U$ and Player-II chooses the pure strategy ζ_k , $k \in \Delta_2$. Then

$$E(\mathbf{u})_k = \langle 1 - \prod_{h=1}^p (1 - \mu_{hk})^{u_h}, \prod_{h=1}^p \omega_{hk}^{u_h}, \prod_{h=1}^p \nu_{hk}^{u_h} \rangle.$$

Let $\rho = \langle \mu, \omega, \nu \rangle$ is the minimum of $E(\mathbf{u})_k$. Then, following Definition 3.3, we have

$$\rho = \langle \mu, \omega, \nu \rangle = \langle \min_{k=1}^q \{ 1 - \prod_{h=1}^p (1 - \mu_{hk})^{u_h} \}, \max_{k=1}^q \{ \prod_{h=1}^p \omega_{hk}^{u_h} \}, \max_{k=1}^q \{ \prod_{h=1}^p \nu_{hk}^{u_h} \} \rangle. \tag{1}$$

Obviously ρ is a function of \mathbf{u} . Now to get the maximin strategy $\mathbf{u}^* \in U$ and gain-floor ρ^* , Player-I should choose the mixed strategy $\mathbf{u}^* \in U$ in such a way that ρ is maximized. Therefore following Definition 3.3 and Definition 4.2, we have

$$\rho^* = \langle \mu^*, \omega^*, \nu^* \rangle = \langle \max_{\mathbf{u} \in U} \min_{k=1}^q \{ 1 - \prod_{h=1}^p (1 - \mu_{hk})^{u_h} \}, \min_{\mathbf{u} \in U} \max_{k=1}^q \{ \prod_{h=1}^p \omega_{hk}^{u_h} \}, \min_{\mathbf{u} \in U} \max_{k=1}^q \{ \prod_{h=1}^p \nu_{hk}^{u_h} \} \rangle. \tag{2}$$

Similarly, let $E(\mathbf{v})_h$ is the expected payoff for Player-II, when Player-II chooses the mixed strategy $\mathbf{v} \in V$ against Player-I's pure strategy ϵ_h , $h \in \Delta_1$. Then

$$E(\mathbf{v})_h = \langle 1 - \prod_{k=1}^q (1 - \mu_{hk})^{v_k}, \prod_{k=1}^q \omega_{hk}^{v_k}, \prod_{k=1}^q \nu_{hk}^{v_k} \rangle.$$

Let $\eta = \langle \alpha, \beta, \gamma \rangle$ is the maximum of $E(\mathbf{v})_h$. Then, following Definition 3.3 and Definition 4.2, we have

$$\eta = \langle \alpha, \beta, \gamma \rangle = \langle \max_{h=1}^p \{ 1 - \prod_{k=1}^q (1 - \mu_{hk})^{v_k} \}, \min_{h=1}^p \{ \prod_{k=1}^q \omega_{hk}^{v_k} \}, \min_{h=1}^p \{ \prod_{k=1}^q \nu_{hk}^{v_k} \} \rangle. \tag{3}$$

Clearly, η is a function of \mathbf{v} . Now, to obtain minimax strategy $\mathbf{v}^* \in V$ and loss-ceiling η^* , Player-II should choose the mixed strategy $\mathbf{v}^* \in V$ in such a way that η is minimized. Then following Definition 3.3, we have

$$\eta^* = \langle \alpha^*, \beta^*, \gamma^* \rangle = \langle \min_{\mathbf{v} \in V} \max_{h=1}^p \{ 1 - \prod_{k=1}^q (1 - \mu_{hk})^{v_k} \}, \max_{\mathbf{v} \in V} \min_{h=1}^p \{ \prod_{k=1}^q \omega_{hk}^{v_k} \}, \max_{\mathbf{v} \in V} \min_{h=1}^p \{ \prod_{k=1}^q \nu_{hk}^{v_k} \} \rangle. \tag{4}$$

Theorem 5.1. For SVNN matrix game \tilde{N} , Player-I's gain floor does not exceed Player-II's loss ceiling, i.e., $\rho^* \subseteq \eta^*$.

