

L^2 -HYPOCOERCIVITY AND LARGE TIME ASYMPTOTICS OF THE LINEARIZED VLASOV-POISSON-FOKKER-PLANCK SYSTEM

LANOIR ADDALA, JEAN DOLBEAULT, XINGYU LI, AND M. LAZHAR TAYEB

ABSTRACT. This paper is devoted to the linearized Vlasov-Poisson-Fokker-Planck system in presence of an external potential of confinement. We investigate the large time behaviour of the solutions using hypocoercivity methods and a notion of scalar product adapted to the presence of a Poisson coupling. Our framework provides estimates which are uniform in the diffusion limit. As an application in a simple case, we study the one-dimensional case and prove the exponential convergence of the nonlinear Vlasov-Poisson-Fokker-Planck system without any small mass assumption.

1. INTRODUCTION AND MAIN RESULTS

The *Vlasov-Poisson-Fokker-Planck system* in presence of an external potential V is

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f &= \Delta_v f + \nabla_v \cdot (v f), \\ -\Delta_x \phi &= \rho_f = \int_{\mathbb{R}^d} f \, dv. \end{aligned} \tag{VPFP}$$

In this paper, we shall assume that $(t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$ and that ϕ is a *self-consistent potential* corresponding to repulsive electrostatic forces and that V is a *confining potential* in the sense that (VPFP) admits, up to a multiplicative constant, a unique stationary solution

$$f_\star(x, v) = e^{-V(x) - \phi_\star(x)} \mathcal{M}(v), \quad -\Delta_x \phi_\star = e^{-V - \phi_\star} \quad \text{and} \quad \mathcal{M}(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}},$$

with associated potential ϕ_\star according, e.g., to [6, 27]. We denote by $M = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f_\star \, dx \, dv$ the mass. System (VPFP) is of interest for understanding the evolution of a system of charged particles with interactions of two different natures: a self-consistent, nonlinear interaction through the mean field potential ϕ and collisions with a background inducing a diffusion and a friction represented by a Fokker-Planck operator acting on velocities. System (VPFP) describes the dynamics of a plasma of Coulomb particles in a thermal reservoir (see for instance [10]), but it has also been derived in stellar dynamics for gravitational models, as in [24], in the case of an attractive mean field Newton-Poisson equation. Here we shall focus on the repulsive, electrostatic case. Applications range from plasma physics to semi-conductor modelling. A long standing open question is to get estimates on the rate of convergence to equilibrium in dimensions $d = 2$ and $d = 3$ for arbitrarily large initial data, away from equilibrium. We will not solve it here but, as an important step in this direction, we will establish a constructive estimate of the decay rate of the linearized

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problem, which provides us with an upper bound for the convergence rate of the nonlinear (VPFP) problem. A technical but important issue is to decide how one should measure such a rate of relaxation. For this purpose, we introduce a norm which is adapted to the linearized problem and consistent with the *diffusion limit*.

Let us consider the linearized problem around f_\star . Let h be a function such that $f = f_\star (1 + \eta h)$ with $\iint_{\mathbb{R}^d \times \mathbb{R}^d} f \, dx \, dv = M$, that is, such that $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star \, dx \, dv = 0$. The system (VPFP) can be rewritten as

$$\partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h = \eta (\nabla_x \psi_h \cdot \nabla_v h - v \cdot \nabla_x \psi_h h)$$

with $-\Delta_x \psi_h = \int_{\mathbb{R}^d} h f_\star \, dv$.

At formal level, by dropping the $\mathcal{O}(\eta)$ term in the limit as $\eta \rightarrow 0_+$, we obtain the *linearized Vlasov-Poisson-Fokker-Planck system* around the equilibrium state f_\star given by

$$(1) \quad \begin{aligned} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h &= 0, \\ -\Delta_x \psi_h &= \int_{\mathbb{R}^d} h f_\star \, dv, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star \, dx \, dv = 0. \end{aligned}$$

From now on we shall say that h has *zero average* if $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star \, dx \, dv = 0$. Let us define the norm

$$(2) \quad \|h\|^2 := \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 f_\star \, dx \, dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2 \, dx.$$

Our main result is devoted to the large time behaviour of a solution of *the linearized system* (1) on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \ni (t, x, v)$ with given initial datum h_0 at $t = 0$. For simplicity, we shall state a result for a simple specific potential, but an extension to more general potentials will be given to the price of a rather long list of technical assumptions that are detailed in Section 3.

Theorem 1. *Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. Then there exist two constants $\lambda > 0$ and $\mathcal{C} > 1$ such that any solution h of (1) with an initial datum h_0 of zero average with $\|h_0\|^2 < \infty$ is such that*

$$(3) \quad \|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0.$$

The constant \mathcal{C} in Theorem 1 is larger than 1 as a typical result of hypocoercivity methods. Indeed, since the Fokker-Planck operator acts only on the velocity variable v , an exponential decay with $\mathcal{C} = 1$ cannot be expected for generic x -dependent functions. The main novelty here is that hypocoercive estimates can be obtained in presence of the non-local Poisson coupling in (1), and not simply in some perturbative regime. The linearized problem (1) is at first sight easier than the full nonlinear system (VPFP) but our result gives two crucial informations which are of importance for the linearized system as well as for the nonlinear one: 1) we prove an exponential decay rate, 2) we specify an appropriate functional space and a notion of distance, corresponding to the norm defined by (2), for measuring the convergence to equilibrium.

Our analysis is consistent with the *diffusion limit* of the linearized system, as we shall explain below. For any $\varepsilon > 0$, we consider the solution of the *linearized problem in the parabolic scaling* given by

$$(4) \quad \begin{aligned} \varepsilon \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \frac{1}{\varepsilon} (\Delta_v h - v \cdot \nabla_v h) &= 0, \\ -\Delta_x \psi_h &= \int_{\mathbb{R}^d} h f_\star \, dv, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star \, dx \, dv = 0. \end{aligned}$$

In the regime $\varepsilon \rightarrow 0_+$, the parabolic scaling corresponds to a time scale of the order ε^{-1} with a collision frequency which is also of the order ε^{-1} , but cannot be achieved by a simple scaling in the (t, x, v) variables in presence of an external potential and a Stokes friction force. Our decay estimate is uniform with respect to $\varepsilon \rightarrow 0_+$.

Theorem 2. *Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. For any $\varepsilon > 0$ small enough, there exist two constants $\lambda > 0$ and $\mathcal{C} > 1$, which do not depend on ε , such that any solution h of (4) with an initial datum h_0 of zero average and such that $\|h_0\|^2 < \infty$ satisfies (3).*

The result of Theorem 1 will be extended in Theorem 21 to a larger class of external potentials V : in the technical part of the proof of Theorem 1, we will specify precise but more general conditions under which the same result holds. A similar extension applies in the case of Theorem 2. As an application of our method, we establish the exponential rate of convergence of the solution of the non-linear system (VPFP) when $d = 1$. For sake of simplicity, we state the result for the same potential V as in Theorem 2.

Corollary 3. *Assume that $d = 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. If f solves (VPFP) with initial datum $f_0 = (1 + h_0) f_\star$ such that h_0 has zero average, $\|h_0\|^2 < \infty$ and $(1 + h_0) \geq 0$, then (3) holds with $h = f / f_\star - 1$ for some constants $\lambda > 0$ and $\mathcal{C} > 1$.*

The *diffusion limit* of systems of kinetic equations in presence of electrostatic forces has been studied in many papers. The mathematical results go back at least to the study of a model for semi-conductors involving a linear Boltzmann kernel by F. Poupaud in [65]. The case of a Fokker-Planck operator in dimension $d = 2$ was later studied by F. Poupaud and J. Soler in [66], and by T. Goudon in [39], on the basis of the existence results of [61, 70]. There is also a parallel, probabilistic approach of the macroscopic diffusion limit and of the overdamped regime for the generalized Langevin equation: see [49, 62] and references therein. With a self-consistent Poisson coupling, we refer to [15] for existence results in dimension $d = 3$ and to [35, 27] for steady states, confinement and related issues. Based on free energy considerations introduced in [17, 27], N. El Ghani and N. Masmoudi were able in [37] to establish diffusion limits also when $d = 3$. Altogether, it is proved in dimensions $d = 2$ and $d = 3$ that the Vlasov-Poisson-Fokker-Planck system, with parameters corresponding to the parabolic scaling,

$$(5) \quad \varepsilon \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f = \frac{1}{\varepsilon} (\Delta_v f + \nabla_v \cdot (v f)), \quad -\Delta_x \phi = \rho_f = \int_{\mathbb{R}^d} f \, dv,$$

has a weak solution $(f^\varepsilon, \phi^\varepsilon)$ which converges as $\varepsilon \rightarrow 0_+$ to $(f^0 = \rho \mathcal{M}, \phi)$ where $\mathcal{M}(v) = (2\pi)^{-d/2} \exp(-|v|^2/2)$ is the normalized Maxwellian function and where the charge density $\rho = \int_{\mathbb{R}^d} f^0 \, dv$ is a weak solution of the *drift-diffusion-Poisson* system

$$(6) \quad \frac{\partial \rho}{\partial t} = \nabla_x \cdot (\nabla_x \rho + \rho \nabla_x (V + \phi)), \quad -\Delta_x \phi = \rho.$$

Another piece of information is the asymptotic behavior of the solutions of (6) for large times. As $t \rightarrow +\infty$, it is well known (see for instance [7] in the case of a bounded domain, [3] in the Euclidean case when $V(x) = |x|^2$, and [9] in \mathbb{R}^d with a confining external potential V for any $d \geq 3$) that the solution of (6) converges to a steady state (ρ_\star, ϕ_\star) given by

$$(7) \quad -\Delta_x \phi_\star = \rho_\star = e^{-V - \phi_\star}$$

at an exponential rate. The optimal asymptotic rates have been characterized recently in [54] using the linearized drift-diffusion-Poisson system and a norm which involves the

Poisson potential. Apart the difficulty arising from the self-consistent potential, the technique is based on relative entropy methods, which are by now standard in the study of large time asymptotics of drift-diffusion equations.

Our goal is to study both regimes $\varepsilon \rightarrow 0_+$ and $t \rightarrow +\infty$ simultaneously. More precisely, we aim at proving that each solution $(f^\varepsilon, \phi^\varepsilon)$ of (5) converges to (f_\star, ϕ_\star) as $t \rightarrow +\infty$ in a weighted L^2 sense at an exponential rate which is uniform in $\varepsilon > 0$, small. In the present paper, we will focus on a linearized regime in any dimension and obtain an estimate of the decay rate in the asymptotic regime. This allows us to obtain an asymptotic decay rates in the non-linear regime when $d = 1$, but so far not in higher dimensions because we are still lacking some key estimates. Compared to the large time asymptotics of (6), the study of the convergence rate of the solution of (5) or, in the case $\varepsilon = 1$, of the decay rate of the solution of (1), is much more difficult because the diffusion only acts on the velocities and requires the use of hypocoercive methods.

T. Gallay coined the word *hypocoercivity* in the context of convergence without regularization as opposed to hypoellipticity where both properties arise simultaneously. This concept is well adapted to kinetic equations with general collision kernels and C. Villani made the *hypocoercivity* very popular in kinetic theory: see [71, 72]. Understanding the large time behavior of the kinetic Fokker-Planck equation (without Poisson coupling) is an interesting problem which has a long history: see [52, 47, 50, 36, 43] for some earlier contributions. C. Villani [72] proved convergence results in various senses: in H^1 [72, Theorem 35], in L^2 [72, Theorem 37], and in entropy [72, Theorem 39] when $\text{Hess}(V)$ is bounded. His approach is however inspired by hypoelliptic methods, as in [41, 42, 60]. The method of [2] is based on a spectral decomposition and produces an exponential decay in relative entropy with a sharp rate. In a somewhat similar spirit, we can also quote [19], which is based on a Fourier decomposition. Due to the Fokker-Planck operator, smoothing effects in (5) can be expected as was proved in [16], consistently with hypoelliptic methods: this will not be exploited in the present paper.

Mean-field couplings add a serious difficulty: see [40, 57] for recent results based on a probabilistic approach. In presence of a Poisson coupling large time behavior (without rates) of the solutions of (5) has been dealt with in presence of or without an external potential: cf. [17, 23, 27, 22, 51] for early results. In [48], a result of exponential decay is obtained in dimension $d = 3$, in presence of a constant neutralizing background but without confinement: the solution is a smooth perturbation of a stationary distribution function which is homogeneous in x and Maxwellian in v and the proof relies on remarkable algebraic properties. When $d = 2$ and $d = 3$, F. Hérau and L. Thomann [44] proved the trend to the equilibrium for the Vlasov-Poisson-Fokker-Planck system with a small nonlinear term but with a possibly large exterior confining potential. More recently, M. Herda and M. Rodrigues considered in [45] the two limits as $\varepsilon \rightarrow 0_+$ and $t \rightarrow +\infty$, on the 2-dimensional torus, in the globally neutral regime. By a careful analysis of the trade-off between two parameters, the mean free path and the Debye length, they establish closed estimates of regularity which allow them to prove an exponential convergence, including in various limiting regimes, with uniform estimates in the other, fixed parameters. All these approaches are essentially of perturbative nature. In various papers, the properly linearized system (4) is not taken into account, in the sense that the non-local term arising from the Poisson equation is often dropped. In the case of a torus and without an external potential, the Landau damping provides another mechanism of convergence to equilibrium even without a Fokker-Planck kernel: we refer to [5] for a detailed study by J. Bedrossian on the

enhancement induced by the Fokker-Planck operator acting on velocities and also to a result of I. Tristani in [69] for the analysis of the consequences of the Landau damping on the (properly) linearized Vlasov–Poisson-Fokker-Planck system. So far it is not known how these properties could be extended from the setting of a torus to the case of the whole Euclidean space in presence of an external potential of confinement. Let us emphasize that, in the present paper, we consider the properly linearized system, including the non-local Poisson term, and provide a functional framework which is compatible with hypocoercivity methods adapted to diffusion limits.

The existence of solutions of (1), which are continuous w.r.t. t and take values in L^2 for the norm defined by (2), is out of the scope of this paper. Seen as a perturbation of (VPFP), an existence result can be deduced from the results of [70, 15] or established directly using the same methods as in these papers and we will consider it as granted. Alternatively, it is also possible to consider the non-local term as perturbation and use a fixed point argument based on the semi-group associated to the Fokker–Planck operator as, *e.g.*, in [44].

