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A Generalization in Space of Jung's Theorem

In Florentin Smarandache: "Collected Papers", vol. I (second edition).
Ann Arbor (USA): InfoLearnQuest, 2007.

In this short note we will prove a generalization of Joung's theorem in space.

Theorem. Let us have n points in space such that the maximum distance between any two points is a. Prove that there exists a sphere of radius $r \le a\frac{\sqrt{6}}{4}$ that contains in its interior or on its surface all these points.

Proof:

Let $P_1,...,P_n$. be the points. Let $S_1(O_1, r_1)$ be a sphere of center O_1 and radius r_1, which contains all these points. We note $r_2 = \max_{1\le i\le n} P_iO_1 = P_1O_1$ and construct the sphere $S_2(O_1, r_2)$, $r_2 \le r_1$, with $P_1 \in Fr(S_2)$, where $Fr(S_2) =$ frontier (surface) of S_2.

We apply a homothety H in space, of center P_1, such that the new sphere $H(S_2) = S_3(O_3, r_3)$ has the property: $Fr(S_3)$ contains another point, for example P_2, and of course S_3 contains all points P_i.

1) If P_1, P_2 are diametrically opposite in S_3 then $r_{\min} = \frac{a}{2}$.

If no, we do a rotation R so that $R(S_3) = S_4(O_4, r_4)$ for which $\{P_3, P_2, P_1\} \subset Fr(S_4)$ and S_4 contains all points P_i.

2) If $\{P_1, P_2, P_3\}$ belong to a great circle of S_4 and they are not included in an open semicircle, then $r_{\min} \le \frac{a}{\sqrt{3}}$ (Jung's theorem).

If no, we consider the fascicule of spheres S for which $\{P_1, P_2, P_3\} \subset Fr(S)$ and S contains all points P_i. We choose a sphere S_5 such that $\{P_1, P_2, P_3, P_4\} \subset Fr(S_5)$.

3) If $\{P_1, P_2, P_3, P_4\}$ are not included in an open semisphere of S_5, then the tetrahedron $\{P_1, P_2, P_3, P_4\}$ can be included in a regulated tetrahedron of side a, whence we find that the radius of S_5 is $\le a\frac{\sqrt{6}}{4}$.

If no, let's note. $\max_{1\le i\le j\le 4} P_iP_j = P_1P_4$. Does the sphere S_6 of diameter P_1P_4 contain all points P_i?

If yes, stop (we are in the case 1).

If no, we consider the fascicule of spheres S' such that $\{P_1, P_4\} \subset Fr(S')$ and S' contains all the points P_i. We choose another sphere S_7, for which $P_5 \notin \{P_1, P_2, P_3, P_4\}$ and $P_5 \in Fr(S_7)$.

With these new notations (the points P_1, P_4, P_5 and the sphere S_7) we return to the case 2.

This algorithm is finite; therefore it constructs the required sphere.

[Published in "GAZETA MATEMATICA", Nr. 9-10-11-12, 1992, Bucharest, Romania, p. 352.]