Proof. For any $\mathbf{u} \in U$, it implies that

$$\min_{\mathbf{v} \in V} \{\tilde{E}(\mathbf{u}, \mathbf{v})\} \subseteq \tilde{E}(\mathbf{u}, \mathbf{v}).$$

Similarly, for any $\mathbf{v} \in V$, we have

$$\tilde{E}(\mathbf{u}, \mathbf{v}) \subseteq \max_{\mathbf{u} \in U} \{\tilde{E}(\mathbf{u}, \mathbf{v})\}.$$

Thus, for any $\mathbf{u} \in U$ and $\mathbf{v} \in V$, we obtain

$$\min_{\mathbf{v} \in V} \{\tilde{E}(\mathbf{u}, \mathbf{v})\} \subseteq \max_{\mathbf{u} \in U} \{\tilde{E}(\mathbf{u}, \mathbf{v})\}.$$

Therefore, $\min_{\mathbf{v} \in V} \{\tilde{E}(\mathbf{u}, \mathbf{v})\} \subseteq \min_{\mathbf{v} \in V} \max_{\mathbf{u} \in U} \{\tilde{E}(\mathbf{u}, \mathbf{v})\}$.

Hence,

$$\max_{\mathbf{u} \in U} \min_{\mathbf{v} \in V} \{\tilde{E}(\mathbf{u}, \mathbf{v})\} \subseteq \min_{\mathbf{v} \in V} \max_{\mathbf{u} \in U} \{\tilde{E}(\mathbf{u}, \mathbf{v})\},$$

i.e., $\rho^* \subseteq \eta^*$. \square

Following Equation (1) and Equation (2), maximin strategy \mathbf{u}^* and the gain floor $\rho^* = \langle \mu^*, \omega^*, \nu^* \rangle$ for Player-I can be obtained by solving the following multi-objective programming problem (5)

$$\begin{aligned} & \max\{\mu\}, \min\{\omega\}, \min\{\nu\} \\ \text{s.t.} \quad & 1 - \prod_{h=1}^p (1 - \mu_{hk})^{u_h} \geq \mu, \quad \prod_{h=1}^p \omega_{hk}^{u_h} \leq \omega, \quad \prod_{h=1}^p \nu_{hk}^{u_h} \leq \nu, \quad (k \in \Delta_2) \\ & \sum_{h=1}^p u_h = 1, \quad u_h \geq 0, \\ & \mu \geq 0, \quad \omega \geq 0, \quad \nu \geq 0, \quad 0 \leq \mu + \omega + \nu \leq 3 \end{aligned} \tag{5}$$

where $\mu = \min_{k=1}^q \{1 - \prod_{h=1}^p (1 - \mu_{hk})^{u_h}\}$, $\omega = \max_{k=1}^q \{ \prod_{h=1}^p \omega_{hk}^{u_h} \}$ and $\nu = \max_{k=1}^q \{ \prod_{h=1}^p \nu_{hk}^{u_h} \}$.

Similarly, the following multi-objective programming problem (6) is constructed by following Equation (3) and Equation (4) to obtain the minimax strategy \mathbf{v}^* and the loss-ceiling $\eta^* = \langle \alpha^*, \beta^*, \gamma^* \rangle$ for Player-II.

$$\begin{aligned} & \min\{\alpha\}, \max\{\beta\}, \max\{\gamma\} \\ \text{s.t.} \quad & 1 - \prod_{k=1}^q (1 - \mu_{hk})^{v_k} \leq \alpha, \quad \prod_{k=1}^q \omega_{hk}^{v_k} \geq \beta, \quad \prod_{k=1}^q \nu_{hk}^{v_k} \geq \gamma, \quad (h \in \Delta_1) \\ & \sum_{k=1}^q v_k = 1, \quad v_k \geq 0, \\ & \alpha \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0, \quad 0 \leq \alpha + \beta + \gamma \leq 3, \end{aligned} \tag{6}$$

where $\alpha = \max_{h=1}^p \{1 - \prod_{k=1}^q (1 - \mu_{hk})^{v_k}\}$, $\beta = \min_{h=1}^p \{ \prod_{k=1}^q \omega_{hk}^{v_k} \}$ and $\gamma = \min_{h=1}^p \{ \prod_{k=1}^q \nu_{hk}^{v_k} \}$.