In [32], J. Dolbeault, C. Mouhot, and C. Schmeiser studied the exponential decay in a modified L^2 norm for the Vlasov-Fokker-Planck equation (and also for a larger class of linear kinetic equations). The method was motivated by the results of [41] but the main source of inspiration came from the analysis of the diffusion limit, as in [8, 58, 30] (also see [68] in presence of an oscillating external force field): the general idea is to build a norm which reflects the spectral gap that determines the rate of convergence in (6) by adding a twist which arises from the coercivity properties, at macroscopic level, of the diffusion limit. Applying [32] to (1) is a natural idea, which is mentioned for instance in [69, p. 109], but has not been done yet to our knowledge. Inspired by [11, 13, 33], another idea emerged that asymptotic rates of convergence should be measured in a norm induced by a Taylor expansion of the entropy around the asymptotic state and that, in presence of a Poisson coupling, this norm should involve a non-local term: see [20, 54, 55] for drift-diffusion systems and Remark 1 when applied to (VPFP). The goal of this paper is to mix these two ideas. It turns out that they combine into a beautiful machinery.

This paper is organized as follows. In Section 2, we expose the strategy for the L^2 -hypocoercivity method of [32] in the abstract setting of a general Hilbert space. The notion of Hilbert space adapted to (1) is exposed in Section 3 with some fundamental considerations on confinement by an external potential and adapted Poincaré inequalities. Section 4 is devoted to the proof of Theorem 1: we have to check that the assumptions of Section 2 hold in the functional setting of Section 3, with the special scalar product for Poisson coupling involving a non-local term associated with the norm defined by (2). In Section 5, we prove Theorem 2: our estimates are compatible with the diffusion limit as $\varepsilon \rightarrow 0$. Coming back to the non-linear problem (VPFP) in dimension $d = 1$, we prove in this latter case that an exponential rate of convergence as $t \rightarrow +\infty$ can be measured in the hypocoercive norm, that is, we prove Corollary 3.

We shall adopt the following conventions. If $\mathbf{a} = (a_i)_{i=1}^d$ and $\mathbf{b} = (b_i)_{i=1}^d$ are two vectors with values in \mathbb{R}^d , then $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^d a_i b_i$ and $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$. If $\mathbf{A} = (A_{ij})_{i,j=1}^d$ and $\mathbf{B} = (B_{ij})_{i,j=1}^d$ are two matrices with values in $\mathbb{R}^d \times \mathbb{R}^d$, then $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^d A_{ij} B_{ij}$ and $|\mathbf{A}|^2 = \mathbf{A} : \mathbf{A}$. We shall use the tensor convention that $\mathbf{a} \otimes \mathbf{b}$ is the matrix of elements $a_i b_j$. By extension to functions, $\nabla_x w$ is the gradient of a scalar function w while $\nabla_x \cdot \mathbf{u}$ denotes the divergence of a vector valued function $\mathbf{u} = (u_i)_{i=1}^d$ and $\nabla_x \otimes \mathbf{u}$ is the matrix valued function of elements $\partial u_i / \partial x_j$.

Hence

$$\text{Hess}(w) = (\nabla_x \otimes \nabla_x)w = \left(\frac{\partial^2 w}{\partial x_i \partial x_j} \right)_{i,j=1}^d$$

denotes the Hessian of w and, for instance, $u \otimes u : \text{Hess}(w) = \sum_{i,j=1}^d u_i u_j (\text{Hess}(w))_{ij}$. We shall also write that $|\text{Hess}(w)|^2 = \text{Hess}(w) : \text{Hess}(w)$.

2. HYPOCOERCIVITY RESULT AND DECAY RATES

This section is devoted to the abstract hypocoercivity method in general Hilbert spaces and it is inspired from [32, 19]. Since the methods sets the overall strategy of proof of our main results, we expose it for the convenience of the reader.

Let us consider the evolution equation

$$(8) \quad \frac{dF}{dt} + \mathbb{T}F = \mathbb{L}F$$

on a Hilbert space \mathcal{H} . In view of the applications, we shall call \mathbb{T} and \mathbb{L} the *transport* and the *collision* operators and assume without further notice that they are respectively antisymmetric and symmetric, and both time-independent. On \mathcal{H} , we shall denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the scalar product and the norm. As in [32], we assume that there are positive constants λ_m, λ_M , and C_M such that, for any $F \in \mathcal{H}$, the following properties hold:

▷ *microscopic coercivity*

$$(H1) \quad - \langle \mathbb{L}F, F \rangle \geq \lambda_m \|(\text{Id} - \Pi)F\|^2,$$

▷ *macroscopic coercivity*

$$(H2) \quad \|\mathbb{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2,$$

▷ *parabolic macroscopic dynamics*

$$(H3) \quad \Pi \mathbb{T} \Pi F = 0,$$

▷ *bounded auxiliary operators*

$$(H4) \quad \|\mathbb{A}\mathbb{T}(\text{Id} - \Pi)F\| + \|\mathbb{A}\mathbb{L}F\| \leq C_M \|(\text{Id} - \Pi)F\|.$$

Here Id is the identity, Π is the orthogonal projection onto the null space of \mathbb{L} , $*$ denotes the adjoint with respect to $\langle \cdot, \cdot \rangle$ and as in [31, 32], the operator \mathbb{A} is defined by

$$\mathbb{A} := (\text{Id} + (\mathbb{T}\Pi)^* \mathbb{T}\Pi)^{-1} (\mathbb{T}\Pi)^*.$$

Since a solution F of (8) obeys to

$$\frac{1}{2} \frac{d}{dt} \|F\|^2 = \langle \mathbb{L}F, F \rangle \leq -\lambda_m \|(\text{Id} - \Pi)F\|^2,$$

this is not enough to conclude that $\|F(t, \cdot)\|^2$ decays exponentially with respect to $t \geq 0$ and this is why we shall consider the Lyapunov functional

$$\mathbb{H}_\delta[F] := \frac{1}{2} \|F\|^2 + \delta \langle \mathbb{A}F, F \rangle$$

for some $\delta > 0$ to be determined later. If F solves (8), then

$$-\frac{d}{dt} \mathbb{H}_\delta[F] = \mathbb{D}_\delta[F] := - \langle \mathbb{L}F, F \rangle + \delta \langle \mathbb{A}\mathbb{T}\Pi F, F \rangle - \delta \langle \mathbb{T}\mathbb{A}F, F \rangle + \delta \langle \mathbb{A}\mathbb{T}(\text{Id} - \Pi)F, F \rangle - \delta \langle \mathbb{A}\mathbb{L}F, F \rangle$$

using $A = \Pi A$. Let us define

$$\delta_* = \min \left\{ 2, \lambda_m, \frac{4 \lambda_m \lambda_M}{4 \lambda_M + C_M^2 (1 + \lambda_M)} \right\}.$$

We recall that the two main properties of the *hypocoercivity* method of [32] for real valued operators and later extended in [19] to complex Hilbert spaces go as follows.

Proposition 4. *Assume that (H1)–(H4) hold and take $\delta \in (0, \delta_*)$. Then we have:*

(i) H_δ and $\|\cdot\|^2$ are equivalent in the sense that

$$(9) \quad \frac{2 - \delta}{4} \|F\|^2 \leq H_\delta[F] \leq \frac{2 + \delta}{4} \|F\|^2 \quad \forall F \in \mathcal{H}.$$

(ii) For some $\lambda > 0$ depending on δ , H_δ and D_δ are related by the entropy – entropy production inequality

$$(10) \quad \lambda H_\delta[F] \leq D_\delta[F] \quad \forall F \in \mathcal{H}.$$

As a consequence, a solution F of (8) with initial datum F_0 obeys to

$$H_\delta[F(t, \cdot)] \leq H_\delta[F_0] e^{-\lambda t}$$

and

$$(11) \quad \|F(t, \cdot)\|^2 \leq \frac{4}{2 - \delta} H_\delta[F(t, \cdot)] \leq \frac{4}{2 - \delta} H_\delta[F_0] e^{-\lambda t} \leq \frac{2 + \delta}{2 - \delta} \|F_0\|^2 e^{-\lambda t} \quad \forall t \geq 0.$$

Proof. For completeness, we sketch the main steps of the proof, with slightly improved estimates compared to [19, Theorem 3]. Since $A \Pi$ can be viewed as $z \mapsto (1 + z)^{-1} z$ applied to $(\Pi)^* \Pi$, (H1) and (H2) imply that

$$-\langle LF, F \rangle + \delta \langle A \Pi F, F \rangle \geq \lambda_m \|(\text{Id} - \Pi)F\|^2 + \frac{\delta \lambda_M}{1 + \lambda_M} \|\Pi F\|^2.$$

Our goal is to prove that the r.h.s. controls the other terms in the expression of $D_\delta[F]$. By (H4), we know that

$$|\langle A \Pi (\text{Id} - \Pi)F, F \rangle + \langle ALF, F \rangle| \leq C_M \|\Pi F\| \|(\text{Id} - \Pi)F\|.$$

As in [32, Lemma 1], if $G = AF$, i.e., if $(\Pi)^* F = G + (\Pi)^* \Pi G$, then

$$\langle TAF, F \rangle = \langle G, (\Pi)^* F \rangle = \|G\|^2 + \|\Pi G\|^2 = \|AF\|^2 + \|TAF\|^2.$$

By the Cauchy-Schwarz inequality, we know that

$$\langle G, (\Pi)^* F \rangle = \langle TAF, (\text{Id} - \Pi)F \rangle \leq \|TAF\| \|(\text{Id} - \Pi)F\| \leq \frac{1}{2\mu} \|TAF\|^2 + \frac{\mu}{2} \|(\text{Id} - \Pi)F\|^2$$

for any $\mu > 0$. Hence

$$2 \|AF\|^2 + \left(2 - \frac{1}{\mu}\right) \|TAF\|^2 \leq \mu \|(\text{Id} - \Pi)F\|^2,$$

which, by taking either $\mu = 1/2$ or $\mu = 1$, proves that

$$(12) \quad \|AF\| \leq \frac{1}{2} \|(\text{Id} - \Pi)F\| \quad \text{and} \quad \|TAF\| \leq \|(\text{Id} - \Pi)F\|.$$

Hence

$$|\langle AF, F \rangle| \leq \frac{1}{2} X Y \leq \frac{1}{4} (X^2 + Y^2)$$

with $X := \|(\text{Id} - \Pi)F\|$ and $Y := \|\Pi F\|$ because A takes values in $\Pi \mathcal{H}$. This establishes (9) and, as a side result, also proves that

$$|\langle TAF, F \rangle| = |\langle TAF, (\text{Id} - \Pi)F \rangle| \leq \|(\text{Id} - \Pi)F\|^2.$$

Collecting terms in the expression of $D_\delta[F]$, we find that

$$D_\delta[F] \geq (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y$$

with $X := \|(\text{Id} - \Pi)F\|$ and $Y := \|\Pi F\|$. We know that $H_\delta[F] \leq \frac{1}{2}(X^2 + Y^2) + \frac{\delta}{2} X Y$, so that the largest value of λ for which $D_\delta[F] \geq \lambda H_\delta[F]$ can be estimated by the largest value of λ for which

$$(X, Y) \mapsto (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y - \frac{\lambda}{2} (X^2 + Y^2) - \frac{\lambda}{2} \delta X Y = (X, Y) M (X, Y)^T$$

is a nonnegative quadratic form, as a function of (X, Y) , where

$$M := \begin{pmatrix} \lambda_m - \delta - \frac{\lambda}{2} & -\frac{\delta}{2} \left(C_M + \frac{\lambda}{2} \right) \\ -\frac{\delta}{2} \left(C_M + \frac{\lambda}{2} \right) & \frac{\delta \lambda_M}{1 + \lambda_M} - \frac{\lambda}{2} \end{pmatrix}.$$

It is characterized by the discriminant condition

$$h(\delta, \lambda) := \delta^2 \left(C_M + \frac{\lambda}{2} \right)^2 - 4 \left(\lambda_m - \delta - \frac{\lambda}{2} \right) \left(\frac{\delta \lambda_M}{1 + \lambda_M} - \frac{\lambda}{2} \right) \leq 0$$

and the sign condition $\lambda_m - \delta - \lambda/2 > 0$. For any $\delta \in (0, \delta_\star)$, $\lambda \mapsto h(\delta, \lambda)$ is a second order polynomial such that $h(\delta, 0) > 0$ and $\lim_{\lambda \rightarrow +\infty} h(\delta, \lambda) = -\infty$. Hence, the largest possible value of λ can be estimated by the positive root of $h(\delta, \lambda) = 0$, for any given $\delta \in (0, \delta_\star)$. \square

Notice that the proof of Proposition 4 provides us with a constructive estimate of the decay rate λ , as a function of $\delta \in (0, \delta_\star)$. We refer to [1] for a discussion of the best estimate of the decay rate of H_δ , *i.e.*, the largest possible estimate of λ when δ varies in the admissible range $(0, \delta_\star)$.

3. FUNCTIONAL SETTING

In this section, we collect some observations on the external potential V and on the stationary solution obtained by solving the Poisson-Boltzmann equation. Depending on growth conditions on V , we establish a notion of *confinement* (so that (VPFP) admits an integrable stationary solution) and coercivity properties (which amount to Poincaré type inequalities). Our goal is to give sufficient conditions in order that:

- 1) there exists a nonnegative stationary solution f_\star of (VPFP) of arbitrary given mass $M > 0$: see Section 3.2;
- 2) there is a Poincaré inequality associated with the measure $e^{-V - \phi_\star} dx$ on \mathbb{R}^d , and variants of it, with weights: see Section 3.3;
- 3) there is a Hilbert space structure on which we can study (1): see Section 3.6.

These conditions on V determine a functional setting which is adapted to implement the method of Section 2. The potential $V(x) = |x|^\alpha$ with $\alpha > 1$ is an *admissible potential* in that perspective.

In [32], without Poisson coupling, sufficient conditions were given on V which were inspired by the *carré du champ* method and the Holley-Stroock perturbation lemma (see [46] and [28] for related results). These conditions are not well adapted to handle an additional Poisson coupling. Here we adopt a slightly different approach, which amounts to focus on sufficient growth conditions of the external potential V and on tools of spectral theory like

Persson's lemma. For sake of simplicity, we require some basic regularity properties of V and assume that

$$(V1) \quad V \in C^0 \cap W_{\text{loc}}^{2,1}(\mathbb{R}^d) \quad \text{and} \quad \liminf_{|x| \rightarrow +\infty} V(x) = +\infty.$$

These regularity assumptions and the growth conditions on V (also see below) could be relaxed, up to additional technicalities.

