A hybrid nonlinear-Kalman ensemble transform filter for data assimilation in systems with different degrees of nonlinearity

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PDAF Parallel Data Assimilation Framework

1. Overview

The nonlinear filter NETF (Nonlinear Ensemble Transform Filter) is combined with Local Ensemble Transform Kalman Filter (LETKF) to build a hybrid filter algorithm. This LKNETF filter combines the stability of the LETKF with the nonlinear properties of the NETF to obtain improved assimilation results for small ensemble sizes. Both filter components are localized in a consistent way so that the filter can be applied with high-dimensional models.

The degree of filter nonlinearity is defined by a hybrid weight which shifts the analysis between the LETKF and NETF. Since the NETF is more sensitive to sampling errors than the LETKF, the LETKF should be preferred in linear Gaussian cases. Accordingly the adaptive hybrid weight is defined based on the nonlinearity of the system so that the adaptivity yields a good filter performance in linear and nonlinear situations.

6. Hybrid Weight γ

Here, we define different rules to compute the hybrid weight γ adaptively.

A) Using the effective sample size $N_{eff} = \sum (w^i)^{-2}$ **:**

γ_α Choose γ_α so that N_{eff} is as small as possible while fulfilling $\alpha \geq N_{eff}/N$ [see 3].

New alternative linear dependence

γ_{lin} $$\gamma_{lin} = 1 - N_{eff}/N$$

Note: It is known that particle filters do not work well if N_{eff} is close to 1. However, this does not imply that a PF or NETF is better than the LETKF for higher N_{eff}.

B) Using absolute mean skewness and kurtosis of the observed ensemble:

Kalman filters assume that distributions are Gaussian. In this case the LETKF is preferable. We use the mean absolute skewness (*skew*) & kurtosis (*kurt*) of the observed ensemble to quantify the non-Gaussianity.

In general, *skew* and *kurt* are not bounded. However, we can normalize them by

$$skew' = skew/\sqrt{\kappa} \qquad kurt' = kurt/\kappa$$

where $\sqrt{\kappa} \approx N$. Now we define

$$\gamma_{sk} = \min(1 - |kurt'|, 1 - |skew'|)$$

To avoid too low N_{eff} we define combined rules

$\gamma_{sk,lin}$ $$\gamma_{sk,lin} = max[\gamma_{sk},\ \gamma_{lin}]$$

$\gamma_{sk,\alpha}$ $$\gamma_{sk,\alpha} = max[\gamma_{sk},\ \gamma_{\alpha}]$$

2. Linear and Nonlinear Filters

The transformation of the ensemble mean and ensemble perturbations for ensemble size N can be written in the generic form:

$$\overline{\mathbf{x}}^a = \overline{\mathbf{x}}^f + \mathbf{X}'^f \tilde{\mathbf{w}}$$
$$\mathbf{X}'^a = \mathbf{X}'^f \mathbf{W}$$

Ensemble Kalman & nonlinear filters just use different definitions of the

- weight vector $\tilde{\mathbf{w}}$ (dimension N)
- Transform matrix $\mathbf{W}$ (dimension $N \times N$)

3. NETF

The NETF [1, 2] is a second-order exact particle filter. We compute the normalized weight vector $\tilde{\mathbf{w}} = (w^{(1)}, \ldots, w^{(N)}) / \sum w^{(i)}$ using likelihood weights. For Gaussian observation errors this is

$$w^i \sim \exp\left(-0.5(\mathbf{y} - \mathbf{H}\mathbf{x}_i^f)^T \mathbf{R}^{-1} (\mathbf{y} - \mathbf{H}\mathbf{x}_i^f)\right)$$

The weights are also used for the transform matrix

$$\mathbf{W} = \sqrt{N}\left[\mathrm{diag}(\tilde{\mathbf{w}}) - \tilde{\mathbf{w}}\tilde{\mathbf{w}}^T\right]^{1/2} \Lambda$$

Here, Λ is the identity or a mean preserving random matrix that can be applied to stabilize the filter.

4. LETKF

In the LETKF we compute a local update of the ensemble mean and perturbations. The weight vector is computed according to the Kalman filter, which always assumes that the errors are Gaussian. Using the transform matrix

$$\mathbf{A}^{-1} = \rho(N-1)\mathbf{I} + (\mathbf{H}\mathbf{X}'^f)^T \mathbf{R}^{-1} \mathbf{H}\mathbf{X}'^f$$

that results from the equations of the Kalman filter and always assumes Gaussian errors we have:

$$\tilde{\mathbf{w}} = \mathbf{A}(\mathbf{H}\mathbf{X}'^f)^T \mathbf{R}^{-1}\left(\mathbf{y} - \mathbf{H}\overline{\mathbf{x}^f}\right)$$
$$\mathbf{W} = \sqrt{N-1}\ \mathbf{A}^{1/2}\mathbf{\Lambda}$$

5. Hybrid Filter LKNETF

Since NETF and ETKF are very similar one can easily combine both filters into a hybrid analysis step. Different hybrid schemes can be formulated:

1-step update (*HSync*)

$$\mathbf{X}^a_{HSync} = \overline{\mathbf{X}}^f + (1-\gamma)\Delta\mathbf{X}_{NETF} + \gamma\Delta\mathbf{X}_{ETKF}$$

Here the analysis increments $\Delta\mathbf{X}$ of both filters are computed and then a weighted average of both is used.