In Problem (5), the objective functions ω and ν have the same importance, so we take the average of these two functions. Again maximization of μ is equivalent to minimization of $1 - \mu$ and therefore Problem (5) reduces to the following Problem (7) as follows.

$$\begin{aligned}
 & \min\{1 - \mu\}, \min\left\{\frac{\omega + \nu}{2}\right\}, \\
 \text{s.t.} \quad & 1 - \prod_{h=1}^p (1 - \mu_{hk})^{u_h} \geq \mu, \prod_{h=1}^p \omega_{hk}^{u_h} \leq \omega, \prod_{h=1}^p \nu_{hk}^{u_h} \leq \nu, (k \in \Delta_2) \\
 & \sum_{h=1}^p u_h = 1, u_h \geq 0, \\
 & \mu \geq 0, \omega \geq 0, \nu \geq 0, 0 \leq \mu + \omega + \nu \leq 3
 \end{aligned} \tag{7}$$

Again, in Problem (6), the objective functions β and γ have the same importance, so we take the average of these two functions. Again minimization of α is equivalent to the maximization of $1 - \alpha$ and therefore Problem (6) changes to Problem (8) as follows.

$$\begin{aligned}
 & \max\{1 - \alpha\}, \max\left\{\frac{\beta + \gamma}{2}\right\} \\
 \text{s.t.} \quad & 1 - \prod_{k=1}^q (1 - \mu_{hk})^{v_k} \leq \alpha, \prod_{k=1}^q \omega_{hk}^{v_k} \geq \beta, \prod_{k=1}^q \nu_{hk}^{v_k} \geq \gamma, (h \in \Delta_1) \\
 & \sum_{k=1}^q v_k = 1, v_k \geq 0, \\
 & \alpha \geq 0, \beta \geq 0, \gamma \geq 0, 0 \leq \alpha + \beta + \gamma \leq 3
 \end{aligned} \tag{8}$$

Clearly, Problem (7) and Problem (8) are bi-objective programming problems. There are several methods to solve such problems. The notion of Pareto optimal is commonly-used to solve such problems. Here, we use the Lexicographic method [14] to solve Problem (7) and Problem (8).

For Problem (7), the following Problem (9) is constructed first.

$$\begin{aligned}
 & \min\{1 - \mu\} \\
 \text{s.t.} \quad & 1 - \prod_{h=1}^p (1 - \mu_{hk})^{u_h} \geq \mu, \prod_{h=1}^p \omega_{hk}^{u_h} \leq \omega, \prod_{h=1}^p \nu_{hk}^{u_h} \leq \nu, (k \in \Delta_2) \\
 & \sum_{h=1}^p u_h = 1, u_h \geq 0, \\
 & \mu \geq 0, \omega \geq 0, \nu \geq 0, 0 \leq \mu + \omega + \nu \leq 3
 \end{aligned} \tag{9}$$

Let the solution of the non-linear programming Problem (9) is denoted by $(\mathbf{u}', \mu', \omega', \nu')$.

Following the Lexicographic approach, Problem (10) is constructed by combining the Problem (7) and the solution of Problem (9) as follows.

$$\begin{aligned}
 & \min \left\{ \frac{\omega + \nu}{2} \right\} \\
 \text{s.t.} \quad & 1 - \prod_{i=1}^p (1 - \mu_{hk})^{u_h} \geq \mu, \quad \prod_{h=1}^p \omega_{hk}^{u_h} \leq \omega, \quad \prod_{h=1}^p \nu_{hk}^{u_h} \leq \nu, \quad (k \in \Delta_2) \\
 & \sum_{h=1}^p u_h = 1, \quad u_h \geq 0, \\
 & \mu \geq 0, \quad \omega \geq 0, \quad \nu \geq 0, \quad 0 \leq \mu + \omega + \nu \leq 3 \\
 & 1 - \mu \leq 1 - \mu', \quad \omega \leq \omega', \quad \nu \leq \nu'
 \end{aligned} \tag{10}$$

The optimal solution $(\mathbf{u}^*, \mu^*, \omega^*, \nu^*)$ will be obtained by solving Problem (10). Then it is obvious to prove that $(\mathbf{u}^*, \mu^*, \omega^*, \nu^*)$ is a Pareto optimal solution of Problem (7).