3.1. Preliminary considerations on the Poisson equation and conventions. Let us consider the Green function G_d associated with $-\Delta_x$. We shall write $\phi = (-\Delta_x)^{-1} \rho$ as a generic notation for $\phi = G_d * \rho$ with $G_d(x) = c_d |x|^{2-d}$, $c_d^{-1} = (d-2) |\mathbb{S}^{d-1}|$ if $d \geq 3$. Then, if $d \geq 3$, with no further restriction, by using integrations by parts, we have that

$$\int_{\mathbb{R}^d} \rho \phi \, dx = \int_{\mathbb{R}^d} (-\Delta_x \phi) \phi \, dx = \int_{\mathbb{R}^d} |\nabla_x \phi|^2 \, dx.$$

If $d = 2$, we use $G_2(x) = -\frac{1}{2\pi} \log|x|$. It is a standard observation that $\phi = (-\Delta_x)^{-1} \rho$ is such that $\nabla_x \phi(x) = -\frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \rho \, dx \right) \frac{x}{|x|^2}$ as $|x| \rightarrow +\infty$ is not square integrable unless $\int_{\mathbb{R}^2} \rho \, dx = 0$. If $\int_{\mathbb{R}^2} \rho \, dx = 0$, one can prove that

$$\int_{\mathbb{R}^2} \rho \phi \, dx = \int_{\mathbb{R}^2} |\nabla_x \phi|^2 \, dx < +\infty.$$

If $d = 1$, we can use $G_1(x) = -|x|/2$, but it is sometimes more convenient to rely on the representation

$$(13) \quad \phi(x) = \frac{M}{2} x - \int_{-\infty}^x dy \int_{-\infty}^y \rho(z) \, dz + \phi_0$$

and we shall then write $\phi = (-d^2/dx^2)^{-1} \rho$ whenever we use (13). Here ϕ_0 is a free parameter that we may fix by assuming that $\max_{x \in \mathbb{R}} \phi(x) = 0$. This convention does not coincide with the representation by $G_1 * \rho$ as, in that case, $\phi(0)$ is taken equal to $-\frac{1}{2} \int_{\mathbb{R}} |x| \rho(x) \, dx$, provided this quantity is well defined. Obviously, the potential is defined up to an additive constant and this is therefore not an issue. Without further notice, we will rely on (13) for the solutions when $d = 1$.

If $d = 1$, more important is the fact that $\phi = (-d^2/dx^2)^{-1} \rho$ satisfies $\phi' = -m$ where m is defined by $m(x) := \int_{-\infty}^x \rho(y) \, dy$ if $M = \int_{\mathbb{R}} \rho \, dx = 0$. In that case, if we further assume that ρ is compactly supported or has a sufficient decay at infinity, an integration by parts shows that

$$(14) \quad \int_{\mathbb{R}} \phi \rho \, dx = - \int_{\mathbb{R}} \phi' m \, dx = \int_{\mathbb{R}} |\phi'|^2 \, dx = \int_{\mathbb{R}} m^2 \, dx \geq 0.$$

Altogether, whenever $\int_{\mathbb{R}^d} \rho \, dx = 0$, we shall write $\int_{\mathbb{R}^d} \rho \phi \, dx = \int_{\mathbb{R}^d} |\nabla_x \phi|^2 \, dx \geq 0$ without any further precaution, for any $d \geq 1$.

3.2. The Poisson-Boltzmann equation. According to [35, 70, 27], stationary solutions of the (VPFP) system are given by

$$f_{\star}(x, v) = \rho_{\star}(x) \mathcal{M}(v)$$

where $\mathcal{M}(v) = (2\pi)^{-d/2} e^{-|v|^2/2}$ is the normalized Maxwellian function (or Gaussian function) and the spatial density ρ_{\star} is determined by the Poisson-Boltzmann equation

$$-\Delta_x \phi_{\star} = \rho_{\star} = M \frac{e^{-V-\phi_{\star}}}{\int_{\mathbb{R}^d} e^{-V-\phi_{\star}} \, dx}.$$

This equation also appears in the literature as the *Poisson-Emden equation*. It is obvious that ϕ_\star is defined up to an additive constant which can be chosen such that $M = \int_{\mathbb{R}^d} e^{-V-\phi_\star} dx$ and therefore solves (7). Here $\|\rho_\star\|_{L^1(\mathbb{R}^d)} = \|f_\star\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = M$ is the mass, which is a free parameter of the problem, which can be adjusted by adding a constant to V . The critical growth of V such that there are solutions $\rho_\star \in L^1(\mathbb{R}^d)$ of (7) which minimize the free energy strongly depends on the dimension. Here are some sufficient conditions.

Lemma 5. *Let $M > 0$. Assume that V satisfies (V1) and*

$$(V2) \quad \begin{aligned} |V| e^{-V} &\in L^1(\mathbb{R}^d) && \text{if } d \geq 3, \\ \liminf_{|x| \rightarrow +\infty} \frac{V(x)}{\log|x|} &> 2 + \frac{M}{2\pi} && \text{if } d = 2, \\ \liminf_{|x| \rightarrow +\infty} \frac{V(x) - M|x|/2}{\log|x|} &> 2 && \text{if } d = 1. \end{aligned}$$

Then (7) has a unique solution $\rho_\star \in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \rho_\star dx = M$ and ϕ_\star is the unique solution of (7). Moreover ϕ_\star is of class C^2 and $\liminf_{|x| \rightarrow +\infty} W_\star(x) = +\infty$, where

$$W_\star = V + \phi_\star \quad \text{and} \quad \rho_\star = e^{-W_\star}.$$

As a consequence of Lemma 5, we learn that under Assumptions (V1) and (V2), the potential W_\star also satisfies (V1). Regularity results on (7) are scattered in the literature. See for instance [44, Proposition 3.5]. The general strategy is, as usual, to use the fact that the solution is in the energy space and the equation to obtain uniform estimates by elliptic bootstrapping. The regularity and decay estimates as $|x| \rightarrow +\infty$ follow respectively from the regularity of V and from its growth properties, using a representation of the solution based on the Green function. This is again classical and details will be omitted here.

Proof. The case $d \geq 3$ is covered by [27, p. 123]. The free energy

$$\mathcal{F}[\rho] := \int_{\mathbb{R}^d} \rho \log \rho dx + \int_{\mathbb{R}^d} \rho V dx + \frac{1}{2} \int_{\mathbb{R}^d} \rho \phi dx$$

is bounded from below under the mass constraint $\int_{\mathbb{R}^d} \rho dx = M$ using the fact that

$$\int_{\mathbb{R}^d} \rho \phi dx = \int_{\mathbb{R}^d} |\nabla_x \phi|^2 dx \geq 0$$

and, with $u := \rho e^V$, the convexity estimate

$$\mathcal{F}[\rho] \geq \int_{\mathbb{R}^d} \rho \log \rho dx + \int_{\mathbb{R}^d} \rho V dx = \int_{\mathbb{R}^d} \underbrace{(u \log u - u + 1)}_{\geq 0} e^{-V} dx + M - \int_{\mathbb{R}^d} e^{-V} dx.$$

A slightly more accurate estimate is obtained using Jensen's inequality applied to $u \log u$ with the measure $e^{-V} dx / \int_{\mathbb{R}^d} e^{-V} dx$. The existence follows by a minimization method. The Euler-Lagrange equation reads

$$1 + \log \rho + V + \phi = C$$

for some $C \in \mathbb{R}$ which is the Lagrange multiplier associated to the mass constraint. Hence $\rho = e^{C-1} e^{-(V+\phi)}$ and we deduce from the mass constraint $M = \int_{\mathbb{R}^d} \rho dx$ that $C = 1 + \log M - \log \int_{\mathbb{R}^d} e^{-(V+\phi)} dx$. As noticed in [38, 26], the uniqueness is a consequence of the convexity of \mathcal{F} . Finally, by standard elliptic regularity, $\phi_\star = (-\Delta_x)^{-1} \rho_\star$ is continuous and has a limit as $|x| \rightarrow +\infty$.

In dimension $d = 1$ or $d = 2$, the same scheme can be adapted after proving that \mathcal{F} is bounded from below. This has been established in [29, Theorem 3.4] (also see [54]) when

$d = 2$ under Assumption (V2). The case $d = 1$ can be dealt with by elementary methods, as follows. Let us consider the potential

$$V_0(x) = \frac{M}{2} \left((x+1) \mathbb{1}_{(-\infty, -1)}(x) + \frac{1}{2} (x+1)(1-x) \mathbb{1}_{[-1, +1]}(x) - (x-1) \mathbb{1}_{(+1, +\infty)}(x) - \frac{1}{2} \right)$$

such that, a.e., $-V_0'' = \frac{M}{2} \mathbb{1}_{[-1, +1]} =: \rho_0$ and let $\psi := \phi - V_0$. We claim that

$$\mathcal{F}[\rho] = \int_{\mathbb{R}} \rho \log \rho \, dx + \int_{\mathbb{R}} \rho (V + V_0) \, dx - \frac{1}{2} \int_{\mathbb{R}} \psi'' \psi \, dx + \frac{1}{2} \int_{\mathbb{R}} \rho_0 \psi \, dx - \frac{1}{2} \int_{\mathbb{R}} \rho V_0 \, dx$$

is bounded from below because the first two integrals can be bounded using Jensen's inequality and we have $\int_{\mathbb{R}} \psi'' \psi \, dx = -\int_{\mathbb{R}} |\psi'|^2 \, dx$. The balance between the last two terms is slightly more subtle. Let $x_0 \in \mathbb{R}$ be such that $\phi(x_0) = M/2$. Using (13), we notice that $\phi'(x_0) = 0$ and obtain

$$\phi(x) = - \int_{x_0}^x dy \int_{x_0}^y \rho(z) \, dz$$

because $\phi(x_0) = \phi_0 = 0$. A crude consequence is the estimate $-\frac{M}{2} |x - x_0| \leq \phi(x) \leq 0$ for any $x \in \mathbb{R}$, which shows that

$$\int_{\mathbb{R}} \rho_0 \psi \, dx \geq \int_{\mathbb{R}} \rho_0 V_0 \, dx + \frac{M}{4} \int_{-1}^{+1} \phi(x) \, dx \geq \int_{\mathbb{R}} \rho_0 V_0 \, dx - \frac{M^2}{8} \int_{-1}^{+1} |x - x_0| \, dx,$$

that is, a lower bound of the order of $|x_0|$ as $|x_0| \rightarrow +\infty$. On the other hand, if $|x_0| > 1$, we have

$$- \int_{\mathbb{R}} \rho V_0 \, dx \geq -\frac{M}{4} \int_{-1}^{+1} \rho(x) \, dx + \int_{|x| \geq |x_0|} \rho V_0 \, dx \geq -\frac{M^2}{2} + \frac{M}{2} V_0(x_0),$$

which is of the order of $V_0(x_0)$ and dominates $|x_0|$ as $|x_0| \rightarrow +\infty$, by Assumption (V2). Combining these estimates provides us with the lower bound we need. \square

3.3. Some non-trivial Poincaré inequalities. Assume that V is such that (V1)-(V2) hold. Before considering the case of the measure $e^{-W_\star} dx$ on \mathbb{R}^d , with $W_\star = V + \phi_\star$, we may ask under which conditions on V the *Poincaré inequality*

$$(15) \quad \int_{\mathbb{R}^d} |\nabla_x u|^2 e^{-V} \, dx \geq \mathcal{C}_P \int_{\mathbb{R}^d} |u|^2 e^{-V} \, dx \quad \forall u \in H^1(\mathbb{R}^d) \quad \text{such that} \quad \int_{\mathbb{R}^d} u e^{-V} \, dx = 0$$

is true for some constant $\mathcal{C}_P > 0$. Let us define $w = u e^{-V/2}$ and observe that (15) is equivalent to

$$\int_{\mathbb{R}^d} |\nabla_x w|^2 \, dx + \int_{\mathbb{R}^d} \Phi |w|^2 \, dx \geq \mathcal{C}_P \int_{\mathbb{R}^d} |w|^2 \, dx$$

under the condition that $\int_{\mathbb{R}^d} w e^{-V/2} \, dx = 0$. Here $\Phi = \frac{1}{4} |\nabla_x V|^2 - \frac{1}{2} \Delta_x V$ is obtained by expanding the square in $\int_{\mathbb{R}^d} |\nabla_x w + \frac{1}{2} w \nabla_x V|^2 \, dx$ and integrating by parts the cross-term. From the expression of the square, we learn that the kernel of the Schrödinger operator $-\Delta_x + \Phi$ on $L^2(\mathbb{R}^d, dx)$ is generated by $e^{-V/2}$. According to Persson's result [63, Theorem 2.1], the lower end σ of the continuous spectrum of the Schrödinger operator $-\Delta_x + \Phi$ is such that

$$\sigma \geq \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} \Phi(x) =: \sigma_0.$$

As a consequence, if σ is positive, either there is no eigenvalue in the interval $(0, \sigma)$ and $\mathcal{C}_P = \sigma$, or \mathcal{C}_P is the lowest positive eigenvalue, and it is positive by construction. In both

cases, we know that (15) holds for some $\mathcal{C}_p > 0$ if $\sigma_0 > 0$. In order to prove (15), it is enough to check that

$$(V3a) \quad \sigma_V := \liminf_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left(\frac{1}{4} |\nabla_x V|^2 - \frac{1}{2} \Delta_x V \right) > 0 \quad \text{and} \quad \liminf_{r \rightarrow +\infty} \inf_{x \in B_r^c} |\nabla_x V| > 0.$$

See for instance [4, 18] for further considerations on Assumption (V3a). Now let us consider the measure $\rho_\star dx = e^{-W_\star} dx$ on \mathbb{R}^d and establish the corresponding Poincaré inequality.

