The Real Algebraic Expression of the Collatz Conjecture

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Abstract

The Collatz conjecture is a very intriguing topic since a simple $3n + 1$ algebraic expression can create an endless 4 to 2 to 1 loop for any positive odd integer. Not surprisingly, the famous $3n + 1$ is not the only algebraic expression that can create an infinite loop when the same conditions for the Collatz conjecture is applied.

This paper details the real algebraic expression discovered for the Collatz conjecture as well as the pattern of the infinite loops created by each specific algebraic expression. Section 2 reveals that the real algebraic expression of the Collatz conjecture is $3n + (3^k)$ and the loop follows the pattern of $[1 \text{ x } 3^k$, $4 \text{ x } 3^k$, $2 \text{ x } 3^k]$ for any positive odd integer, where k is any positive integer ranging from 0 to 31. Moreover, the algebraic expression of the Collatz conjecture for any negative odd integer is $3n - (3^k)$ and the loop follows the pattern of $[1 \text{ x } (-(3^k))$, $4 \text{ x } (-(3^k))$, $2 \text{ x } (-(3^k))]$, where k is any positive integer ranging from 0 to 31.

Section 3 details the key to the pattern of the Collatz conjecture's infinite loop, while section 4 and 5 provides proof for the key mentioned in section 3. Section 6 reviews the pattern of the infinite loops created in section 4 and 5. Section 7 provides an alternative method for determining the pattern of the infinite loop created by a specific algebraic expression of the Collatz conjecture using the key mentioned in section 3.

1. Introduction

Collatz conjecture simply states:

a. Let n be any positive odd integer greater than 0
b. If n is odd, use $3n + 1$
c. If n is even, use $\frac{n}{2}$
d. Repeat the loop until 1 is reached

2. Revealing the real algebraic expression of the Collatz conjecture

While studying the properties of the $3n + 1$ algebraic expression, it was discovered that the real algebraic expression of the Collatz conjecture is $3n + (3^k)$ and the loop follows the pattern of [1 x 3^k, 4 x 3^k, 2 x 3^k] for any positive odd integer, where k is any positive integer ranging from 0 to 31. Ranges from 32 onwards for k no longer works in the python program except for $k = 33$, thus no further examinations were made. See section 4 for proof.

As can be expected, the real algebraic expression also works for any negative odd integer by simply replacing addition with subtraction. Thus, the algebraic expression of the Collatz conjecture is $3n - (3^k)$ and the loop follows the pattern of [1 x $(-(3^k))$, 4 x $(-(3^k))$, 2 x $(-(3^k))$] for any negative odd integer, where k is any positive integer ranging from 0 to 31. See section 5 for proof.

3. The key to the pattern of the Collatz conjecture's infinite loop

The key to determining the pattern of the Collatz conjecture's infinite loop is by observing the pattern created by the $3n + (3^k)$ algebraic expression when (3^k) is used as n inside the iterative loop. Thus, using $3n + 1$ as an example, the only pattern that can be produced when 1 is used as n inside the iterative loop is 4 to 2 to 1 then back to 4 loops. See section 4 for proof.

4. Proof that $n = (3^k)$ is the key to the pattern of the infinite loop created by the Collatz conjecture for any positive odd integer

Modify k to change the $3n + (3^k)$ algebraic expression and at the same time determine the pattern of the infinite loop when $n = (3^k)$.

Example results for $3n + (3^k)$:

Let $n = (3^k)$

- $k = 0;\ 3n + 1$; loop pattern starts with: [1, 4, 2]
- $k = 1;\ 3n + 3$; loop pattern starts with: [3, 12, 6]
- $k = 2;\ 3n + 9$; loop pattern starts with: [9, 36, 18]
- $k = 3;\ 3n + 27$; loop pattern starts with: [27, 108, 54]
- all the way to
- $k = 31;\ 3n + 617673396283947$; loop pattern starts with: [617673396283947, 2470693585135788, 1235346792567894]