Similarly, Problem (8) turns into solving Problem (11) at first according to the Lexicographic method.

$$\begin{aligned}
 & \max \{1 - \alpha\} \\
 \text{s.t.} \quad & 1 - \prod_{k=1}^q (1 - \mu_{hk})^{v_k} \leq \alpha, \quad \prod_{k=1}^q \omega_{hk}^{v_k} \geq \beta, \quad \prod_{k=1}^q \nu_{hk}^{v_k} \geq \gamma, \quad (h \in \Delta_1) \\
 & \sum_{k=1}^q v_k = 1, \quad v_k \geq 0 \\
 & \alpha \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0, \quad 0 \leq \alpha + \beta + \gamma \leq 3
 \end{aligned} \tag{11}$$

Let $(\mathbf{v}', \alpha', \beta', \gamma')$ is the optimal solution of Problem (11). Then Problem (12) is formulated by following the lexicographic approach as follows.

$$\begin{aligned}
 & \max \left\{ \frac{\beta + \gamma}{2} \right\} \\
 \text{s.t.} \quad & 1 - \prod_{k=1}^q (1 - \mu_{hk})^{v_k} \leq \alpha, \quad \prod_{k=1}^q \omega_{hk}^{v_k} \geq \beta, \quad \prod_{k=1}^q \nu_{hk}^{v_k} \geq \gamma, \quad (h \in \Delta_1) \\
 & \sum_{k=1}^q v_k = 1, \quad v_k \geq 0 \\
 & \alpha \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0, \quad 0 \leq \alpha + \beta + \gamma \leq 3 \\
 & 1 - \alpha \geq 1 - \alpha', \quad \beta \geq \beta', \quad \gamma \geq \gamma'
 \end{aligned} \tag{12}$$

Let $(\mathbf{v}^*, \alpha^*, \beta^*, \gamma^*)$ is the optimal solution of Problem (12). Then obviously the solution $(\mathbf{v}^*, \alpha^*, \beta^*, \gamma^*)$ will be the Pareto optimal solution of Problem (8).

5.1. Algorithm

The algorithm for solving SVNN matrix game by the proposed approach is abstracted as follows.

Step-1: Consider a matrix game with payoffs of SVNNs.

Step-2: To solve the game, two nonlinear multi-objective programming problems are formulated.

Step-3: Considering the same preference to the DOI and DONM, the multi-objective programming problems are converted to corresponding bi-objective programming problems.

Step-4: Lexicographic method is used to counter the bi-objective programming problems and the optimal strategies for the players are obtained.

The corresponding algorithmic flowchart of the proposed approach is presented in Figure 1.

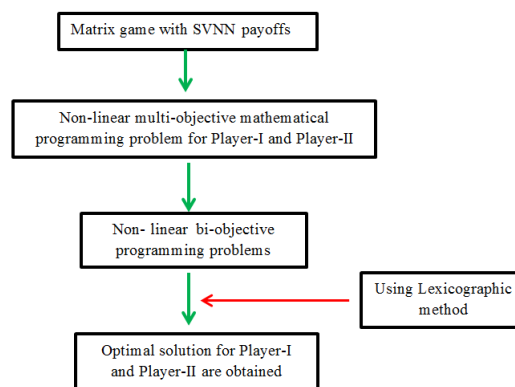


FIGURE 1. Algorithmic flowchart of the proposed approach

6. Numerical Example

This subsection provides a market share problem to illustrate the solution procedure of a SVNN matrix game.

6.1. New factory set up-management problem

Assume that a renowned foreign car manufacturing company 'X' is going to launch a new factory in India. For smooth functioning of the factory, the industrial manufacturing organization of the company 'X' suggests setting up the new factory either in the state Gujrat or in Hariyana, after considering various government policies and environmental conditions. The respective Ministry of Labour and Welfare department looks into the matter on behalf of their state governments and they jump to take the opportunity for a new industrial agreement, though their resources are limited. In that case, the respective Ministry of both states acts

wisely and anticipates opponents' moves. The relationship of the two departments of the states can be regarded as the two players of the game. Here, we assume that the Ministry for Gujrat and Hariyana as Player-I and Player-II, respectively, and the opportunity to get the agreement as payoff values. Player-I and II has limited resources but they frame the game problem as to develop respective strategies and maximize the opportunity to build the new factory. In a zero-sum game, the opportunity gain by Player-I is positive and negative for Player-II. It is unrealistic for the decision-maker (DM) to get accurate and complete information on the payoff values. In this case, the SVNNS are used to express uncertainty.