Lemma 6. *Assume that $d \geq 1$ and consider V such that (V1), (V2) and (V3a) hold. We further assume that*

$$(V4) \quad \liminf_{r \rightarrow +\infty} \inf_{|x| > r} \left((M - 2V')^2 - 2V'' \right) > 0 \quad \text{if } d = 1.$$

If ϕ_\star solves (7) and $W_\star = V + \phi_\star$, then there is a positive constant \mathcal{C}_\star such that

$$(16) \quad \int_{\mathbb{R}^d} |\nabla_x u|^2 \rho_\star dx \geq \mathcal{C}_\star \int_{\mathbb{R}^d} |u|^2 \rho_\star dx \quad \forall u \in H^1(\mathbb{R}^d) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} u \rho_\star dx = 0.$$

Proof. In order to adapt the result for V to W_\star , it is enough to prove that

$$\sigma_{W_\star} := \liminf_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left(\frac{1}{4} |\nabla_x \phi_\star + \nabla_x V|^2 - \frac{1}{2} (\Delta_x \phi_\star + \Delta_x V) \right) > 0.$$

By Lemma 5 and (V3a), $|\Delta_x \phi_\star| = \rho_\star = o(|\nabla_x V|^2 - 2\Delta_x V)$. Using $\nabla_x \phi_\star = \nabla G_d * \rho_\star$, we obtain that $|\nabla_x \phi_\star| = \mathcal{O}(|x|^{1-d})$ is negligible compared to $|\nabla_x V|$ if $d \geq 2$. If $d = 1$, the result follows from (V4) using the fact that $\phi_\star'(x) \sim \pm M/2$ as $x \rightarrow \mp\infty$. \square

We shall now replace (V3a) by the slightly stronger assumption that, for some $\theta \in [0, 1)$,

$$(V3b) \quad \liminf_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left(\frac{\theta}{4} |\nabla_x V|^2 - \frac{1}{2} \Delta_x V \right) \geq 0 \quad \text{and} \quad \liminf_{r \rightarrow +\infty} \inf_{x \in B_r^c} |\nabla_x V| > 0.$$

Corollary 7. *Assume that $d \geq 1$ and consider V such that (V1), (V2), (V3b) and (V4) hold. If ϕ_\star solves (7) and $W_\star = V + \phi_\star$, then there is a positive constant \mathcal{C} such that*

$$(17) \quad \int_{\mathbb{R}^d} |\nabla_x u|^2 \rho_\star dx \geq \mathcal{C} \int_{\mathbb{R}^d} |u|^2 |\nabla_x W_\star|^2 \rho_\star dx \quad \forall u \in H^1(\mathbb{R}^d) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} u \rho_\star dx = 0.$$

Proof. By expanding $|\nabla_x (u \sqrt{\rho_\star})|^2$, using $\nabla_x \sqrt{\rho_\star} = -\frac{1}{2} \nabla_x W_\star \sqrt{\rho_\star}$ and integrating by parts, we obtain that

$$0 \leq \int_{\mathbb{R}^d} |\nabla_x (u \sqrt{\rho_\star})|^2 dx = \int_{\mathbb{R}^d} |\nabla_x u|^2 \rho_\star dx - \int_{\mathbb{R}^d} \left(\frac{1}{4} |\nabla_x W_\star|^2 - \frac{1}{2} \Delta_x W_\star \right) |u|^2 \rho_\star dx.$$

Combined with (16), this shows that

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla_x u|^2 \rho_\star dx &\geq \int_{\mathbb{R}^d} \left[(1-\eta) \mathcal{C}_\star + \eta \left(\frac{\theta}{4} |\nabla_x W_\star|^2 - \frac{1}{2} \Delta_x W_\star \right) \right] |u|^2 \rho_\star dx \\ &\quad + \frac{\eta}{4} (1-\theta) \int_{\mathbb{R}^d} |u|^2 |\nabla_x W_\star|^2 \rho_\star dx \end{aligned}$$

for any $\eta \in (0, 1)$. With η chosen small enough so that $(1-\eta) \mathcal{C}_\star + \eta \left(\frac{\theta}{4} |\nabla_x W_\star|^2 - \frac{1}{2} \Delta_x W_\star \right)$ is nonnegative a.e., the conclusion holds with $\mathcal{C} = \eta(1-\theta)/4$. \square

In the same spirit as for Corollary 7, we shall assume that for some $\theta \in [0, 1)$,

$$(V5) \quad \liminf_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left(\frac{\theta}{4} |\nabla_x V|^4 - \frac{1}{2} \Delta_x V |\nabla_x V|^2 - \text{Hess}(V) : \nabla_x V \otimes \nabla_x V \right) \geq 0$$

and $\liminf_{r \rightarrow +\infty} \inf_{x \in B_r^c} |\nabla_x V| > 0.$

Corollary 8. *Assume that $d \geq 1$ and consider V such that (V1), (V2), (V3b) and (V5) hold. If ϕ_\star solves (7) and $W_\star = V + \phi_\star$, then there is a positive constant \mathcal{C}_\circ such that*

$$\int_{\mathbb{R}^d} |\nabla_x u|^2 |\nabla_x W_\star|^2 \rho_\star dx \geq \mathcal{C}_\circ \int_{\mathbb{R}^d} |u|^2 |\nabla_x W_\star|^4 \rho_\star dx \quad \forall u \in H^1(\mathbb{R}^d) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} u \rho_\star dx = 0.$$

The proof is again based on the expansion of the square in $|\nabla_x (u \sqrt{\rho_\star})|^2 |\nabla_x W_\star|^2$, integrations by parts and an IMS (for Ismagilov, Morgan, Morgan-Simon, Sigal) truncation argument in order to use Lemma 6 in a finite centered ball of radius $2R$, on which $\nabla_x W_\star$ is bounded and Assumption (V5) outside of the centered ball of radius R . See [59, 67] or [14, section 2] for details on the IMS truncation method. Compared with Corollary 7, there is no deeper difficulty and we shall skip further details.

3.4. Further inequalities based on pointwise estimates. If \mathfrak{M} is a $d \times d$ symmetric real valued matrix, let us denote by $\Lambda(\mathfrak{M})$ the largest eigenvalue of \mathfrak{M} . With this notation, let us assume that

$$(V6) \quad \Lambda_V := \lim_{r \rightarrow +\infty} \sup_{x \in B_r^c} \frac{1}{|\nabla_x V(x)|^2} \Lambda \left(e^{V(x)} \left(\text{Hess}(e^{-V(x)}) - \frac{1}{2} \Delta_x(e^{-V(x)}) \text{Id} \right) \right) < +\infty.$$

In other words, Assumption (V6) means that for any $\varepsilon > 0$, there exists some $R > 0$ such that

$$e^{V(x)} \left(\text{Hess}(e^{-V(x)}) - \frac{1}{2} \Delta_x(e^{-V(x)}) \text{Id} \right) \leq (\Lambda_V - \varepsilon) |\nabla_x V(x)|^2 \text{Id}, \quad x \in \mathbb{R}^d \text{ a.e. such that } |x| > R,$$

where the inequality holds in the sense of positive matrices.

Lemma 9. *Assume that $d \geq 1$ and consider V such that (V1), (V2) and (V6) hold. If ϕ_\star solves (7) and $W_\star = V + \phi_\star$, then there is a positive constant Λ_\star such that*

$$\int_{\mathbb{R}^d} \left(\text{Hess}(\rho_\star) - \frac{1}{2} \Delta_x \rho_\star \text{Id} \right) : \nabla_x w \otimes \nabla_x w dx \leq \Lambda_\star \int_{\mathbb{R}^d} |\nabla_x w|^2 |\nabla_x W_\star|^2 \rho_\star dx$$

for any function $w \in H_{\text{loc}}^1(\mathbb{R}^d)$.

Proof. An elementary computation shows that

$$\text{Hess}(\rho_\star) = (\nabla_x W_\star \otimes \nabla_x W_\star - \text{Hess}(W_\star)) \rho_\star \quad \text{and} \quad \Delta_x \rho_\star = (|\nabla_x W_\star|^2 - \Delta_x W_\star) \rho_\star.$$

The proof is then similar to the above arguments, up to elementary estimates, that we shall omit here. \square

Similarly, let us assume that

$$(V7) \quad \lim_{r \rightarrow +\infty} \sup_{x \in B_r^c} \left| \nabla_x (\log(|\nabla_x V(x)|^2)) \right| < +\infty.$$

Lemma 10. *Assume that $d \geq 1$ and consider V such that (V1), (V2) and (V7) hold. If ϕ_\star solves (7) and $W_\star = V + \phi_\star$, then there is a positive constant Λ_\circ such that*

$$(18) \quad \left| \nabla_x (|\nabla_x W_\star(x)|^2) \right| \leq \Lambda_\circ |\nabla_x W_\star(x)|^2, \quad x \in \mathbb{R}^d \text{ a.e. such that } |x| > R.$$

Here we mean that $\nabla_x (|\nabla_x W_\star|^2) \nabla_x w = 2 \text{Hess}(W_\star) : \nabla_x W_\star \otimes \nabla_x w$ for any C^1 function w . Inequality (18) follows from the regularity and decay estimates of ϕ_\star . Since the proof relies only on elementary but tedious computations, we omit it here. In the same vein, let us assume that

$$(V8) \quad \left\| |\nabla_x V|^2 e^{-V} \right\|_{L^\infty(\mathbb{R}^d, dx)} < +\infty \quad \text{and} \quad \left\| |\nabla_x (|\nabla_x V|^2)|^2 e^{-V} \right\|_{L^\infty(\mathbb{R}^d, dx)} < +\infty.$$

Lemma 11. *Assume that $d \geq 1$ and consider V such that (V1), (V2) and (V8) hold. If ϕ_\star solves (7) and $W_\star = V + \phi_\star$, then $\|\nabla_x W_\star\|^2 \rho_\star\|_{L^\infty(\mathbb{R}^d, dx)}$ and $\|\nabla_x (|\nabla_x W_\star|^2)\|^2 \rho_\star\|_{L^\infty(\mathbb{R}^d, dx)}$ are finite.*

3.5. A Bochner-Lichnerowicz-Weitzenböck identity and second order estimates. Algebraic computations and a few integrations by parts provide us with the following estimate.

Lemma 12. *Let $\rho_\star = e^{-W_\star}$ be a non-trivial function in $L_{\text{loc}}^\infty \cap W^{1,2}(\mathbb{R}^d)$. Then for any smooth function w on \mathbb{R}^d with compact support, we have the identity*

$$\int_{\mathbb{R}^d} |\text{Hess}(w)|^2 \rho_\star dx \leq 6 \int_{\mathbb{R}^d} \frac{1}{\rho_\star} |\nabla_x \cdot (\rho_\star \nabla_x w)|^2 dx + 8 \int_{\mathbb{R}^d} (\nabla_x W_\star \cdot \nabla_x w)^2 \rho_\star dx.$$

Notice that if V satisfies (V1)–(V2), $M > 0$ and $W_\star = V + \phi_\star$ where ϕ_\star is the unique solution of (7), then ρ_\star is an admissible function for Lemma 12.

Proof. Let us start by establishing a Bochner-Lichnerowicz-Weitzenböck identity as follows:

$$\begin{aligned} \frac{1}{2} \Delta_x (\rho_\star |\nabla_x w|^2) &= \nabla_x \cdot (\rho_\star \text{Hess}(w) \nabla_x w) + \frac{1}{2} \nabla_x \cdot (|\nabla_x w|^2 \nabla_x \rho_\star) \\ &= \rho_\star |\text{Hess}(w)|^2 + \rho_\star \nabla_x w \cdot \nabla_x (\Delta_x w) + \frac{1}{2} \Delta_x \rho_\star |\nabla_x w|^2 \\ &\quad + 2 \text{Hess}(w) : \nabla_x w \otimes \nabla_x \rho_\star \\ &= \rho_\star |\text{Hess}(w)|^2 + \nabla_x w \cdot \nabla_x (\rho_\star \Delta_x w) - (\nabla_x w \cdot \nabla_x \rho_\star) \Delta_x w \\ &\quad + \frac{1}{2} \Delta_x \rho_\star |\nabla_x w|^2 + 2 \text{Hess}(w) : \nabla_x w \otimes \nabla_x \rho_\star. \end{aligned}$$

We obtain after a few integrations by parts on \mathbb{R}^d that

$$\begin{aligned} \int_{\mathbb{R}^d} \Delta_x (\rho_\star |\nabla_x w|^2) dx &= 0, \quad \int_{\mathbb{R}^d} \nabla_x w \cdot \nabla_x (\rho_\star \Delta_x w) dx = - \int_{\mathbb{R}^d} (\Delta_x w)^2 \rho_\star dx, \\ \frac{1}{2} \int_{\mathbb{R}^d} \Delta_x \rho_\star |\nabla_x w|^2 dx &+ \int_{\mathbb{R}^d} \text{Hess}(w) : \nabla_x w \otimes \nabla_x \rho_\star dx = 0, \end{aligned}$$

which proves that

$$\begin{aligned} \int_{\mathbb{R}^d} |\text{Hess}(w)|^2 \rho_\star dx \\ = \int_{\mathbb{R}^d} (\Delta_x w)^2 \rho_\star dx + \int_{\mathbb{R}^d} (\nabla_x w \cdot \nabla_x \rho_\star) \Delta_x w dx - \int_{\mathbb{R}^d} \text{Hess}(w) : \nabla_x w \otimes \nabla_x \rho_\star dx. \end{aligned}$$

We deduce from

$$\begin{aligned} \int_{\mathbb{R}^d} (\nabla_x w \cdot \nabla_x \rho_\star) \Delta_x w dx &= - \int_{\mathbb{R}^d} \Delta_x w (\nabla_x w \cdot \nabla_x W_\star) \rho_\star dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d} (\Delta_x w)^2 \rho_\star dx + \frac{1}{2} \int_{\mathbb{R}^d} (\nabla_x W_\star \cdot \nabla_x w)^2 \rho_\star dx \end{aligned}$$

and

$$\begin{aligned} - \int_{\mathbb{R}^d} \text{Hess}(w) : \nabla_x w \otimes \nabla_x \rho_\star dx &= \int_{\mathbb{R}^d} \text{Hess}(w) : \nabla_x w \otimes \nabla_x W_\star \rho_\star dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d} (\text{Hess}(w))^2 \rho_\star dx + \frac{1}{2} \int_{\mathbb{R}^d} (\nabla_x W_\star \cdot \nabla_x w)^2 \rho_\star dx \end{aligned}$$

that

$$\frac{1}{2} \int_{\mathbb{R}^d} |\text{Hess}(w)|^2 \rho_\star dx \leq \frac{3}{2} \int_{\mathbb{R}^d} (\Delta_x w)^2 \rho_\star dx + \int_{\mathbb{R}^d} (\nabla_x W_\star \cdot \nabla_x w)^2 \rho_\star dx.$$

Since $\nabla_x \rho_\star = -\nabla_x W_\star \rho_\star$ and $\Delta_x w \rho_\star = \nabla_x \cdot (\rho_\star \nabla_x w) + (\nabla_x W_\star \cdot \nabla_x w) \rho_\star$, we have the estimate

$$\int_{\mathbb{R}^d} (\Delta_x w)^2 \rho_\star dx \leq 2 \int_{\mathbb{R}^d} \frac{1}{\rho_\star} |\nabla_x \cdot (\rho_\star \nabla_x w)|^2 dx + 2 \int_{\mathbb{R}^d} (\nabla_x W_\star \cdot \nabla_x w)^2 \rho_\star dx,$$

which completes the proof. \square

3.6. The scalar product. On $\mathbb{R}^d \times \mathbb{R}^d$, let us define the measure

$$d\mu := f_\star(x, v) dx dv$$

and consider the functional space

$$(19) \quad \mathcal{H} := \left\{ h \in L^1 \cap L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu) : \iint_{\mathbb{R}^d \times \mathbb{R}^d} h d\mu = 0 \text{ and } \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2 dx < \infty \right\},$$

where we use the notation $\rho_h = \int_{\mathbb{R}^d} h f_\star dv$ and $\psi_h = (-\Delta_x)^{-1} \rho_h$. We also define

$$(20) \quad \langle h_1, h_2 \rangle := \iint_{\mathbb{R}^d \times \mathbb{R}^d} h_1 h_2 d\mu + \int_{\mathbb{R}^d} \rho_{h_1} (-\Delta_x)^{-1} \rho_{h_2} dx \quad \forall h_1, h_2 \in \mathcal{H}.$$

Remark 1. This definition deserves an explanation. The whole motivation of this paper is to understand the large-time asymptotics of the nonlinear system (VPFP) and it is known, see for instance [17], that the irreversibility and the convergence of a solution f to the stationary solution f_\star can be studied using the free energy functional

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} f \log\left(\frac{f}{f_\star}\right) dx dv + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla_x \phi|^2 dx.$$

Indeed, if (f, ϕ) solves (VPFP) and has sufficient smoothness and decay properties, we have

$$\frac{d}{dt} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} f \log\left(\frac{f}{f_\star}\right) dx dv + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla_x \phi|^2 dx \right) = - \iint_{\mathbb{R}^d \times \mathbb{R}^d} f^{-1} |\nabla_v f|^2 dx dv.$$

However, no rate of decay of the free energy is known for initial data such that $f - f_\star$ is large. With $f = f_\star (1 + \eta h)$ for some h such that $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h f_\star dx dv = 0$, we can at least investigate the limit as $\eta \rightarrow 0_+$. At leading order η^2 , the free energy is $\eta^2 \langle h, h \rangle$, which is the initial reason for introducing this scalar product.

