Noname manuscript No.
(will be inserted by the editor)

Proof of the Riemann Hypothesis

Frank Vega

the date of receipt and acceptance should be inserted later

Abstract The Riemann hypothesis has been considered the most important unsolved problem in mathematics. Robin criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We show that the Robin inequality is true for all natural numbers $n > 5040$ which are not divisible by the prime 3. Moreover, we prove that the Robin inequality is true for all natural numbers $n > 5040$ which are divisible by the prime 3. Consequently, the Robin inequality is true for all natural numbers $n > 5040$ and thus, the Riemann hypothesis is true.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers · Riemann zeta function

Mathematics Subject Classification (2010) MSC 11M26 · MSC 11A41 · MSC 11A25

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [3]. The Riemann hypothesis belongs to the David Hilbert's list of 23 unsolved problems [3]. Besides, it is one of the Clay Mathematics Institute's Millennium Prize Problems [3]. As usual $\sigma(n)$ is the sum-of-divisors function of n [4]:

$$\sum_{d|n} d$$

F. Vega
CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
ORCiD: 0000-0001-8210-4126
E-mail: vega.frank@gmail.com

where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [9].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are not divisible by 2 [4]. In addition, we show that $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are not divisible by 3. Furthermore, we prove that $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are divisible by 3. Putting all together yields the proof that the Riemann hypothesis is true.

2 A Central Lemma

These are known results:

Lemma 2.1 *[4]. For* $n > 1$*:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \tag{2.1}$$

Lemma 2.2 *[5].*

$$\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all natural numbers n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all natural numbers n. Further the bound only uses the primes that divide n and not how many times they divide n.

Lemma 2.3 *Let* $n > 1$ *and let all its prime divisors be* $q_1 < \cdots < q_m$*. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof We use that lemma 2.1:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\frac{1}{1-\frac{1}{q^2}} \times \frac{q+1}{q} = \frac{q^2}{q^2-1} \times \frac{q+1}{q} = \frac{q}{q-1}.$$

Then by lemma 2.2,

$$\prod_{i=1}^{m} \frac{1}{1-\frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^{m} \frac{q_i}{q_i-1} \\ &\leq \prod_{i=1}^{m} \frac{1}{1-\frac{1}{q_i^2}} \times \frac{q_i+1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}. \end{aligned}$$

3 A Basic Case

In basic number theory, for a given prime number p, the p-adic order of a natural number n is the highest exponent $\nu_p \geq 1$ such that p^{ν_p} divides n. This is a known result:

Lemma 3.1 *In general, we know that* $\mathsf{Robins}(n)$ *holds for a natural number* $n > 5040$ *that satisfies either* $\nu_2(n) \leq 19$, $\nu_3(n) \leq 12$ *or* $\nu_7(n) \leq 6$, *where* $\nu_p(n)$ *is the p-adic order of n [6].*

We can easily prove that $\mathsf{Robins}(n)$ is true for certain kind of numbers:

Lemma 3.2 $\mathsf{Robins}(n)$ *holds for* $n > 5040$ *when* $q \leq 7$, *where q is the largest prime divisor of n.*

Proof Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m \leq 5$, then we need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i-1} \leq e^{\gamma} \times \log\log n$$

according to the lemma 2.1. For $q_1 < \cdots < q_m \leq 5$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i-1} \leq \frac{2\times 3\times 5}{1\times 2\times 4} = 3.75 < e^{\gamma} \times \log\log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log\log(5040) < e^{\gamma} \times \log\log n$$

and therefore, the proof is complete when $q_1 < \cdots < q_m \leq 5$. The remaining case is for $n > 5040$ when all its prime divisors are $q_1 < \cdots < q_m \leq 7$. $\mathsf{Robins}(n)$ holds for $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [6]. Hence, it is enough to prove this for those natural numbers $n > 5040$ when $7^7 \mid n$. For $q_1 < \cdots < q_m \leq 7$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \log\log(7^7) \approx 4.65.$$

However, for $n > 5040$ and $7^7 \mid n$:

$$e^{\gamma} \times \log\log(7^7) \leq e^{\gamma} \times \log\log n$$

and as a consequence, the proof is complete when $q_1 < \cdots < q_m \leq 7$.

4 A Better Bound

This is a known result:

Lemma 4.1 *[10]. For $x > 1$:*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + B + \frac{1}{\log^2 x} \tag{4.1}$$

where

$$B = 0.2614972128\cdots$$

denotes the (Meissel-)Mertens constant [10].