Suppose that the marketing research team (MRT) of the company 'X' receives the information from the respective Ministry of the two states by following mainly three aspects like- Availability of Raw materials (strategy-I), Labour supply- including workers with the right skills (strategy-II) and Grants and financial incentives- usually from the government (strategy-III). These aspects may be considered as strategies of the two different states. Then according to the view of the MRT of the company 'X', the opportunity for Player-I is estimated and evaluated by using linguistic terms as follows.

$$\tilde{N} = \begin{matrix} & \text{I} & \text{II} & \text{III} \\ \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \end{matrix} & \left(\begin{matrix} \text{Very High} & \text{High} & \text{Medium} \\ \text{Low} & \text{Very High} & \text{High} \\ \text{Medium} & \text{Very Low} & \text{Very High} \end{matrix} \right) \end{matrix}.$$

Table 1 shows the corresponding relations between linguistic terms and SVNNS. Then the

TABLE 1. Assigned SVNNS corresponding to linguistic term

Linguistic term	Very High	High	Medium	Low	Very Low
SVNN	$\langle 0.95, 0.07, 0.05 \rangle$	$\langle 0.7, 0.5, 0.25 \rangle$	$\langle 0.5, 0.4, 0.4 \rangle$	$\langle 0.25, 0.3, 0.7 \rangle$	$\langle 0.05, 0.1, 0.95 \rangle$

pay-off matrix \tilde{A} with payoffs of SVNNS may be transformed according to Table 1 as follows.

$$\tilde{A} = \begin{matrix} & \text{I} & \text{II} & \text{III} \\ \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \end{matrix} & \left(\begin{matrix} \langle 0.95, 0.07, 0.05 \rangle & \langle 0.70, 0.50, 0.25 \rangle & \langle 0.50, 0.40, 0.40 \rangle \\ \langle 0.25, 0.30, 0.70 \rangle & \langle 0.95, 0.07, 0.05 \rangle & \langle 0.70, 0.50, 0.25 \rangle \\ \langle 0.50, 0.40, 0.40 \rangle & \langle 0.05, 0.10, 0.95 \rangle & \langle 0.95, 0.07, 0.05 \rangle \end{matrix} \right), \end{matrix}$$

where $\langle 0.95, 0.07, 0.05 \rangle$ in the matrix \tilde{A} is a SVNNS which represents that Player-I will get the opportunity to set up the new project positively by 95%, unable to get the opportunity by 5%. Player-I has an indeterminacy about the establishment of the new project by 7%. These situation occur only when both of the players use the strategy I simultaneously. Other SVNNS in \tilde{A} have also a similar explanation.

6.2. The solution procedure and result discussion

Problem (13) is formulated by following Problem (9) as follows.

$$\begin{aligned}
 & \min\{1 - \mu\} \\
 \text{s.t. } & 1 - 0.05^{u_1}0.75^{u_2}0.5^{u_3} \geq \mu, \quad 1 - 0.3^{u_1}0.05^{u_2}0.95^{u_3} \geq \mu, \quad 1 - 0.5^{u_1}0.3^{u_2}0.05^{u_3} \geq \mu, \\
 & 0.07^{u_1}0.3^{u_2}0.4^{u_3} \leq \omega, \quad 0.5^{u_1}0.07^{u_2}0.1^{u_3} \leq \omega, \quad 0.4^{u_1}0.5^{u_2}0.07^{u_3} \leq \omega, \\
 & 0.05^{u_1}0.7^{u_2}0.4^{u_3} \leq \nu, \quad 0.25^{u_1}0.05^{u_2}0.95^{u_3} \leq \nu, \quad 0.4^{u_1}0.25^{u_2}0.05^{u_3} \leq \nu, \\
 & u_1 + u_2 + u_3 = 1, \quad u_1, u_2, u_3 \geq 0, \\
 & 0 \leq \mu + \omega + \nu \leq 3, \quad \mu, \omega, \nu \geq 0.
 \end{aligned} \tag{13}$$

Solving Equation (13), the obtained optimal solution is $\mu' = 0.772989$, $\omega' = 1$ and $\nu' = 1$.