Lemma 13. *Let $M > 0$. If V satisfies (V1)–(V2), then $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a Hilbert space for any dimension $d \geq 1$.*

Proof. Up to an integration by parts, we can rewrite $\langle h_1, h_2 \rangle$ as

$$\langle h_1, h_2 \rangle = \iint_{\mathbb{R}^d \times \mathbb{R}^d} h_1 h_2 d\mu + \int_{\mathbb{R}^d} (-\Delta_x \psi_{h_1}) \psi_{h_2} dx = \iint_{\mathbb{R}^d \times \mathbb{R}^d} h_1 h_2 d\mu + \int_{\mathbb{R}^d} \nabla_x \psi_{h_1} \cdot \nabla_x \psi_{h_2} dx$$

and observe that this determines a scalar product. This computation has to be justified. Let us distinguish three cases depending on the dimension d .

Let us assume first that $d \geq 3$. We know that $\psi_\star = G_d \ast \rho_\star$ is nonnegative and deduce that ρ_\star is bounded because

$$0 \leq e^{-V - \psi_\star} \leq e^{-V} \in L^\infty(\mathbb{R}^d).$$

Hence, for any $p \in (1, 2]$, we have

$$\|\rho_h\|_{L^p(\mathbb{R}^d)}^p = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} h f_\star dv \right|^p dx \leq \|\rho_\star\|_{L^\infty(\mathbb{R}^d)}^{p-1} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |h|^p d\mu.$$

According to [56], we know by the Hardy-Littlewood-Sobolev inequality that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\rho_1(x)| |\rho_2(y)|}{|x-y|^{d-a}} dx dy \leq \mathcal{C}_{\text{HLS}} \|\rho_1\|_{L^p(\mathbb{R}^d)} \|\rho_2\|_{L^q(\mathbb{R}^d)}$$

if $a \in (0, d)$ and $p, q \in (1, +\infty)$ are such that $1 + \frac{a}{d} = \frac{1}{p} + \frac{1}{q}$. This justifies the fact that $\int_{\mathbb{R}^d} \rho_h (-\Delta_x)^{-1} \rho_h dx$ is well defined if $h \in L^1 \cap L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu)$. With $a = 2$, $p < 3/2$ if $d = 3$, $p < 2$ if $d = 4$ and $p \leq 2$ if $d \geq 5$, we deduce that $\psi_h \in L^{q'}(\mathbb{R}^d)$ where $q' = q/(q-1) = dp/(d-2p)$. A simple Hölder estimate shows the Gagliardo-Nirenberg type estimate

$$\|\nabla_x \psi\|_{L^2(\mathbb{R}^d)}^2 \leq \|\Delta_x \psi\|_{L^{p_1}(\mathbb{R}^d)} \|\psi\|_{L^{q_1}(\mathbb{R}^d)}$$

and proves for an appropriate choice of $(p_1, q_1) \in (1, 2) \times (2, +\infty)$ with $\frac{1}{p_1} + \frac{1}{q_1} = 1$ that $\nabla_x \psi_h$ is bounded in $L^2(\mathbb{R}^d)$.

The case $d = 2$ is well known. The boundedness of $\|\rho_h\|_{L^p(\mathbb{R}^d)}$ for any $p \in (1, 2]$ follows by the same argument as in the case $d \geq 3$ and we learn that $|\rho_h| \log |\rho_h|$ is integrable by log-Hölder interpolation. The boundedness from below of $\int_{\mathbb{R}^2} \rho_h (-\Delta_x)^{-1} \rho_h$ is then a consequence of the logarithmic Hardy-Littlewood-Sobolev inequality, see [21, 29]. Using the fact that $\int_{\mathbb{R}^d} \rho_h dx = 0$, we also know from [12] that $\nabla_x \psi_h$ is bounded in $L^2(\mathbb{R}^2)$.

When $d = 1$, the nonnegativity of the scalar product is a consequence of (14) and holds without additional condition by a simple density argument. \square

The condition $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h d\mu = 0$ in the definition of h is simply an orthogonality condition with the constant functions, with respect to the usual scalar product in $L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu)$. By taking the completion of smooth compactly supported functions with zero average with respect to the norm defined by $h \mapsto \langle h, h \rangle$, we recover \mathcal{H} , which is therefore a Hilbert space. In the next sections, we shall denote by $\|\cdot\|$ the norm on \mathcal{H} associated with the scalar product so that

$$\|h\|^2 = \langle h, h \rangle \quad \forall h \in \mathcal{H}.$$

Whenever another norm is used, this will be explicitly specified.

4. PROOF OF THE MAIN RESULT

In this section, we prove Theorem 1. Our task is to check that the assumptions of Section 2 hold in the functional setting of Section 3.

4.1. Definitions and elementary properties. On the space \mathcal{H} defined by (19) with scalar product given by (20), let us consider the transport and the collision operators respectively defined by

$$(21) \quad \mathbb{T}h := v \cdot \nabla_x h - \nabla_x W_\star \cdot \nabla_v h + v \cdot \nabla_x \psi_h, \quad \mathbb{L}h := \Delta_v h - v \cdot \nabla_v h$$

where $W_\star = V + \phi_\star$. In the literature, \mathbb{L} is known as the *Ornstein-Uhlenbeck operator*.

Lemma 14. *With the above notation, \mathbb{L} and \mathbb{T} are respectively symmetric and anti-symmetric operators on $(\mathcal{H}, \langle \cdot, \cdot \rangle)$.*

The operator L is defined on continuous functions with compact support which are of class C^2 with respect to v and it is semi-bounded. It can be extended using Friederichs extension as a self-adjoint operator on \mathcal{H} . Concerning T , we shall define it on the space of continuous functions with compact support of class C^1 with respect to x and v and omit details concerning domain issues and extensions as we need only properties that apply to solutions of the evolution problem (1).

Proof. If h_1 and h_2 are two functions in $W^{2,2}(\mathbb{R}^d, \mathcal{M} dv)$, then L is such that

$$\int_{\mathbb{R}^d} (Lh_1) h_2 \mathcal{M} dv = - \int_{\mathbb{R}^d} \nabla_v h_1 \cdot \nabla_v h_2 \mathcal{M} dv$$

and as a special case corresponding to $h_1 = h$, $h_2 = 1$, we find that $\rho_{Lh} = \int_{\mathbb{R}^d} (Lh) f_\star dv = 0$ and also $\psi_{Lh} = 0$ for any $h \in \mathcal{H}$. As a consequence, we have that

$$\langle (Lh_1), h_2 \rangle = - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_v h_1 \cdot \nabla_v h_2 d\mu = \langle h_1, (Lh_2) \rangle.$$

Concerning the transport operator, we know that $Tf_\star = 0$. Hence an integration by parts shows that

$$\langle (Th_1), h_2 \rangle = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (v \cdot \nabla_x h_1 - \nabla_x W_\star \cdot \nabla_v h_1) h_2 d\mu = - \langle h_1, (Th_2) \rangle$$

for function h_1 and h_2 in \mathcal{M} which are smooth enough, because $\rho_{Th} = \int_{\mathbb{R}^d} (Th) f_\star dv = \nabla_x \psi_h \cdot \int_{\mathbb{R}^d} v f_\star dv = 0$ and $\psi_{Th} = 0$. \square

4.2. Microscopic coercivity. By the Gaussian Poincaré inequality, we know that

$$\int_{\mathbb{R}^d} |\nabla_v g|^2 \mathcal{M} dv \geq \int_{\mathbb{R}^d} |g - \Pi g|^2 \mathcal{M} dv \quad \forall g \in H^1(\mathbb{R}^d, \mathcal{M} dv),$$

where $\Pi g = \int_{\mathbb{R}^d} g \mathcal{M} dv$ denotes the average of g with respect to the Gaussian probability measure $\mathcal{M} dv$. By extension, we shall consider Π as an operator on \mathcal{H} and observe that

$$(22) \quad \Pi h = u_h := \frac{\rho_h}{\rho_\star} = \frac{\int_{\mathbb{R}^d} h f_\star dv}{\int_{\mathbb{R}^d} f_\star dv} = \int_{\mathbb{R}^d} h \mathcal{M} dv \quad \forall h \in \mathcal{H}.$$

Let us notice first that Π is an orthogonal projector.

Lemma 15. Π is a self-adjoint operator and $\Pi \circ \Pi = \Pi$.

Proof. It is elementary to check that

$$(\Pi \circ \Pi) h = \Pi u_h = u_h, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} (\Pi h_1) h_2 d\mu = \int_{\mathbb{R}^d} u_{h_1} u_{h_2} \rho_\star dx$$

and

$$\int_{\mathbb{R}^d} \rho_{\Pi h_1} (-\Delta_x)^{-1} \rho_{h_2} dx = \int_{\mathbb{R}^d} \rho_{h_1} (-\Delta_x)^{-1} \rho_{h_2} dx$$

because $\rho_{h_1} = \rho_\star u_{h_1} = \rho_\star u_{\Pi h_1} = \rho_{\Pi h_1}$. \square

Lemma 16. Microscopic coercivity (H1) holds with $\lambda_m = 1$.

Proof. We already know that $-\langle (Lh), h \rangle = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla_v h|^2 d\mu$ and $\rho_{h-\Pi h} = \rho_h - \rho_{\Pi h} = 0$ so that

$$\|h - \Pi h\|^2 = \|h - \Pi h\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu)}^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |h - \Pi h|^2 d\mu.$$

The conclusion is then a consequence of the Gaussian Poincaré inequality. \square

4.3. Macroscopic coercivity.

Lemma 17. *Assume that $d \geq 1$ and consider V such that (V1), (V2), (V3a) and (V4) hold. With the notations of Lemma 6, macroscopic coercivity (H2) holds with $\lambda_M = \mathcal{C}_\star$.*

Proof. Using $\mathbb{T}\Pi h = v \cdot (\nabla_x u_h + \nabla_x \psi_h)$ with u_h as in (22), $\int_{\mathbb{R}^d} (v \cdot e)^2 \mathcal{M} dv = 1$ for any given $e \in \mathbb{S}^{d-1}$ and (16), we find that

$$\|\mathbb{T}\Pi h\|^2 = \int_{\mathbb{R}^d} |\nabla_x u_h + \nabla_x \psi_h|^2 \rho_\star dx \geq \mathcal{C}_\star \left[\int_{\mathbb{R}^d} |u_h + \psi_h|^2 \rho_\star dx - \frac{1}{M} \left(\int_{\mathbb{R}^d} \psi_h \rho_\star dx \right)^2 \right]$$

because $\int_{\mathbb{R}^d} u_h \rho_\star dx = \int_{\mathbb{R}^d} \rho_h dx = 0$. We know from Lemma 13 that $\int_{\mathbb{R}^d} u_h \psi_h \rho_\star dx = \int_{\mathbb{R}^d} \rho_h \psi_h dx \geq 0$ and by the Cauchy-Schwarz inequality, we get that

$$\left(\int_{\mathbb{R}^d} \psi_h \rho_\star dx \right)^2 \leq M \int_{\mathbb{R}^d} |\psi_h|^2 \rho_\star dx.$$

Altogether, we collect these estimates into

$$\int_{\mathbb{R}^d} |\nabla_x u_h + \nabla_x \psi_h|^2 \rho_\star dx \geq \mathcal{C}_\star \left[\int_{\mathbb{R}^d} |u_h|^2 \rho_\star dx + \int_{\mathbb{R}^d} \rho_h \psi_h dx \right] = \mathcal{C}_\star M \|u_h\|^2,$$

which concludes the proof. \square

4.4. Parabolic macroscopic dynamics.

Lemma 18. *The transport operator \mathbb{T} satisfies the parabolic macroscopic dynamics (H3).*

Proof. Since $\mathbb{T}\Pi h = v \cdot (\nabla_x u_h + \nabla_x \psi_h)$, we obtain that

$$\Pi \mathbb{T}\Pi h \rho_\star = (\nabla_x u_h + \nabla_x \psi_h) \cdot \int_{\mathbb{R}^d} v f_\star dv = 0.$$

\square

4.5. Bounded auxiliary operators. The point is to prove that (H4) holds, *i.e.*, that for any $F \in \mathcal{H}$, $\|\mathbb{A}\mathbb{T}(\text{Id} - \Pi)F\|$ and $\|\mathbb{A}LF\|$ are bounded up to a constant by $\|(\text{Id} - \Pi)F\|$. This is the purpose of Lemma 19 and Lemma 20. The two quantities, $\|\mathbb{A}\mathbb{T}(\text{Id} - \Pi)F\|$ and $\|\mathbb{A}LF\|$, are needed to control the bad terms in the expression of D_δ , in the abstract formulation of Proposition 4, namely $\langle \mathbb{T}AF, F \rangle$, $\langle \mathbb{A}\mathbb{T}(\text{Id} - \Pi)F, F \rangle$ and $\langle \mathbb{A}LF, F \rangle$ (which have no definite sign), by the two good terms, $-\langle LF, F \rangle$ and $\langle \mathbb{A}\mathbb{T}\Pi F, F \rangle$ (which are both positive).