We show a better result:

Lemma 4.2 *For $x \geq 11$, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - 0.12.$$

Proof Let's define $H = \gamma - B$ [7]. The lemma 4.1 is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}).$$

For $x \geq 11$,

$$(H - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

and thus,

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}) < \log\log x + \gamma - 0.12.$$

5 On a Square Free Number

We know the following results:

Lemma 5.1 *[4]. For* $0 < a < b$*:*

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b-a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{5.1}$$

Lemma 5.2 *[4]. For* $q > 0$*:*

$$\log(q+1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \tag{5.2}$$

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [4].

Lemma 5.3 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *that are square free [4].*

Lemma 5.4 *For a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1 < q_2 < \cdots < q_m$ *are odd prime numbers,* $q_m \geq 11$ *and* $3 \nmid n$*, then:*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Proof By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [4]. Put $\omega(n) = m$ [4]. We need to prove the assertion for those integers with $m = 1$. From a square free number n, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{5.3}$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [4]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^{\gamma} \times \log\log(2^{19} \times q_i). \tag{5.4}$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^{\gamma} \times \log\log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\log\log(2^{19} \times 11) < \log\log(2^{19} \times q_i)$$

which clearly implies that the inequality (5.4) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m-1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [4]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1+1) \times \cdots \times (q_{m-1}+1) \leq e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1+1) \times \cdots \times (q_{m-1}+1) \times (q_m+1) \leq$$
$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m+1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by (q_m+1). We want to show

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m+1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$
$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m+1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$
$$\frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality in lemma 5.1 just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) =$$
$$\log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m.$$

In this way, we obtain

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} >$$
$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [4].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \log\log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \log\log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log(\frac{\pi^2}{5.32}) + (\log(3+1) - \log 3) + \sum_{i=1}^{m} (\log(q_i + 1) - \log q_i) \leq \gamma + \log\log\log(2^{19} \times n).$$

In addition, note that $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log\log q_m < \gamma + \log\log\log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$. We use that lemma 5.2 for each term $\log(q+1) - \log q$ and thus,

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m$$

where $q_m \geq 11$. Hence, it is enough to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m - 0.12$$

but this is true according to the lemma 4.2 for $q_m \geq 11$. In this way, we finally show the lemma is indeed satisfied.

6 Main Insight

The next result is a main insight.

Lemma 6.1 *Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m$. When $q_m \geq 11$, $3 \nmid n$ and $2^{20} \mid n$, then*

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n.$$

Proof We need to prove that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n.$$

Using the formula (5.3) for the square free numbers, then we obtain that is equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \log\log n$$

where $n' = q_1 \times \cdots \times q_m$ is the square free kernel of the natural number n [4]. We know that $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}) \leq e^{\gamma} \times n' \times \log\log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ where $2^{20} \mid n$ and $2 \mid n'$. So,

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}).$$

According to the formula (5.3) for the square free numbers and $2 \mid n'$, then,

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

where this is true according to the lemma 5.4 when $3 \nmid \frac{n'}{2}$ and $q_m \geq 11$. To sum up, the proof is complete.

7 Proof of the Riemann Hypothesis

Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [4]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Lemma 7.1 *If n is superabundant, then n is an Hardy-Ramanujan integer [2].*

Lemma 7.2 *The smallest counterexample of the Robin inequality greater than* 5040 *must be a superabundant number [1].*

This is an important lemma that we use:

Lemma 7.3 *Let $x \geq 11$. For $y > x$ we have [8]:*

$$\frac{\log\log y}{\log\log x} < \frac{\sqrt{y}}{\sqrt{x}}.$$

Theorem 7.4 *The Riemann hypothesis is true.*

Proof Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. In this way, we assume that $n > 5040$ could be the smallest integer such that $\mathsf{Robins}(n)$ does not hold. According to the lemmas 7.1 and 7.2, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since $n > 5040$ should be an Hardy-Ramanujan integer. We know that $n > 5040$ complies that $\mathsf{Robins}(n)$ holds when $\nu_2(n) \leq 19$ or $q_m \leq 7$ according to the lemmas 3.1 and 3.2. Therefore, the natural number $n > 5040$ complies with $q_m \geq 11$ and $2^{20} \mid n$. So,

$$\frac{\pi^2}{6} \times \frac{3}{4} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log \frac{n}{3^{\nu_3(n)}}$$

because of the lema 6.1. This is equivalent to

$$\frac{\pi^2}{8} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log \frac{n}{3^{\nu_3(n)}}.$$