According to Equation (10), Problem (14) is constructed as follows.

$$\begin{aligned}
 & \min\left\{\frac{\omega + \nu}{2}\right\} \\
 \text{s.t. } & 1 - 0.05^{u_1}0.75^{u_2}0.5^{u_3} \geq \mu, \quad 1 - 0.3^{u_1}0.05^{u_2}0.95^{u_3} \geq \mu, \quad 1 - 0.5^{u_1}0.3^{u_2}0.05^{u_3} \geq \mu, \\
 & 0.07^{u_1}0.3^{u_2}0.4^{u_3} \leq \omega, \quad 0.5^{u_1}0.07^{u_2}0.1^{u_3} \leq \omega, \quad 0.4^{u_1}0.5^{u_2}0.07^{u_3} \leq \omega \\
 & 0.05^{u_1}0.7^{u_2}0.4^{u_3} \leq \nu, \quad 0.25^{u_1}0.05^{u_2}0.95^{u_3} \leq \nu, \quad 0.4^{u_1}0.25^{u_2}0.05^{u_3} \leq \nu \\
 & 1 - \mu \leq 0.227011, \quad \omega \leq 1, \quad \nu \leq 1 \\
 & 0 \leq \mu + \omega + \nu \leq 3, \quad \mu, \omega, \nu \geq 0 \\
 & u_1 + u_2 + u_3 = 1, \quad u_1, u_2, u_3 \geq 0.
 \end{aligned} \tag{14}$$

Solving Problem (14), the obtained optimal solution is $\mathbf{u}^* = (0.3151, 0.3138, 0.3711)$, and $\rho^* = \langle \mu^*, \omega^*, \nu^* \rangle = \langle 0.7228, 0.2476, 0.2476 \rangle$.

Now for Player-II, Problem (15) is constructed by following Equation (11), as follows.

$$\begin{aligned}
 & \max\{1 - \alpha\} \\
 \text{s.t. } & 1 - 0.05^{v_1}0.3^{v_2}0.5^{v_3} \leq \alpha, \quad 1 - 0.75^{v_1}0.05^{v_2}0.3^{v_3} \leq \alpha, \quad 1 - 0.5^{v_1}0.95^{v_2}0.05^{v_3} \leq \alpha, \\
 & 0.07^{v_1}0.5^{v_2}0.4^{v_3} \geq \beta, \quad 0.3^{v_1}0.07^{v_2}0.5^{v_3} \geq \beta, \quad 0.4^{v_1}0.1^{v_2}0.07^{v_3} \geq \beta, \\
 & 0.05^{v_1}0.25^{v_2}0.4^{v_3} \geq \gamma, \quad 0.7^{v_1}0.05^{v_2}0.25^{v_3} \geq \gamma, \quad 0.4^{v_1}0.95^{v_2}0.05^{v_3} \geq \gamma, \\
 & v_1 + v_2 + v_3 = 1, \quad v_1, v_2, v_3 \geq 0, \\
 & 0 \leq \alpha + \beta + \gamma \leq 3, \quad \alpha, \beta, \gamma \geq 0.
 \end{aligned} \tag{15}$$

Solving Equation (15), the obtained optimal solution is $\alpha' = 0.7346452$, $\beta' = 0$ and $\gamma' = 0$.