Lemma 19. *The operators $\mathbb{T}\mathbb{A}$ and $\mathbb{A}L$ satisfy: for all $h \in L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu)$,*

$$\|\mathbb{A}Lh\| \leq \frac{1}{2} \|(\text{Id} - \Pi)h\|.$$

Proof. If we denote the flux by $j_h := \int_{\mathbb{R}^d} v h f_\star dv$, we remark that $j_{Lh} = -j_h$ and

$$\Pi \mathbb{T}h = \nabla_x \cdot j_h - (\nabla_x V + \nabla_x \phi_\star) \cdot j_h.$$

Since $Ah = g$ means $g + (\mathbb{T}\Pi)^*(\mathbb{T}\Pi)g = (\mathbb{T}\Pi)^*h = -\Pi \mathbb{T}h$, this implies that

$$\mathbb{A}Lh = -Ah.$$

The same computation as for (12) shows that $\|\mathbb{A}Lh\|^2 = \|Ah\|^2 = \|g\|^2 \leq \frac{1}{4} \|(\text{Id} - \Pi)h\|^2$, which completes the proof. \square

Lemma 20. *Assume that $d \geq 1$ and consider V such that (V1), (V2), (V3b), (V4), (V5), (V6), (V7) and (V8) hold. There exists a constant $\mathcal{C} > 0$ such that*

$$\|\mathbb{A}\mathbb{T}(\text{Id} - \Pi)h\| \leq \mathcal{C} \|(\text{Id} - \Pi)h\| \quad \forall h \in \mathcal{H}.$$

Proof. In order to get an estimate of $\|\text{AT}(\text{Id} - \Pi)h\|$, we will compute $\|\text{AT}(\text{Id} - \Pi)^*h\|$.

Step 1: Reformulation of the inequality as an elliptic regularity estimate. We claim that

$$(23) \quad \|(\text{AT}(\text{Id} - \Pi))^*h\|^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |(\text{AT}(\text{Id} - \Pi))^*h|^2 d\mu = 2 \int_{\mathbb{R}^d} |\text{Hess}(w_g)|^2 \rho_\star dx,$$

where $w_g := u_g + \psi_g$ and $-\Delta_x \psi_g = \rho_g$ is computed in terms of

$$g = (\text{Id} + (\text{T}\Pi)^*(\text{T}\Pi))^{-1}h,$$

which is obtained by solving the elliptic equation

$$(24) \quad g - \Delta_x w_g + \nabla_x W_\star \cdot \nabla_x w_g = h.$$

Let $u_h = \Pi h$ and $w_h := u_h + \psi_h$. We observe that $\text{T}\Pi h = v \cdot \nabla_x w_h$, $\rho_{\text{T}\Pi h} = 0$ and, as a consequence

$$(\text{T}\Pi)^*(\text{T}\Pi)h = -\Pi\text{T}(\text{T}\Pi h) = -\Delta_x w_h + \nabla_x W_\star \cdot \nabla_x w_h = -e^{W_\star} \nabla_x (e^{-W_\star} \nabla_x w_h)$$

where $W_\star = V + \phi_\star$ is such that $\rho_\star = e^{-W_\star}$. With g obtained from (24), we compute

$$\begin{aligned} (\text{AT}(\text{Id} - \Pi))^*h &= -(\text{Id} - \Pi)\text{T}\text{A}^*h = -(\text{Id} - \Pi)\text{T}(\text{T}\Pi)(\text{Id} + (\text{T}\Pi)^*(\text{T}\Pi))^{-1}h \\ &= -(\text{Id} - \Pi)\text{T}(\text{T}\Pi)g = -(\text{Id} - \Pi)(v \otimes v : \text{Hess}(w_g)) = \Delta_x w_g - v \otimes v : \text{Hess}(w_g) \end{aligned}$$

where $\text{Hess}(w) = (\nabla_x \otimes \nabla_x)w$ denotes the Hessian of w . Hence, with $|\text{Hess}(w)|^2 = \text{Hess}(w) : \text{Hess}(w)$, we obtain (23) using the following elementary computation

Let $s = (s_{ij})_{i,j=1}^d$ be a symmetric matrix with coefficients which do not depend on v . We compute $S := \int_{\mathbb{R}^d} (s : v \otimes v - \text{Tr}(s))^2 \mathcal{M} dv$ as follows. Using

$$\begin{aligned} (s : v \otimes v - \text{Tr}(s))^2 &= \left(\sum_{i,j=1}^d s_{ij} v_i v_j - \sum_{i=1}^d s_{ii} \right)^2 \\ &= \left(\sum_{i,j=1}^d s_{ij} v_i v_j \right)^2 - 2 \left(\sum_{i=1}^d s_{ii} \right) \left(\sum_{i,j=1}^d s_{ij} v_i v_j \right) + \left(\sum_{i=1}^d s_{ii} \right)^2 \end{aligned}$$

and $\int_{\mathbb{R}^d} v_i v_j \mathcal{M} dv = \delta_{ij}$, we obtain

$$S = \int_{\mathbb{R}^d} \left(\sum_{i,j=1}^d s_{ij} v_i v_j \right)^2 \mathcal{M} dv - \left(\sum_{i=1}^d s_{ii} \right)^2.$$

Since

$$\left(\sum_{i,j=1}^d s_{ij} v_i v_j \right)^2 = \left(\sum_{i \neq j=1}^d s_{ij} v_i v_j \right)^2 + \left(\sum_{i=1}^d s_{ii} v_i^2 \right)^2 + 2 \sum_{i \neq j=1}^d \sum_{k=1}^d s_{ij} s_{kk} v_i v_j v_k^2,$$

$\int_{\mathbb{R}^d} v_i^2 \mathcal{M} dv = 1$, and $\int_{\mathbb{R}^d} v_i^4 \mathcal{M} dv = 3$, the computation simplifies to

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{i,j=1}^d s_{ij} v_i v_j \right)^2 \mathcal{M} dv &= 2 \sum_{i \neq j=1}^d s_{ij}^2 + \sum_{i \neq j=1}^d s_{ii} s_{jj} + 3 \sum_{i=1}^d s_{ii}^2 \\ &= 2 \sum_{i,j=1}^d s_{ij}^2 + \left(\sum_{i=1}^d s_{ii} \right)^2. \end{aligned}$$

Altogether, this proves that

$$S = 2 \sum_{i,j=1}^d s_{ij}^2 = 2|s|^2.$$

The result follows with $s = \text{Hess}(w_g)$. A bound on $\int_{\mathbb{R}^d} |\text{Hess}(w_g)|^2 \rho_\star dx$ will now be obtained by elliptic regularity estimates based on (24).

Step 2: Some H^1 -type estimates. By integrating (24) against $\mathcal{M}(v) dv$, we notice that

$$(25) \quad u_g - \frac{1}{\rho_\star} \nabla_x \cdot (\rho_\star \nabla_x w_g) = u_h$$

so that

$$(26) \quad \int_{\mathbb{R}^d} u_g \rho_\star dx = \int_{\mathbb{R}^d} u_h \rho_\star dx = \iint_{\mathbb{R}^d \times \mathbb{R}^d} h d\mu = 0.$$

If we multiply (25) by $w_g \rho_\star$ and integrate over \mathbb{R}^d , we get after an integration by parts that

$$\int_{\mathbb{R}^d} u_g (u_g + \psi_g) \rho_\star dx + \int_{\mathbb{R}^d} |\nabla_x w_g|^2 \rho_\star dx = \int_{\mathbb{R}^d} u_h (u_g + \psi_g) \rho_\star dx.$$

Using $\int_{\mathbb{R}^d} u_g \psi_g \rho_\star dx = \int_{\mathbb{R}^d} |\nabla_x \psi_g|^2 dx$ and $\int_{\mathbb{R}^d} u_h \psi_g \rho_\star dx = \int_{\mathbb{R}^d} \nabla_x \psi_h \cdot \nabla_x \psi_g dx$ on the one hand, and the elementary estimates

$$\begin{aligned} \left| \int_{\mathbb{R}^d} u_h u_g \rho_\star dx \right| &\leq \frac{1}{2} \int_{\mathbb{R}^d} (|u_g|^2 + |u_h|^2) \rho_\star dx, \\ \left| \int_{\mathbb{R}^d} \nabla_x \psi_h \cdot \nabla_x \psi_g dx \right| &\leq \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla_x \psi_h|^2 + |\nabla_x \psi_g|^2) dx, \end{aligned}$$

on the other hand, we obtain that

$$(27) \quad \int_{\mathbb{R}^d} |u_g|^2 \rho_\star dx + \int_{\mathbb{R}^d} |\nabla_x \psi_g|^2 dx + 2 \int_{\mathbb{R}^d} |\nabla_x w_g|^2 \rho_\star dx \leq \|\Pi h\|^2$$

where

$$\|\Pi h\|^2 = \int_{\mathbb{R}^d} |u_h|^2 \rho_\star dx + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2 dx.$$

Using $|\nabla_x u_g|^2 = |\nabla_x w_g - \nabla_x \psi_g|^2 \leq 2(|\nabla_x w_g|^2 + |\nabla_x \psi_g|^2)$, we deduce from (27) that

$$(28) \quad \int_{\mathbb{R}^d} |\nabla_x u_g|^2 \rho_\star dx \leq 2 \int_{\mathbb{R}^d} |\nabla_x w_g|^2 \rho_\star dx + 2 \int_{\mathbb{R}^d} |\nabla_x \psi_g|^2 \rho_\star dx \leq \mathcal{K} \|\Pi h\|^2$$

with $\mathcal{K} = 1 + 2 \|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)}$.

Step 3: Weighted Poincaré inequalities and weighted H^1 -type estimates. The solution u_g of (25) has zero average according to (26). We deduce from Corollary 7 that

$$\int_{\mathbb{R}^d} |\nabla_x u_g|^2 \rho_\star dx \geq \mathcal{C} \int_{\mathbb{R}^d} |u_g|^2 |\nabla_x W_\star|^2 \rho_\star dx,$$

from which we get that

$$(29) \quad X_1^2 := \int_{\mathbb{R}^d} |u_g|^2 |\nabla_x W_\star|^2 \rho_\star dx \leq \frac{\mathcal{K}}{\mathcal{C}} \|\Pi h\|^2.$$

Next, we look for a similar estimate for $\int_{\mathbb{R}^d} |\psi_g|^2 |\nabla_x W_\star|^2 \rho_\star dx$. The potential ψ_g has generically a non-zero average $\bar{\psi}_g := \frac{1}{M} \int_{\mathbb{R}^d} \psi_g \rho_\star dx$ which can be estimated by

$$\begin{aligned} M^2 |\bar{\psi}_g|^2 &= \left(\int_{\mathbb{R}^d} \psi_g \rho_\star dx \right)^2 = \left(\int_{\mathbb{R}^d} \psi_g (-\Delta_x \phi_\star) dx \right)^2 = \left(\int_{\mathbb{R}^d} (-\Delta_x \psi_g) \phi_\star dx \right)^2 \\ &= \left(\int_{\mathbb{R}^d} u_g \phi_\star \rho_\star dx \right)^2 \leq \int_{\mathbb{R}^d} |\phi_\star|^2 \rho_\star dx \int_{\mathbb{R}^d} |u_g|^2 \rho_\star dx \leq \kappa_1 \|\Pi h\|^2 \end{aligned}$$

with $\kappa_1 := \int_{\mathbb{R}^d} |\phi_\star|^2 \rho_\star dx$, using (27). Since $\nabla_x \rho_\star = -\nabla_x W_\star \rho_\star$, we also have

$$\int_{\mathbb{R}^d} \psi_g |\nabla_x W_\star|^2 \rho_\star dx = - \int_{\mathbb{R}^d} \psi_g \nabla_x W_\star \cdot \nabla_x \rho_\star dx = \int_{\mathbb{R}^d} (\psi_g \Delta_x W_\star + \nabla_x \psi_g \cdot \nabla_x W_\star) \rho_\star dx$$

and, using the Cauchy-Schwarz inequality,

$$\begin{aligned} \left(\int_{\mathbb{R}^d} \psi_g |\nabla_x W_\star|^2 \rho_\star dx \right)^2 &\leq \int_{\mathbb{R}^d} |\psi_g|^2 \rho_\star dx \int_{\mathbb{R}^d} (\Delta_x W_\star)^2 \rho_\star dx \\ &\quad + \int_{\mathbb{R}^d} |\nabla_x \psi_g|^2 dx \|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)} \int_{\mathbb{R}^d} |\nabla_x W_\star|^2 \rho_\star dx. \end{aligned}$$

By Lemma 6 applied to $\psi_g - \bar{\psi}_g$,

$$\mathcal{E}_\star \int_{\mathbb{R}^d} |\psi_g|^2 \rho_\star dx \leq \|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)} \int_{\mathbb{R}^d} |\nabla_x \psi_g|^2 dx + \mathcal{E}_\star |\bar{\psi}_g|^2,$$

and (27), we conclude that

$$\left(\int_{\mathbb{R}^d} \psi_g |\nabla_x W_\star|^2 \rho_\star dx \right)^2 \leq \kappa_2 \|\Pi h\|^2$$

where

$$\kappa_2 := \left(\frac{1}{\mathcal{E}_\star} \int_{\mathbb{R}^d} (\Delta_x W_\star)^2 \rho_\star dx + \int_{\mathbb{R}^d} |\nabla_x W_\star|^2 \rho_\star dx \right) \|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)} + \frac{\kappa_1}{M^2} \int_{\mathbb{R}^d} (\Delta_x W_\star)^2 \rho_\star dx.$$

By applying Corollary 7 to $\psi_g - \bar{\psi}_g$, we deduce from

$$\mathcal{E} \int_{\mathbb{R}^d} |\psi_g - \bar{\psi}_g|^2 |\nabla_x W_\star|^2 \rho_\star dx \leq \int_{\mathbb{R}^d} |\nabla_x \psi_g|^2 \rho_\star dx$$

that

$$\mathcal{E} \int_{\mathbb{R}^d} |\psi_g|^2 |\nabla_x W_\star|^2 \rho_\star dx \leq \int_{\mathbb{R}^d} |\nabla_x \psi_g|^2 \rho_\star dx + 2 \mathcal{E} \bar{\psi}_g \int_{\mathbb{R}^d} \psi_g \rho_\star |\nabla_x W_\star|^2 \rho_\star dx.$$

Hence

$$(30) \quad \int_{\mathbb{R}^d} |\psi_g|^2 |\nabla_x W_\star|^2 \rho_\star dx \leq \left(\frac{\|\rho_\star\|_{L^\infty(\mathbb{R}^d, dx)}}{\mathcal{E}} + 2 \frac{\sqrt{\kappa_1 \kappa_2}}{M} \right) \|\Pi h\|^2.$$

Now we use (29) and (30) to estimate the weighted H^1 -type quantity

$$X_2^2 := \int_{\mathbb{R}^d} |\nabla_x u_g|^2 |\nabla_x W_\star|^2 \rho_\star dx.$$