See the link for the python code in Appendix A. filename: 3n_plus_3k_whileloop.py

```python
# Using a while loop for the Collatz conjecture
# created by Glenn Patrick King Ang 10/23/2021
# ===================
# n = 1 or any positive odd integer > 0
n = 1
# k can be set from 0 to 31
# k = 0; 3n+3**k = 3n+1
# k = 1; 3n+3
k = 1
# ===================

previous_n = 0
loop = 0
while n > 0:
    previous_n = n
    if previous_n == 2 * 3 ** k:
        loop += 1
        print(f"========== Infinite Loop:{loop} from n = 1")
        print(f"========== Algebraic expression: 3n + {3**k}")
    if previous_n == 2 * 3 ** k and loop == 3:
        print("end loop")
        break
    elif n % 2 == 1:
        n = (3 * n) + 3 ** k
    elif n % 2 == 0:
        n = n / 2
    print(int(n))
```

In this python code:

- k can be set from 0 to 31

algebraic expression:

$$3n + (3^k)$$

Loop pattern:

$$[1 \text{ x } 3^k,$$
$$4 \text{ x } 3^k,$$
$$2 \text{ x } 3^k]$$

Therefore, for k = 0, the Collatz conjecture is going to loop 4 to 2 to 1 endlessly because the full $3n + 1$ algebraic expression used in the loop was designed not only to reduce any positive odd integer into 2, but to also make sure that once the positive even integer is further reduced to 1, the next positive even number created is 4.

5. Proof that $n = -(3^k)$ is the key to the pattern of the infinite loop created by the Collatz conjecture for any negative odd integer

Modify k to change the $3n - (3^k)$ algebraic expression and at the same time determine the infinite loop pattern when $n = -(3^k)$.

Example results for $3n - (3^k)$:

Let $n = -(3^k)$

- $k = 0;\ 3n - 1$; loop pattern starts with: $[-1, -4, -2]$
- $k = 1;\ 3n - 3$; loop pattern starts with: $[-3, -12, -6]$
- $k = 2;\ 3n - 9$; loop pattern starts with: $[-9, -36, -18]$
- $k = 3;\ 3n - 27$; loop pattern starts with: $[-27, -108, -54]$
- all the way to
- $k = 31;\ 3n - 617673396283947$; loop pattern starts with: $[-617673396283947, -2470693585135788, -1235346792567894]$

See the link for the python code in Appendix A. filename: 3n_minus_3k_whileloop.py

```
# Using a while loop for the Collatz conjecture
# created by Glenn Patrick King Ang 10/23/2021
# ===================
# n = -1 or any negative odd integer < 0
n = -3
# k can be set from 0 to 31
# k = 0; 3n-3**k = 3n-1
# k = 1; 3n-3
k = 1
# ===================

previous_n = 0
loop = 0
while n < 0:
    previous_n = n
    if previous_n == -2 * 3 ** k:
        loop += 1
        print(f"========== Infinite Loop:{loop} from
n = -1 ")
        print(f"========== Algebraic expression: 3n
- {3**k}")
    if previous_n == -2 * 3 ** k and loop == 3:
        print("end loop")
        break
    elif n % 2 == 1:
        n = (3 * n) - 3 ** k
    elif n % 2 == 0:
        n = n / 2
    print(int(n))
```

In this python code:

- k can be set from 0 to 31

algebraic expression:

$$3n - (3^k)$$

Loop pattern:

$$[1 \text{ x } (-(3^k)),\ 4 \text{ x } (-(3^k)),\ 2 \text{ x } (-(3^k))]$$

6. Reviewing the pattern of the infinite loops

Creating a table for the observed pattern of the infinite loops when $n = (3^k)$:

k	Algebraic Expression	Pattern of the infinite loop
0	$3n + 1$	1, 4, 2
1	$3n + 3$	3, 12, 6
1	$3n + 9$	9, 36, 18
3	$3n + 27$	27,108, 54
31	$3n + 617673396283947$	617673396283947, 2470693585135788, 1235346792567894

The $3n + (3^k)$ algebraic expression was proven to follow a specific pattern of infinite loop based on [1 x 3^k, 4 x 3^k, 2 x 3^k], where k is any positive odd integer from 0 to 31. As shown in section 4, the infinite loop is formed based on the number added to $3n$ or simply (3^k).

Creating a table for the observed pattern of the infinite loops when $n = -(3^k)$:

k	Algebraic Expression	Pattern of the infinite loop
0	$3n - 1$	−1, −4, −2
1	$3n - 3$	−3, −12, −6
1	$3n - 9$	−9, −36, −18
3	$3n - 27$	−27, −108, −54
31	$3n - 617673396283947$	−617673396283947, −2470693585135788, −1235346792567894

The $3n - (3^k)$ algebraic expression was proven to follow a specific pattern of infinite loop based on [1 x $(-(3^k))$, 4 x $(-(3^k))$, 2 x $(-(3^k))$], where k is any positive odd integer from 0 to 31. As shown in section 5, the infinite loop of a specific algebraic expression is formed based on $-(3^k)$.