Then following Equation (12), Problem (16) is constructed as follows.

$$\begin{aligned}
 & \max \left\{ \frac{\beta + \gamma}{2} \right\} \\
 \text{s.t.} \quad & 1 - 0.05^{v_1} 0.3^{v_2} 0.5^{v_3} \leq \alpha, \quad 1 - 0.75^{v_1} 0.05^{v_2} 0.3^{v_3} \leq \alpha, \quad 1 - 0.5^{v_1} 0.95^{v_2} 0.05^{v_3} \leq \alpha, \\
 & 0.07^{v_1} 0.5^{v_2} 0.4^{v_3} \geq \beta, \quad 0.3^{v_1} 0.07^{v_2} 0.5^{v_3} \geq \beta, \quad 0.4^{v_1} 0.1^{v_2} 0.07^{v_3} \geq \beta, \\
 & 0.05^{v_1} 0.25^{v_2} 0.4^{v_3} \geq \gamma, \quad 0.7^{v_1} 0.05^{v_2} 0.25^{v_3} \geq \gamma, \quad 0.4^{v_1} 0.95^{v_2} 0.05^{v_3} \geq \gamma, \\
 & 1 - \alpha \geq 0.2653548, \quad 0 \leq \alpha + \beta + \gamma \leq 3, \quad \alpha, \beta, \gamma \geq 0, \\
 & v_1 + v_2 + v_3 = 1, \quad v_1, v_2, v_3 \geq 0.
 \end{aligned} \tag{16}$$

Solving Problem (16), the optimal obtained solution is $\mathbf{v}^* = (0.1807, 0.4255, 0.3938)$, and $\eta^* = \langle \alpha^*, \beta^*, \gamma^* \rangle = \langle 0.7346, 0.1116, 0.1518 \rangle$.

The following observations can be made for the results obtained.

- (1) $\rho^* = \langle \mu^*, \omega^*, \nu^* \rangle = \langle 0.7228, 0.2476, 0.2476 \rangle$ represents that the state Gujrat has the opportunity to set up the new factory positively by 72.28% and unable to get the opportunity by 24.76%. Also, it is indeterminate to assume that the state Gujrat will get the opportunity by 24.76%. In this case, strategies I, II and III are chosen with probability 0.3151, 0.3138 and 0.3711, respectively.
- (2) $\eta^* = \langle \alpha^*, \beta^*, \gamma^* \rangle = \langle 0.7346, 0.1116, 0.1518 \rangle$ represents that the state Hariyana has the opportunity to set up the new factory positively by 73.46% and unable to get the opportunity by 11.16%. Whereas, it is indeterminate to say that the state Hariyana will get the opportunity by 15.18%. In this case, strategies I, II and III are chosen with probability 0.1807, 0.4255, and 0.3938, respectively.
- (3) ρ^* and η^* are obtained as SVNN, which is desirable.
- (4) It is clear that $\mu^* \leq \alpha^*$, $\omega^* \geq \beta^*$, $\nu^* \geq \gamma^*$, then $\rho^* \subseteq \eta^*$, which follows Theorem 5.1.
- (5) For the maximin strategy $\mathbf{u}^* = (0.3151, 0.3138, 0.3711)$ and minimax strategy $\mathbf{v}^* = (0.1807, 0.4255, 0.3938)$, the expected payoff for Player-I is $E(\mathbf{u}^*, \mathbf{v}^*) = \langle 0.7713, 0.1862, 0.2083 \rangle$, which is the value of SVNN matrix game \tilde{N} . From Table 1, it is to conclude that the expected payoff for Player-I is between “Very High” and “High” in terms of linguistic terms.

6.3. Analysis and comparison of results with Li and Nan [13] approach

NS takes into consideration the indeterminacy together with the membership and non-membership whereas intuitionistic fuzzy set (IFS) consider membership and non-membership and ignore indeterminacy of the elements. So NS is the generalization of IFS. Therefore Atanassov’s intuitionistic fuzzy number [2] is a particular case of SVNN as we can simply consider the sum of three independent functions as equal to 1.