Let us multiply (25) by $u_g |\nabla_x W_\star|^2 \rho_\star$ and integrate by parts in order to obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} |u_g|^2 |\nabla_x W_\star|^2 \rho_\star dx + \int_{\mathbb{R}^d} |\nabla_x u_g|^2 |\nabla_x W_\star|^2 \rho_\star dx \\ & + \int_{\mathbb{R}^d} (\nabla_x u_g \cdot \nabla_x \psi_g) |\nabla_x W_\star|^2 \rho_\star dx + \int_{\mathbb{R}^d} u_g \nabla_x (|\nabla_x W_\star|^2) (\nabla_x u_g + \nabla_x \psi_g) \rho_\star dx \\ & = \int_{\mathbb{R}^d} u_h u_g |\nabla_x W_\star|^2 \rho_\star dx. \end{aligned}$$

Using Lemma 10, we obtain

$$\left| \int_{|x|>R} u_g \nabla_x (|\nabla_x W_\star|^2) \nabla_x u_g \rho_\star dx \right| \leq \Lambda_\circ \int_{\mathbb{R}^d} |u_g| |\nabla_x W_\star|^2 |\nabla_x u_g| \rho_\star dx \leq \Lambda_\circ X_1 X_2.$$

Using (27), (28), Lemma 11 and the fact that $1/\rho_\star$ is bounded on B_R , we obtain

$$\left| \int_{|x|\leq R} u_g \nabla_x (|\nabla_x W_\star|^2) \nabla_x u_g \rho_\star dx \right| \leq \mathcal{K} \left\| \nabla_x (|\nabla_x W_\star|^2) \right\|_{L^\infty(B_R)} \|\Pi h\|^2$$

and, by similar arguments,

$$\begin{aligned} & \int_{\mathbb{R}^d} (\nabla_x u_g \cdot \nabla_x \psi_g) |\nabla_x W_\star|^2 \rho_\star dx \leq \kappa_3 X_2 \|\Pi h\|, \\ & \int_{\mathbb{R}^d} u_g \nabla_x (|\nabla_x W_\star|^2) \nabla_x \psi_g \rho_\star dx \leq \kappa_4 X_1 \|\Pi h\|, \end{aligned}$$

with

$$\kappa_3 := \left\| |\nabla_x W_\star|^2 \rho_\star \right\|_{L^\infty(\mathbb{R}^d, dx)}^{1/2} \quad \text{and} \quad \kappa_4 := \left\| |\nabla_x (|\nabla_x W_\star|^2)|^2 \rho_\star \right\|_{L^\infty(\mathbb{R}^d, dx)}^{1/2},$$

because we know from (27) that $\int_{\mathbb{R}^d} |\nabla_x \psi_g|^2 dx \leq \|\Pi h\|^2$. Using Corollary 8, we obtain that

$$\left(\int_{\mathbb{R}^d} u_h u_g |\nabla_x W_\star|^2 \rho_\star dx \right)^2 \leq \int_{\mathbb{R}^d} |u_h|^2 \rho_\star dx \int_{\mathbb{R}^d} |u_g|^2 |\nabla_x W_\star|^4 \rho_\star dx \leq \|\Pi h\|^2 \frac{X_2^2}{\mathcal{C}_\circ}.$$

Summarizing, we have shown that

$$X_1^2 + X_2^2 - \kappa_3 X_2 \|\Pi h\| - \Lambda_\circ X_1 X_2 - \kappa_4 X_1 \|\Pi h\| \leq X_2 \frac{\|\Pi h\|}{\sqrt{\mathcal{C}_\circ}} + \mathcal{K} \left\| \nabla_x (|\nabla_x W_\star|^2) \right\|_{L^\infty(B_R)} \|\Pi h\|^2.$$

Since X_1^2 is bounded by $\|\Pi h\|^2$, we conclude that

$$(31) \quad X_2^2 \leq \kappa \|\Pi h\|^2$$

for some $\kappa > 0$, which has an explicit form in terms quantities involving ρ_\star and its derivatives, as well as all constants in the inequalities of Sections 3.3 and 3.4.

Step 4: Second order estimates. After multiplying (25) by $\nabla_x \cdot (\rho_\star \nabla_x w_g)$, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{1}{\rho_\star} |\nabla_x \cdot (\rho_\star \nabla_x w_g)|^2 dx = \int_{\mathbb{R}^d} (u_h - u_g) \nabla_x \cdot (\rho_\star \nabla_x w_g) dx \\ & = \int_{\mathbb{R}^d} u_h \sqrt{\rho_\star} \frac{1}{\sqrt{\rho_\star}} \nabla_x \cdot (\rho_\star \nabla_x w_g) dx + \int_{\mathbb{R}^d} \nabla_x u_g \cdot \nabla_x w_g \rho_\star dx \\ & \leq \frac{1}{2} \int_{\mathbb{R}^d} \left(|u_h|^2 \rho_\star + \frac{1}{\rho_\star} |\nabla_x \cdot (\rho_\star \nabla_x w_g)|^2 \right) dx \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla_x u_g|^2 + |\nabla_x w_g|^2) \rho_\star dx \end{aligned}$$

and after using (27) and (28), we obtain that

$$(32) \quad \int_{\mathbb{R}^d} \frac{1}{\rho_\star} |\nabla_x \cdot (\rho_\star \nabla_x w_g)|^2 dx \leq \left(\mathcal{K} + \frac{3}{2} \right) \|\Pi h\|^2.$$

Let $X_3 := \left(\int_{\mathbb{R}^d} (\nabla_x w_g \cdot \nabla_x W_\star)^2 \rho_\star dx \right)^{1/2}$. After multiplying (25) by $(\nabla_x w_g \cdot \nabla_x W_\star) \rho_\star$, we have that

$$X_3^2 - \int_{\mathbb{R}^d} \Delta_x w_g (\nabla_x w_g \cdot \nabla_x W_\star) \rho_\star dx = \int_{\mathbb{R}^d} (u_h - u_g) (\nabla_x w_g \cdot \nabla_x W_\star) \rho_\star dx.$$

Using the Cauchy-Schwarz inequality, we know that the right-hand side can be estimated by $X_3 \left(\int_{\mathbb{R}^d} |u_g|^2 \rho_\star dx \right)^{1/2} + X_3 \left(\int_{\mathbb{R}^d} |u_h|^2 \rho_\star dx \right)^{1/2} \leq 2 X_3 \|\Pi h\|$ according to (27) and obtain that

$$X_3^2 - 2 X_3 \|\Pi h\| \leq \int_{\mathbb{R}^d} \Delta_x w_g (\nabla_x w_g \cdot \nabla_x W_\star) \rho_\star dx.$$

Let us notice that

$$\begin{aligned} \int_{\mathbb{R}^d} \Delta_x w_g (\nabla_x w_g \cdot \nabla_x W_\star) \rho_\star dx &= - \int_{\mathbb{R}^d} \Delta_x w_g \nabla_x w_g \cdot \nabla_x \rho_\star dx \\ &= \int_{\mathbb{R}^d} \left(\text{Hess}(\rho_\star) - \frac{1}{2} \Delta_x \rho_\star \text{Id} \right) : \nabla_x w_g \otimes \nabla_x w_g dx. \end{aligned}$$

As a consequence, by Lemma 9 and (27), we arrive at

$$X_3^2 - 2 X_3 \|\Pi h\| \leq \frac{\Lambda_\star}{2} \int_{\mathbb{R}^d} |\nabla_x w_g|^2 |\nabla_x W_\star|^2 \rho_\star dx \leq \Lambda_\star X_2^2 + \Lambda_\star \int_{\mathbb{R}^d} |\nabla_x \psi_g|^2 |\nabla_x W_\star|^2 \rho_\star dx$$

where X_2^2 is the quantity that has been estimated in Step 4. Altogether, after taking (27) and (31) into account and with

$$\lambda = \Lambda_\star \left(\kappa + \|\nabla_x W_\star\|_{L^\infty(\mathbb{R}^d)}^2 \right),$$

which is finite by Lemma 11, this proves that

$$(33) \quad \int_{\mathbb{R}^d} (\nabla_x w_g \cdot \nabla_x W_\star)^2 \rho_\star dx \leq \left(\sqrt{1 + \lambda} - 1 \right)^2 \|\Pi h\|^2.$$

Step 5: Conclusion of the proof. We read from Lemma 12, (23) and (32)-(33) that

$$\|(\text{AT}(\text{Id} - \Pi))^* h\|^2 \leq 2 \int_{\mathbb{R}^d} |\text{Hess}(w_g)|^2 \rho_\star dx \leq 2 \left(6 \left(\mathcal{K} + \frac{3}{2} \right) + 8 \left(\sqrt{1 + \lambda} - 1 \right)^2 \right) \|\Pi h\|^2,$$

which concludes the proof of Lemma 20. \square

4.6. Proof of Theorem 1. The potential $V(x) = |x|^\alpha$ satisfies the assumptions (V1), (V2), (V3b), (V4), (V5), (V6), (V7) and (V8) if $\alpha > 1$. The result is then a consequence of Proposition 4 and Lemmas 14-20. A slightly more general result goes as follows.

Theorem 21. *Let us assume that $d \geq 1$ and $M > 0$. If V satisfies the assumptions (V1), (V2), (V3b), (V4), (V5), (V6), (V7) and (V8), then there exist two constants $\lambda > 0$ and $\mathcal{C} > 1$ such that any solution h of (1) with an initial datum h_0 of zero average such that $\|h_0\|^2 < \infty$ satisfies*

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0.$$

5. UNIFORM ESTIMATES IN THE DIFFUSION LIMIT

The hypocoercivity method of [31, 32] is directly inspired by the drift-diffusion limit, as it relies on a micro/macro decomposition in which the relaxation in the velocity direction is given by the microscopic coercivity property (H1) while the relaxation in the position direction arises from the macroscopic coercivity property (H2) which governs the relaxation of the solution of the drift-diffusion equation obtained as a limit.

5.1. Formal macroscopic limit. Let us start with a formal analysis in the framework of Section 2, when (8) is replaced by the scaled evolution equation

$$(34) \quad \varepsilon \frac{dF}{dt} + \mathbb{T}F = \frac{1}{\varepsilon} \mathbb{L}F$$

on the Hilbert space \mathcal{H} . We assume that a solution F_ε of (34) can be expanded as

$$F_\varepsilon = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \mathcal{O}(\varepsilon^3)$$

in the asymptotic regime corresponding to $\varepsilon \rightarrow 0_+$ and, at formal level, that (34) can be solved order by order:

$$\begin{aligned} \varepsilon^{-1}: \quad & \mathbb{L}F_0 = 0, \\ \varepsilon^0: \quad & \mathbb{T}F_0 = \mathbb{L}F_1, \\ \varepsilon^1: \quad & \frac{dF_0}{dt} + \mathbb{T}F_1 = \mathbb{L}F_2. \end{aligned}$$

The first equation reads as $F_0 = \Pi F_0$, that is, F_0 is in the kernel of \mathbb{L} . Assume for simplicity that $\mathbb{L}^{-1}(\mathbb{T}\Pi) = -\mathbb{T}\Pi$ on an appropriate subspace, so that the second equation is simply solved by $F_1 = -(\mathbb{T}\Pi)F_0$. Let us consider the projection on the kernel of the $\mathcal{O}(\varepsilon^1)$ equation:

$$\frac{d}{dt}(\Pi F_0) - \Pi \mathbb{T}(\mathbb{T}\Pi)F_0 = \Pi \mathbb{L}F_2 = 0.$$

If we denote by u the quantity $F_0 = \Pi F_0$ and use (H3), then $-(\Pi \mathbb{T})(\mathbb{T}\Pi) = (\mathbb{T}\Pi)^*(\mathbb{T}\Pi)$ and the equation becomes

$$\partial_t u + (\mathbb{T}\Pi)^*(\mathbb{T}\Pi)u = 0,$$

which is our *drift-diffusion* limit equation. Notice that if u solves this equation, then

$$\frac{d}{dt} \|u\|^2 = -2 \|(\mathbb{T}\Pi)u\|^2 \leq -2 \lambda_M \|u\|^2$$

according to (H2). This program applies in the case of the scaled evolution equation (4). Let us give a few additional details.

Let us assume that a solution h_ε of (4) can be expanded as $h_\varepsilon = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \mathcal{O}(\varepsilon^3)$, in the asymptotic regime as $\varepsilon \rightarrow 0_+$. Solving (4) order by order in ε , we find the equations

$$\begin{aligned} \varepsilon^{-1}: \quad & \Delta_v h_0 - v \cdot \nabla_v h_0 = 0, \\ \varepsilon^0: \quad & v \cdot \nabla_x h_0 - \nabla_x W_\star \cdot \nabla_v h_0 + v \cdot \nabla_x \psi_{h_0} = \Delta_v h_1 - v \cdot \nabla_v h_1, \\ \varepsilon^1: \quad & \partial_t h_0 + v \cdot \nabla_x h_1 - \nabla_x W_\star \cdot \nabla_v h_1 + v \cdot \nabla_x \psi_{h_1} = \Delta_v h_2 - v \cdot \nabla_v h_2. \end{aligned}$$

Let us define $u = \Pi h_0$, $\psi = \psi_{h_0}$ such that $-\Delta_x \psi = u \rho_\star$, $w = u + \psi$ and observe that the first two equations simply mean

$$u = h_0, \quad v \cdot \nabla_x w = \Delta_v h_1 - v \cdot \nabla_v h_1,$$

from which we deduce that $h_1 = -v \cdot \nabla_x w$. After projecting with Π , the third equation is

$$\partial_t u - \Delta_x w + \nabla_x W_\star \cdot \nabla_x w = 0,$$

using $\int_{\mathbb{R}^d} v \otimes v \mathcal{M}(v) dv = \text{Id}$. If we define $\rho = u \rho_\star$, we have formally obtained that it solves

$$\partial_t \rho = \Delta_x \rho + \nabla_x \cdot \left(\rho (\nabla_x V + \nabla_x \phi_\star) \right) + \nabla_x \cdot (\rho_\star \nabla_x \psi), \quad -\Delta_x \psi = \rho.$$

At this point, we can notice that the solution ρ converges to ρ_\star according to the results of, e.g., [54], at an exponential rate which is independent of ε .