7. Alternative method to prove the pattern of the infinite loop created by a specific algebraic expression of the Collatz conjecture

(Ang 2021) provided some proofs that explains how the $n + 1$ in the $2n + n + 1$ or more commonly known as the $3n + 1$ algebraic expression ensures all positive odd integer will eventually be reduced to a positive even integer 2 inside an iterative loop. The positive even number 2 will then be further reduced to 1 by simply following the conditions of the Collatz conjecture. Through this observation, he was able to prove why all positive odd integers always ends up in the 4 to 2 to 1 then back to 4 loops for the $3n + 1$ algebraic expression.

Although the process he used to prove the infinite loop using $n + 1$ worked on $3n + 1$, it was determined that the same process does not work on $3n + (3^k)$ where k ranges from 1 onwards. Thus, another process should exist to determine and prove which number a specific algebraic expression is designed to reduce any positive odd integer into.

Since he was able to prove that the algebraic expression was designed to reduce any positive odd integer into a specific positive even integer, it is now possible to easily prove why a specific algebraic expression loops endlessly in a specific pattern. By carefully observing the patterns formed by the iterative loop in section 6, it is now safe to infer that the infinite loop is formed based on $(3^k)\ or\ (-(3^k))$.

a. Alternative method to determine which even integer a specific algebraic expression of the Collatz conjecture is designed to reduce any odd integer into.

1. **Proof A:** Multiplying the $(3^k)\ or\ (-(3^k))$ by 2
   - The 2 came from $\frac{n}{2}$ of the Collatz conjecture

For $3n + (3^k)$: $\mathbf{(3^k)}$ **x 2**

k	$3n + (3^k)$	Positive odd integer will be reduced to
0	$3n + 1$	$2 \text{ x } 1 = 2$
1	$3n + 3$	$2 \text{ x } 3 = 6$
2	$3n + 9$	$2 \text{ x } 9 = 18$
3	$3n + 27$	$2 \text{ x } 27 = 54$
	all the way to	
31	$3n + 617673396283947$	$2 \text{ x } 617673396283947$ $= 1235346792567894$

For $3n-(3^k)$: $(-(\mathbf{3^k}))$ **x 2**

k	$3n-(3^k)$	Negative odd integer will be reduced to
0	$3n-1$	$2 \text{ x} -1 = -2$
1	$3n-3$	$2 \text{ x} -3 = -6$
2	$3n-9$	$2 \text{ x} -9 = -18$
3	$3n-27$	$2 \text{ x} -27 = -54$
	all the way to	
31	$3n-617673396283947$	$2 \text{ x} -617673396283947$ $= -1235346792567894$

2. **Proof B:** Dividing the (3^k) or $-(3^k)$ by 3
   - The 3 came from $3n$ of the Collatz conjecture
   - Then plugging the quotient back into the algebraic expression

For $3n+(3^k)$: $(\mathbf{3^k})$ **/ 3**

k	$3n+(3^k)$	Dividing the (3^k) by 3	Plugging the quotient back
0	$3n+1$	$1/3 = 1/3$	$3(1/3)+1 = 2$
1	$3n+3$	$3/3 = 1$	$3(1)+3 = 6$
2	$3n+9$	$9/3 = 3$	$3(3)+9 = 18$
3	$3n+27$	$27/3 = 9$	$3(9)+27 = 54$
	all the way to		
31	$3n+617673396283947$	$617673396283947/3$ $= 205891132094649$	$3(205891132094649)$ $+617673396283947$ $= 1235346792567894$

For $3n-(3^k)$: $-(\mathbf{3^k})$ **/ 3**

k	$3n-(3^k)$	Dividing the $-(3^k)$ by 3	Plugging the quotient back
0	$3n-1$	$-1/3 = -1/3$	$3(-1/3)-1 = -2$
1	$3n-3$	$-3/3 = -1$	$3(-1)-3 = -6$
2	$3n-9$	$-9/3 = -3$	$3(-3)-9 = -18$
3	$3n-27$	$-27/3 = -9$	$3(-9)-27 = -54$
	all the way to		
31	$3n-617673396283947$	$-617673396283947/3$ $= -205891132094649$	$3(-205891132094649)$ -617673396283947 $= -1235346792567894$

b. Method to determine the highest even integer of the infinite loop for any specific algebraic expression of the Collatz conjecture