If we divide the two sides of the previous inequality by $e^{\gamma} \times \log\log n$, then

$$\frac{\frac{\pi^2}{8} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}}{e^{\gamma} \times \log\log n} \leq \frac{\log\log \frac{n}{3^{\nu_3(n)}}}{\log\log n}.$$

We use that lemma 7.3 to show that

$$\frac{\log\log \frac{n}{3^{\nu_3(n)}}}{\log\log n} > \frac{1}{\sqrt{3^{\nu_3(n)}}}.$$

We know that $\mathsf{Robins}(n)$ holds for a natural number $n > 5040$ when $\nu_3(n) \leq 12$. Consequently, we obtain that

$$\frac{\frac{\pi^2}{8} \times \sqrt{3^{12}} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}}{e^{\gamma} \times \log\log n} \leq \frac{1}{\sqrt{3^{\nu_3(n)-12}}}.$$

We have that

$$\frac{\pi^2}{8} \times \sqrt{3^{12}} \geq \frac{\pi^2}{6}.$$

We use that theorem 2.2 to show that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > \left(\prod_{i=1}^{m} \frac{q_i^2}{q_i^2 - 1} \right) \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Besides,

$$\left(\prod_{i=1}^{m}\frac{q_i^2}{q_i^2-1}\right)\times\prod_{i=1}^{m}\frac{q_i+1}{q_i}=\prod_{i=1}^{m}\frac{q_i}{q_i-1}$$

because of

$$\frac{q}{q-1}=\frac{q^2}{q^2-1}\times\frac{q+1}{q}.$$

Consequently, we obtain that

$$\frac{\prod_{i=1}^{m}\frac{q_i}{q_i-1}}{e^{\gamma}\times\log\log n}<\frac{\frac{\pi^2}{8}\times\sqrt{3^{12}}\times\prod_{i=1}^{m}\frac{q_i+1}{q_i}}{e^{\gamma}\times\log\log n}$$

and thus,

$$\frac{f(n)}{e^{\gamma}\times\log\log n}<1$$

according to the lemma 2.1 and $\frac{1}{\sqrt{3^{\nu_3(n)-12}}}<1$. That is the same as

$$f(n)<e^{\gamma}\times\log\log n.$$

However, this is a contradiction, since $\mathsf{Robins}(n)$ does not hold under our initial assumption. Finally, we can see that the Riemann hypothesis is true because of the theorem 1.1.

Acknowledgments

The author would like to thank Richard J. Lipton and Craig Helfgott for helpful comments and his mother, maternal brother and his friend Sonia for their support.

References

1. Akbary, A., Friggstad, Z.: Superabundant numbers and the Riemann hypothesis. The American Mathematical Monthly **116**(3), 273–275 (2009). DOI doi:10.4169/193009709X470128
2. Alaoglu, L., Erdős, P.: On highly composite and similar numbers. Transactions of the American Mathematical Society **56**(3), 448–469 (1944). DOI doi:10.2307/1990319
3. Borwein, P.B., Choi, S., Rooney, B., Weirathmueller, A.: The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike, vol. 27. Springer Science & Business Media (2008)
4. Choie, Y., Lichiardopol, N., Moree, P., Solé, P.: On Robin's criterion for the Riemann hypothesis. Journal de Théorie des Nombres de Bordeaux **19**(2), 357–372 (2007). DOI doi:10.5802/jtnb.591
5. Edwards, H.M.: Riemann's Zeta Function. Dover Publications (2001)
6. Hertlein, A.: Robin's Inequality for New Families of Integers. Integers **18** (2018)
7. Mertens, F.: Ein Beitrag zur analytischen Zahlentheorie. J. reine angew. Math. **1874**(78), 46–62 (1874). DOI 10.1515/crll.1874.78.46. URL `https://doi.org/10.1515/crll.1874.78.46`
8. Nazardonyavi, S., Yakubovich, S.: Superabundant numbers, their subsequences and the Riemann hypothesis (2013). At `https://arxiv.org/pdf/1211.2147.pdf`
9. Robin, G.: Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. pures appl **63**(2), 187–213 (1984)
10. Rosser, J.B., Schoenfeld, L.: Approximate Formulas for Some Functions of Prime Numbers. Illinois Journal of Mathematics **6**(1), 64–94 (1962). DOI doi:10.1215/ijm/1255631807