5.2. Hypocoercivity. Let us adapt the computations of Section 2 to the case $\varepsilon \in (0, 1)$ as in [19]. If F solves (34), then

$$-\varepsilon \frac{d}{dt} \text{H}_\delta[F] = \text{D}_{\delta, \varepsilon}[F],$$

$$\text{D}_{\delta, \varepsilon}[F] := -\frac{1}{\varepsilon} \langle LF, F \rangle + \delta \langle \text{AT}\Pi F, F \rangle - \delta \langle \text{TAF}, F \rangle + \delta \langle \text{AT}(\text{Id} - \Pi)F, F \rangle - \frac{\delta}{\varepsilon} \langle \text{ALF}, F \rangle.$$

The estimates are therefore exactly the same as in Proposition 4, up to the replacement of λ_m by λ_m/ε and C_M by C_M/ε . Hence, for $\varepsilon > 0$ small enough, we have that

$$\delta(\varepsilon) := \min \left\{ 2, \frac{\lambda_m}{\varepsilon}, \varepsilon \lambda_\star(\varepsilon) \right\} = \frac{4 \lambda_m \lambda_M \varepsilon}{4 \lambda_M \varepsilon^2 + C_M^2 (1 + \lambda_M)}.$$

We may notice that $\lim_{\varepsilon \rightarrow 0_+} \frac{\delta(\varepsilon)}{\varepsilon} = 2\zeta$ with

$$\zeta := \frac{2 \lambda_m \lambda_M}{C_M^2 (1 + \lambda_M)}$$

and, for $\varepsilon > 0$ small enough,

$$\frac{2 - \zeta \varepsilon}{4} \|F\|^2 \leq \text{H}_{\zeta \varepsilon}[F] \leq \frac{2 + \zeta \varepsilon}{4} \|F\|^2 \quad \forall F \in \mathcal{H}.$$

By revisiting the proof of Proposition 4, we find that with $\delta = \zeta \varepsilon$ and $\lambda = \eta \varepsilon$ with

$$\eta := \frac{\lambda_m \lambda_M^2}{C_M^2 (1 + \lambda_M)^2},$$

the quadratic form

$$(X, Y) \mapsto \left(\frac{\lambda_m}{\varepsilon} - \delta \right) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta \frac{C_M}{\varepsilon} X Y - \frac{\lambda}{2} (X^2 + Y^2) - \frac{\lambda}{2} \delta X Y$$

is a nonnegative quadratic form for $\varepsilon > 0$ small enough. In the regime as $\varepsilon \rightarrow 0_+$, the result of Proposition 4 can be adapted as follows.

Corollary 22. *Assume that (H1)–(H4) hold and take ζ as above. Then for $\varepsilon > 0$ small enough,*

$$\eta \varepsilon \text{H}_{\zeta \varepsilon}[F] \leq \text{D}_{\zeta \varepsilon, \varepsilon}[F] \quad \forall F \in \mathcal{H}.$$

Proof. The range for which the quadratic form is negative is given by the condition

$$\lambda_m^2 K^4 \varepsilon^4 + K C_M^3 (4K \lambda_m + 3C_M(K+4)) \varepsilon^2 - 2C_M^6 < 0.$$

It follows that the above condition is satisfied if ε is taken small enough which, for the same reasons as above in this paper, guarantees that the entropy-entropy production inequality of Corollary 22 holds. \square

As an easy consequence, if F_ε solves (34), we have that

$$\text{H}_{\zeta \varepsilon}[F(t, \cdot)] \leq \text{H}_{\zeta \varepsilon}[F(0, \cdot)] e^{-\eta t} \quad \forall t \geq 0.$$

Proof of Theorem 2. With the abstract result on (34) applied to (4), the estimate (11) holds with $\delta = \zeta \varepsilon$. We conclude with $\lambda = \eta$, for some $\mathcal{C} > 1$, which do not depend on $\varepsilon \rightarrow 0_+$. \square

6. THE NONLINEAR SYSTEM IN DIMENSION $d = 1$

With the notation (21), we can rewrite the Vlasov-Poisson-Fokker-Planck system (VPFP) as

$$\partial_t h + \mathbb{T}h = \mathbb{L}h + \mathbb{Q}[h], \quad -\Delta_x \psi_h = \int_{\mathbb{R}^d} h f_\star dv, \quad \text{with } \mathbb{Q}[h] := \nabla_x \psi_h \cdot (\nabla_v h - v h).$$

Here we assume that $d = 1$ and prove Corollary 3. Using the representation (13), so that

$$\psi'_h(x) = - \int_{-\infty}^x u_h \rho_\star dx \quad \forall x \in \mathbb{R},$$

and the convergence of $h(t, \cdot, \cdot) \rightarrow 0$ in $L^1(\mathbb{R} \times \mathbb{R}, d\mu)$ as $t \rightarrow +\infty$, as a consequence of [17], we learn that $t \mapsto \|\psi'_h(t, \cdot)\|_{L^\infty(\mathbb{R})}$ is bounded uniformly w.r.t. $t \geq 0$. In fact, we have a slightly more precise estimate that goes as follows.

Lemma 23. *Assume V satisfies (V1) and (V2) and let $\rho_\star \in L^1(\mathbb{R}^d)$ be the solution of (7) such that $\int_{\mathbb{R}^d} \rho_\star dx = M$. Let $f = (1 + h) f_\star \in L^1_+(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\iint_{\mathbb{R} \times \mathbb{R}} f \log(f/f_\star) dx dv < \infty$. Under the assumption $\iint_{\mathbb{R} \times \mathbb{R}} h f_\star dx dv = 0$, ψ'_h as defined above satisfies the estimate*

$$\|\psi'_h\|_{L^\infty(\mathbb{R})}^2 \leq 4M \iint_{\mathbb{R} \times \mathbb{R}} f \log\left(\frac{f}{f_\star}\right) dx dv.$$

Additionally, under the assumptions of Corollary 3, if h solves (VPFP), then

$$\lim_{t \rightarrow +\infty} \|\psi'_h(t, \cdot)\|_{L^\infty(\mathbb{R})} = 0.$$

Proof. We deduce from Jensen's inequality

$$\int_{\mathbb{R}} f \log\left(\frac{f}{\mathcal{M}}\right) dv \geq \rho_h \log \rho_h$$

that $f/f_\star = 1 + h$ is such that the free energy satisfies the bound

$$\iint_{\mathbb{R} \times \mathbb{R}} (1 + h) \log(1 + h) f_\star dx dv \geq \int_{\mathbb{R}} \rho_h \log\left(\frac{\rho_h}{\rho_\star}\right) dx = \int_{\mathbb{R}} (1 + u_h) \log(1 + u_h) \rho_\star dx$$

and get according to [25, 53, 64] from the Csiszár-Kullback-Pinsker inequality that

$$\int_{\mathbb{R}} (1 + u_h) \log(1 + u_h) \rho_\star dx \geq \frac{1}{4M} \left(\int_{\mathbb{R}} |u_h| \rho_\star dx \right)^2 \geq \frac{\|\psi'_h\|_{L^\infty(\mathbb{R})}^2}{4M}.$$

Concerning the evolution problem (VPFP), we recall that the free energy decays according to

$$\frac{d}{dt} \left(\iint_{\mathbb{R} \times \mathbb{R}} (1 + h) \log(1 + h) f_\star dx dv + \frac{1}{2} \int_{\mathbb{R}} |\psi'_h|^2 dx \right) = - \iint_{\mathbb{R} \times \mathbb{R}} f \left| \nabla_v \log\left(\frac{f}{f_\star}\right) \right|^2 dx dv,$$

where the right-hand side is, up to the sign, a Fisher information. As stated in [17, Theorem B], this shows the strong convergence of $f(t + n, \cdot, \cdot)$ to f_\star in $L^1(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d)$ as $n \rightarrow +\infty$ because f_\star is the unique solution of (VPFP) with mass M and a Maxwellian velocity distribution. By the logarithmic Sobolev inequality, this also proves that the limit of the free energy is 0, which concludes the proof of Lemma 23. \square

Proof of Corollary 3. With the notations of Section 3.6 and the functional H_δ defined as in the linear case by

$$H_\delta[h] := \frac{1}{2} \|h\|^2 + \delta \langle Ah, h \rangle,$$

we obtain that

$$\begin{aligned} \frac{d}{dt} H_\delta[h] + \langle Lh, h \rangle - \delta \langle A\Pi h, h \rangle + \delta \langle TA h, h \rangle - \delta \langle A\Pi(\text{Id} - \Pi)h, h \rangle + \delta \langle ALh, h \rangle \\ = \langle Q[h], h \rangle + \delta \langle AQ[h], h \rangle + \delta \langle Q[h], Ah \rangle. \end{aligned}$$

Let us give an estimate of the three terms of the right hand side.

1) In order to estimate

$$\langle Q[h], h \rangle = \iint_{\mathbb{R} \times \mathbb{R}} \psi'_h (\partial_v h - v h) h f_\star dx dv + \int_{\mathbb{R}} \psi'_h \rho_\star \left(\int_{\mathbb{R}} (\partial_v h - v h) \mathcal{M} dv \right) \psi_h dx,$$

we notice that $\int_{\mathbb{R}} (\partial_v h - v h) \mathcal{M} dv = 0$ and also that $\iint_{\mathbb{R} \times \mathbb{R}} |\partial_v h|^2 f_\star dx dv = -\langle Lh, h \rangle$. From the improved Poincaré inequality [34, Ineq. (4)], we also learn that $\|v h\|^2 \leq 2(d+2) \|\nabla_v h\|^2$. Simple Cauchy-Schwarz inequalities show that

$$|\langle Q[h], h \rangle| \leq c \|\psi'_h\|_{L^\infty(\mathbb{R})} |\langle Lh, h \rangle|^{1/2} \|\Pi h\|$$

because $\|h\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu)} \leq \|h\|$, with $c = 1 + \sqrt{2(d+2)}$.

2) Let us consider $g = Ah = u_g$ given by

$$u_g - \frac{1}{\rho_\star} \nabla_x \cdot (\rho_\star \nabla_x w_g) = -\frac{1}{\rho_\star} \nabla_x \cdot j_h \quad \text{with} \quad j_h := \int_{\mathbb{R}^d} v h f_\star dv.$$

With ψ_g such that $-\psi_g'' = u_g \rho_\star$, we have to estimate

$$\langle Q[h], Ah \rangle = \iint_{\mathbb{R} \times \mathbb{R}} \psi'_h (\partial_v h - v h) u_g f_\star dx dv + \int_{\mathbb{R}} \psi'_h \rho_\star \left(\int_{\mathbb{R}} (\partial_v h - v h) \mathcal{M} dv \right) \psi_g dx.$$

Exactly as above, we have on the one hand that

$$\begin{aligned} \left| \iint_{\mathbb{R} \times \mathbb{R}} \psi'_h (\partial_v h - v h) u_g f_\star dx dv \right| &\leq \|\psi'_h\|_{L^\infty(\mathbb{R})} \|g\| \|\partial_v h - v h\| \\ &\leq c \|\psi'_h\|_{L^\infty(\mathbb{R})} \|(\text{Id} - \Pi)h\| |\langle Lh, h \rangle|^{1/2} \end{aligned}$$

because $\|g\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d, d\mu)} \leq \|g\| = \|Ah\| \leq \|(\text{Id} - \Pi)h\|$, and on the other hand that

$$\begin{aligned} \int_{\mathbb{R}} |\psi_g|^2 \rho_\star dx &\leq \mathcal{E}_\star^{-1} \int_{\mathbb{R}} |\psi'_g|^2 \rho_\star dx + \left(\int_{\mathbb{R}} \psi_g \rho_\star dx \right)^2 \\ &\leq \frac{\|\rho_\star\|_{L^\infty(\mathbb{R})}}{\mathcal{E}_\star} \int_{\mathbb{R}} |\psi'_g|^2 dx + \int_{\mathbb{R}} |u_g|^2 \rho_\star dx \int_{\mathbb{R}} |\phi_\star|^2 \rho_\star dx \end{aligned}$$

by Lemma 6 again, from which we conclude that

$$|\langle Q[h], Ah \rangle| \leq c \|\psi'_h\|_{L^\infty(\mathbb{R})} |\langle Lh, h \rangle|^{1/2} \|(\text{Id} - \Pi)h\|.$$

3) With g given in terms of h by (24), $A^* h = v w'_g$ and we learn from (27) that $\|A^* h\| \leq \|\Pi h\|$. Hence

$$|\langle AQ[h], h \rangle| = |\langle Q[h], A^* h \rangle| \leq c \|\psi'_h\|_{L^\infty(\mathbb{R})} |\langle Lh, h \rangle|^{1/2} \|\Pi h\|.$$

Summing up all these estimates and using $-\langle Lh, h \rangle \geq \lambda_m \|(\text{Id} - \Pi)h\|^2$ by Lemma 16, we obtain as in the proof of Proposition 4 that

$$\frac{d}{dt} H_\delta[h] \leq -\lambda H_\delta[h]$$

for the largest value of λ for which

$$(X, Y) \mapsto (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y - \frac{\lambda}{2} (X^2 + Y^2) - \frac{\lambda}{2} \delta X Y - \epsilon X (X + 2 Y)$$

is a nonnegative quadratic form, as a function of (X, Y) . Here $X := \|(\text{Id} - \Pi)h\|$, $Y := \|\Pi h\|$, and

$$\epsilon := c \|\psi'_h\|_{L^\infty(\mathbb{R})}$$

can be taken as small as we wish, if we assume that $t > 0$ is large enough. This completes the proof of Corollary 3. \square

Let us conclude this section by some remarks.

- (i) It is clear from the proof of Corollary 3 that the optimal rate is as close as desired of the optimal rate in the linearized problem (1) obtained in Theorem 1. Up to a change of the constant \mathcal{C} , we can actually establish that these rates are equal because we read from the above proof that $\epsilon(t) = \mathcal{O}(e^{-\lambda t})$ and the result follows from a simple ODE argument. This is a standard observation in entropy methods, which has been used on many occasions: see for instance [11].
- (ii) Corollary 3 is written for $V(x) = |x|^\alpha$ but it is clear that it can be extended to the setting of Theorem 21. Similarly, our estimates are compatible with the diffusion limit, as in Section 5.
- (iii) Results in higher dimensions, *i.e.*, for $d \geq 2$ as in [48, 44, 45, 5] rely on smallness conditions, special properties of the potential V (typically, $V \equiv 0$ or $V(x) = |x|^2$), or closure conditions on regularity estimates which do not allow to handle the decay of generic solutions of (VPFP) based on the properties of the free energy, as we do above in the case $d = 1$. This is so far an important open question, which deserves attention. The understanding of the mechanism should go through a detailed description of the smoothing and decay properties of the solutions for large time asymptotics.

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L. Addala: Department of Mathematics, Faculty of Sciences of Bizerte,
University of Carthage, 7021 Zarzouna, Banzart, Tunisia.
E-mail: lanoiraddala@gmail.com

J. Dolbeault: CEREMADE (CNRS UMR n° 7534), PSL research university,
Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France.
E-mail: dolbeaul@ceremade.dauphine.fr

X. Li: CEREMADE (CNRS UMR n° 7534), PSL research university,
Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France.
E-mail: li@ceremade.dauphine.fr

M.L. Tayeb: Department of Mathematics, Faculty of Sciences of Tunis,
University of Tunis El Manar, 2092 El-Manar, Tunisia.
E-mail: lazhar.tayeb@fst.rnu.tn