2-step updates (HNK and HKN)

In the 2-step update we can compute the NETF first followed by the ETKF, both with increased observation errors according to the hybrid weight (Variant *HNK*):

Step 1: $$\tilde{\mathbf{X}}^a_{HNK} = \mathbf{X}^a_{NETF}[\mathbf{X}^f, (1-\gamma)\mathbf{R}^{-1}]$$

Step 2: $$\mathbf{X}^a_{HNK} = \mathbf{X}^a_{ETKF}[\tilde{\mathbf{X}}^a_{HNK}, \gamma\mathbf{R}^{-1}]$$

Alternatively, we can compute the ETKF update before the NETF (Variant *HKN*).

7. Experiments

Lorenz-63

For the Lorenz-63 model, the default parameters are used. All 3 state variables are observed. The nonlinearity increases with the forecast length.

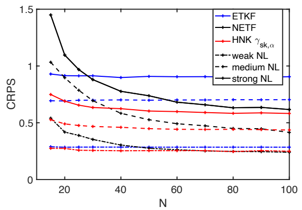


Fig. 1: CRPS for 3 different nonlinearities (NL). The ETKF shows little dependence on *N*. The NETF yields decreasing errors for growing *N*. The HNK filter yields the smallest errors. The effect of the hybridization is particularly large for small ensembles.

Fig. 2: CRPS for ETKF, NETF, and the hybrid filter variant HNK for different choices of γ and *N*=25. The nonlinearity is varied. $\gamma_{sk,lin}$ leads to the smallest errors with an error reduction of up to 32% compared to the ETKF.

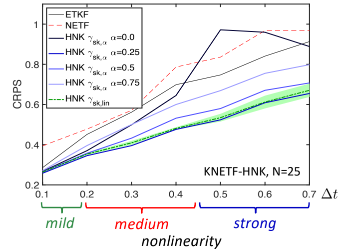
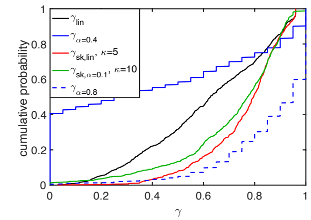


Fig. 3: The hybrid filter needs to use at each analysis a 'good' value of γ. Shown is the **probability of some value of γ over the course of the experiment for HNK and different hybrid rules.** Similar CRPS are obtained for different distributions (solid lines). The rule γ_α with α=0.8 (dashed) yields a larger error.



Lorenz-96

The results for the Lorenz-96 model (40 grid points, F=8) are shown for a forecast duration of 8 time steps. Each second grid point is observed.

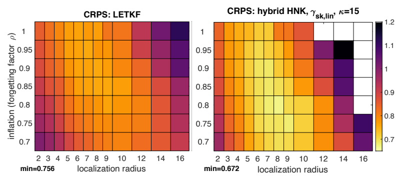


Fig. 4: CRPS in dependence on localization radius and ensemble inflation for LETKF and LKNETF-HNK. The hybrid filter yields smaller errors.

Table 1: Maximum error reduction in % compared to the LETKF for the different hybrid filter variants and γ rules for the Lorenz-96 model. The smallest errors are obtained for the variant HNK and $\gamma_{sk,\alpha}$ or $\gamma_{sk,lin}$. The variant HKN shows the smallest effects.

	γ_α	γ_{lin}	$\gamma_{sk,\alpha}$	$\gamma_{sk,lin}$	γ_α	γ_{lin}	$\gamma_{sk,\alpha}$	$\gamma_{sk,lin}$
	N=15				***N=40***			
HNK	8.6	9.3	10.9	**11.2**	19.6	17.6	**21.5**	17.6
HKN	2.7	2.0	*3.2*	2.7	4.7	2.0	*4.9*	1.4
HSync	6.5	4.9	*6.6*	6.0	8.4	6.0	*10.6*	6.4

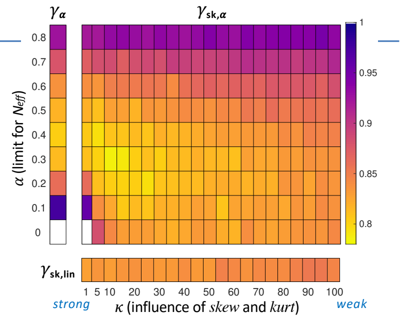


Fig 5: CRPS in dependence of α and κ for different γ-rules for HNK. Minimum CRPS is obtained for $\kappa \approx 10$.

8. Summary

The hybrid ensemble filter LKNETF combines the stable LETKF with the nonlinear filter NETF. Different variants of the hybrid filter are introduced.

The assimilation experiments for both models are implemented using PDAF [4,5] so that identical filter implementations are used. The lowest estimation errors are obtained using the hybrid variant HNK that applies the NETF first followed by the LETKF. The hybrid rules utilizing skewness and kurtosis ($\gamma_{sk,\alpha}$ and $\gamma_{sk,lin}$) yield very stable results and the lowest errors.

Yet unknown is which realistic problems have sufficient nonlinearity so that the hybrid filter can yield improved results. In initial twin experiments with the ocean model NEMO at a resolution of 0.25°, the LKNETF-HNK leads to error reductions of up to 3%. Here, the nonlinearity (or non-Gaussianity) is not sufficient to yield a larger effect.

References

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[4] Nerger, L. & W. Hiller. *Computers & Geosciences* (2013) **55**: 110–118

[5] http://pdaf.awi.de