1. **Proof C:** Using the (3^k) or $-(3^k)$ as n

For $3n + (3^k)$:

k	$3n + (3^k)$	$n = (3^k)$	Highest integer of the infinite loop
0	$3n + 1$	1	$3(1) + 1 = 4$ (1 x 4)
1	$3n + 3$	3	$3(3) + 3 = 12$ (3 x 4)
2	$3n + 9$	9	$3(9) + 9 = 36$ (9 x 4)
3	$3n + 27$	27	$3(27) + 27 = 108$ (27 x 4)
	$all\ the\ way\ to$		
31	$3n + 617673396283947$	617673396283947	$3(617673396283947)$ $+617673396283947$ $= 2470693585135788$ $(617673396283947 \text{ x } 4)$

For $3n - (3^k)$:

k	$3n - (3^k)$	$n = -(3^k)$	Highest integer of the infinite loop
0	$3n - 1$	-1	$3(-1) - 1 = -4$ $(-1$ x 4)
1	$3n - 3$	-3	$3(-3) - 3 = -12$ $(-3$ x 4)
2	$3n - 9$	-9	$3(-9) - 9 = -36$ $(-9$ x 4)
3	$3n - 27$	-27	$3(-27) - 27 = -108$ $(-27$ x 4)
	$all\ the\ way\ to$		
31	$3n - 617673396283947$	-617673396283947	$3(-617673396283947)$ -617673396283947 $= -2470693585135788$ $(-617673396283947 \text{ x } 4)$

This provides a conclusive proof that the $3n + 1$ or the Collatz conjecture's real algebraic expression $3n + (3^k)$, where k ranges from 0 to 31, was designed not only to reduce any positive integer into (3^k) x 2, but to also make sure that once the positive even integer is further reduced back to the positive odd integer (3^k), the next positive even number created is (3^k) x 4.

8. Conclusion

Discovering other methods that can be used to determine the pattern of the infinite loop for a specific algebraic expression of the Collatz conjecture is a lot easier once it was established how the Collatz conjecture's infinite loop was being created in the first place.

If the Collatz conjecture used other algebraic expressions like $3n + 9$ or $3n + 27$, it might not have been possible to infer that the algebraic expressions themselves were designed to reduce any positive odd integer into a specific positive even integer inside an iterative loop with the end goal of reaching (3^k).

From all the proof that has been given, it is now possible to confirm that:

A. the real algebraic expression of the Collatz conjecture:
   - for any positive odd integer: $3n + (3^k)$
     - the infinite loop follows the pattern of $[1\ x\ 3^k, 4\ x\ 3^k, 2\ x\ 3^k]$
     - where k is any positive integer ranging from 0 to 31

   - for any negative odd integer: $3n - (3^k)$
     - the infinite loop follows the pattern of $[1 \text{ x } (-(3^k)), 4 \text{ x } (-(3^k)),$ $2 \text{ x } (-(3^k))]$
     - where k is any positive integer ranging from 0 to 31

B. The key to the pattern of the infinite loop is by observing the pattern created by the algebraic expression when (3^k) for any positive odd integer or $-(3^k)$ for any negative odd integer is used as n inside the iterative loop.

C. There exists an alternative method to prove the pattern of the infinite loop created by a specific algebraic expression of the Collatz conjecture using (3^k) for any positive odd integer or $-(3^k)$ for any negative odd integer as shown in section 7.

Modifying the Collatz conjecture for $3n + (3^k)$, where k ranges from 0 to 31:

a. Let n be any positive odd integer greater than 0
b. If n is odd, use $3n + (3^k)$
c. If n is even, use $\frac{n}{2}$
d. Repeat the loop until you reach (3^k)

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Reference:

Ang, Glenn Patrick King. (2021). Deciphering the Collatz Conjecture Through Recursion. Zenodo. https://doi.org/10.5281/zenodo.5710439

Appendix A:

Link to the GitHub repository containing the python codes used in this research paper:

https://github.com/07231985/collatzconjecture