To verify the efficiency of the proposed approach, at first, we transfer the payoff matrix \tilde{N} with payoffs as SVNN to a payoff matrix \tilde{N}' with Atanassov's intuitionistic fuzzy sets by considering the DOI as zero, where

$$\tilde{N}' = \begin{matrix} & \text{I} & \text{II} & \text{III} \\ \text{I} & \langle 0.95, 0.05 \rangle & \langle 0.70, 0.25 \rangle & \langle 0.50, 0.4 \rangle \\ \text{II} & \langle 0.25, 0.70 \rangle & \langle 0.95, 0.05 \rangle & \langle 0.70, 0.25 \rangle \\ \text{III} & \langle 0.50, 0.40 \rangle & \langle 0.05, 0.95 \rangle & \langle 0.95, 0.05 \rangle \end{matrix}.$$

Here, the payoff matrix \tilde{N}' is the same as the payoff matrix considered in the paper of Li and Nan [13]. Li and Nan [13] proposed a nonlinear programming approach to solve matrix game with payoffs of Atanassov's intuitionistic fuzzy sets. Li and Nan [13] use the weighted average method to solve a pair of nonlinear programming models which are derived from two auxiliary nonlinear bi-objective programming models. We use the proposed technique for the payoff matrix \tilde{N}' and obtained results are shown in Table 2. Table 2 also shows the results obtained by Li and Nan [13] approach for the mid-value 0.5 of the interval of the weight λ .

TABLE 2. Results for the matrix game with pay-off matrix \tilde{A}'

Articles	\mathbf{u}^*	\mathbf{v}^*	$E(\mathbf{u}^*, \mathbf{v}^*)$
Li and Nan [13] (for $\lambda = 0.5$)	(0.4076,0.3325,0.2599)	(0.2681,0.2958,0.4361)	$\langle 0.7729, 0.2035 \rangle$
Our proposed technique	(0.4009,0.3292,0.2699)	(0.2770,0.2972,0.4258)	$\langle 0.7730, 0.2037 \rangle$

Table 2 shows that the optimal expected value obtained by the proposed technique discussed in this paper is $\langle 0.7729, 0.2035 \rangle$ and obtained by Li and Nan [13] approach is $\langle 0.7730, 0.2037 \rangle$, which implies the fact that the optimal value for both of the players obtained by the proposed method and Li and Nan [13] approach are approximately the same. Table 2 shows that Player-I chooses three strategies (I, II, and III) with probability 0.4076,0.3325, 0.2599 respectively which is obtained through the proposed approach, whereas Player-I choose the same strategies with probability 0.4009, 0.3292, and 0.2699 respectively which is obtained through Li and Nan [13] approach. This implies that the optimal strategies for both players are very similar. The approach discussed in this paper considers DOI together with the DOM and the DONM. Therefore, the proposed method is an extension of Li and Nan [13] method and counter uncertainty in a larger sense. This manifests the effectiveness and validity of the proposed technique.

7. Conclusion

SVNN is a vital tool to tackle uncertainty in decision-making problems. This paper uses SVNNs to represent the imprecise payoffs so that players can consider the neutrality of the

elements better. In this paper, a solution procedure is established to solve a new matrix game where the payoffs are represented by SVNNS. Two non-linear multi-objective programming problems are formulated to obtain the optimal values and optimal strategies for the players. These multi-objective programming problems are transformed to bi-objective programming problems by considering the same importance of the objective functions. Then Lexicographic method is applied to counter the bi-objective programming problems.

The proposed approach is illustrated by solving a market share problem, which implies the validity and effectiveness of the proposed approach. The optimal solutions are obtained in SVNNS form, which is desirable. We consider the matrix game problem from Li and Nan [13] with payoffs as IFSs and the proposed approach is illustrated by considering the DOI as zero. The obtained results are compared with the results obtained by Li and Nan's [13] approach and observed that the optimal strategies for both players are very similar. This demonstrates the reliability of our approach.

The limitation of the proposed approach is that it does not find the solution to the game problem directly, as it considers the construction of multi-objective programming problems. Also, we do not make any conclusion about the existence of the solution of the SVNNS matrix game in the sense discussed in this paper. Therefore these problems need a further investigation in the future.

The proposed methods are indeed able to solve SVNNS matrix games. Moreover, the concept of this work is readily applicable to other games such as two-person non-zero-sum games, multi-objective matrix game problems. Although the discussed approach is applied to solve a market share problem, the proposed approach may be applied in management science, war science, economics, advertising, cyber security related problems. In addition, a pair of nonlinear bi-objective programming problems is derived from the two auxiliary nonlinear multi-objective programming problems and countered by using the Lexicographic method. Therefore, other new methods for solving matrix games with payoffs of the SVNNSs may be investigated in the near future.

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