

## Time-Dependent and Independent Quantum Entropy

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Entropy in quantum mechanics is often based on von Neuman's prescription (1) which yields zero for a pure eigenstate. Some alternative approaches also yield this result (2). In other words, it seems this type of entropy deals with transitions between different energy eigenstates and excludes the intrinsic entropy of a pure energy eigenstate. There is, however, literature (3) dealing with both spatial and momentum entropy of a single energy eigenstate. In particular, Shannon's entropy equation yields  $-W^*W \ln[W^*W]$  and  $-a^*(p)a(p) \ln[a^*(p)a(p)]$  respectively where  $W$ , the wavefunction, is  $\sum_p a(p)\exp(ipx)$ . In a previous note (4), we argued that one may use  $W_n(x) a(p) \exp(-ipx)$  as a probability in Shannon's entropy equation to obtain an entropy which is the sum of Shannon's spatial and momentum entropies for a single energy eigenstate. A third term proportional to  $ipx$  integrates to zero.

In this note, we wish to extend this approach to a time-dependent wavefunction (for example:  $W(x,t) = \sum_n b_n(E_n) \exp(-iE_n t) W_n(x)$ ). We suggest  $W(x,t) b_n(E_n) a_n(p) \exp(iE_n t) \exp(-ipx)$  as a candidate probability for Shannon's entropy and obtain:

$$S = \sum_n b_n(E_n) b_n(E_n) \ln(b_n(E_n)) + \sum_n b_n(E_n) b_n(E_n) \left\{ \sum_p a_n(p) a_n(p) \ln(a_n(p)) \right\} \\ + \sum_{x,t} W^*W \ln(W(x,t)) + \sum_{x,t,n,p} i(E_n - px) W b_n(E_n) a_n(p)$$

For simplicity, we consider  $b_n(E_n)$  and  $a_n(p)$  as being real.

For the case of a single  $n$  present (i.e. a pure eigenstate) one obtains the entropy of such a state and not zero as in the von Neuman or other approaches. We argue that it might be good to have a single entropy formulation which works for both a single energy eigenstate and multiple states and not the two seemingly independent approaches which seem to be generally used. Furthermore, the approach here includes the intrinsic entropy of pure states together with that of transitions between states in the general  $W(x,t)$  (wavefunction) case.

### Pure Energy Eigenstate

A pure energy eigenstate is treated as having entropy in (3), although in (1) and (2) its entropy is treated as 0. In fact in the literature it is treated (3) as having both spatial and momentum entropy given by Shannon's entropy equation:

$$\text{Spatial entropy} = \int dx W^*(x)W(x) \ln[W^*(x)W(x)] \quad ((1a))$$

$$\text{Momentum entropy} = \sum_p a^*(p)a(p) \ln[a^*(p)a(p)] \quad ((1b))$$

Here  $W(x)$  is really  $W_n(x)$  as it is associated with a single energy level and  $a_p$  is  $a_n(p)$ .

In a previous note (5), we argued that one requires a sum of spatial and momentum entropies in order to account for an adiabatic increase in time in the length of a box with infinite potential walls. In such a case, the entropy should remain constant and this requires two entropy terms

In (4), we argued that one may automatically obtain a sum of spatial and momentum entropies using:

Probability  $(x, p) = W(x) b(p) \exp(-ipx)$  ((2)) Here  $W=W_n$   
A cross term proportional to  $ipx$  appears, but integrates to zero.

### Time-Dependent Entropy

We speculate on generalizing the above approach to account for a time-dependent wavefunction, in particular:

$$W(x,t) = \sum_n b_n(E_n) \exp(-iE_n t) W_n(x) \quad ((3))$$

In such a case, one no longer has  $W_n(x)$ , but  $W_n(x,t)$  i.e. the  $x,t$  conditional probabilities are coupled. As a result, we suggest coupling  $b_n(E_n)$  and  $a_n(p)$ . In other words,  $a_n(p)$  is linked to the probability  $a_n(p)a_n(p)$  of finding a particle in a state  $p$  in a pure energy eigenstate associated with  $E_n$ . Thus, we use:

$$P(x,t, E_n, p) = W(x,t) b_n(E_n) a_n(p) \exp(i E_n t) \exp(-i p x) \quad ((4))$$

as the probability in Shannon's entropy. One may then sum over  $x,t, E_n$  and  $p$  using:

$$\sum_p a_n^*(p)a_n(p) = 1 \quad \text{and noting that } \int W_n^*(x)W_{n1}(x) dx = \delta(n, n1)$$

One finds (considering  $b_n(E_n)$  and  $a_n(p)$  to be real):

$$\begin{aligned} \text{Entropy} = & \sum_n b_n(E_n)b_n(E_n) \ln[b_n(E)] \\ & + \sum_n b_n(E_n)b_n(E_n) \{ \sum_p a_n(p)a_n(p) \ln(a_n(p)) \} \\ & + \sum_{x,t} W^*(x,t)W(x,t) \ln(W) \\ & + \sum_{x,t,p,n} i (E_n t - p x) W(x,t) b_n(E_n) a_n(p) \exp(i E_n t) \exp(-ipx) \end{aligned} \quad ((5))$$

In the case of a pure energy eigenstate associated with a single  $E_n$ , ((5)) becomes ((1a))+((1b)) plus a cross term. The  $px$  term of the cross term integrates to zero. If one considers both negative and positive times, the  $E_n t$  term integrates to 0 for the same reasons.

Thus, one essentially obtains a pure eigenstate entropy.

For the time dependent case, one has mixed time and spatial entropy linked with  $W(x,t)$ . Thus, there is a single entropy expression which works for both a pure energy eigenstate and a mixed, time dependent linear combination of energy eigenstates. One does not need two separate approaches to deal with these two.

## Conclusion

In conclusion, we consider modifying a conditional probability  $P(x,p) = W(x)a(p)\exp(-ipx)$  used together with Shannon's entropy expression to obtain the entropy of a pure eigenstate i.e.  $W=W_n$ . In the pure eigenstate case this yields a sum of spatial and momentum entropies plus a cross term which integrates to 0. To generalize to the  $W(x,t)$  case, we consider:  $P(x,t,p, E_n) = W(x,t) b_n(E_n)a_n(p) \exp(i E_n t) \exp(-i p x)$  with  $b_n$  and  $a_n$  being real for simplicity. Given that  $W(x,t)$  couples  $x$  and  $t$ , we couple  $b_n(E_n)a_n(p)$  for the same energy state. This yields an entropy given by ((5)) which contains a spatial/time portion of  $W^*W(x,t) \ln[W^*W]$ , an energy entropy piece:  $\sum b_n(E_n)b_n(E_n) \ln[b_n(E_n)]$  and a weighted momentum entropy term of  $\sum \text{over } n b_n(E_n)b_n(E_n) \{ \sum \text{over } p a_n(p)a_n(p) \ln(a_n(p)) \}$ .

As a result, one obtains an entropy equation which may be used for both a single energy eigenstate or a time-dependent linear combination of energy states. One does not need to use separate approaches e.g. an approach which handles multiple energy states, but yields an entropy of zero for a pure state (1)(2) and a separate approach for a pure energy eigenstate. Furthermore, the  $W(x,t)$  case contains both entropy associated with transitions between  $W_n(x)$  states and the intrinsic  $x$ - $p$  entropies of the  $W_n(x)$  state itself which are not included in treatments which consider a pure state to have an entropy of 0.

## References

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