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The three-dimensional Euler equations: singular or non-singular?

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Abstract
One of the outstanding open questions in modern applied mathematics is whether solutions of the incompressible Euler equations develop a singularity in the vorticity field in a finite time. This paper briefly reviews some of the issues concerning this problem, together with some observations that may suggest that it may be more subtle than first thought.

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1. Introduction

1.1. Opening remarks

The year 2007 marked the 300th anniversary of Leonhard Euler's birth and the 250th anniversary of his foundational paper on fluid dynamics. The meetings held in Aussois and St. Petersburg to celebrate these anniversaries also brought into sharper focus the very great distance needed to travel before it can be said that the behaviour of solutions of the three-dimensional Euler equations are thoroughly understood. Our generation has been raised in an age of ever-larger-scale computations from which has emerged the natural question whether the vorticity field develops a singularity (blow-up) in finite time [1,2]. The general philosophy has been that ever greater computing power, together with careful numerical integration and data handling, will eventually provide a platform where convergence to a 'yes' or 'no' answer to this question might be expected from a range of computations. An extensive literature has arisen [3–22], including those where high symmetry has been imposed [23–32], but there appears to be no conclusive agreement with a variety of contradictory results appearing in recent years [15–22].

Lack of space precludes a full listing of the history of both computations and analysis; indeed two recent reviews cover many aspects of this [33,34]. Nevertheless, a few remarks are in order. The older computations of Kerr [15, 16] together with preliminary newer results by Bustamante and Kerr [17], in which a singularity has been observed, have been contradicted by the recent computations of Hou and Li [18,19] who have seen only super-exponential growth.

Both computations used similar, but not identical, anti-parallel vortex tube initial conditions. Their results largely coincide until a late stage where differences in filtering are noticeable. Another recent contribution is that of Orlandi and Carnevale [20] who have used Lamb dipoles as initial conditions to observe singular behaviour, and that of Grafke *et al* [22]. Analytical approaches have mainly centred around conditional estimates on the magnitude and direction of vorticity that have extended the Beale–Kato–Majda theorem [35] (see section 1.2) to include the direction of vorticity; the most notable papers here are those by Constantin *et al* [44], Deng *et al* [46, 47] and Chae [42]. This approach has added to our local geometric understanding of the problem while falling short of a total solution [35–48].

It is possible, however, that a question requiring a definite 'yes' or 'no' answer to the open question of singularity formation may ignore some of the complexities inherent in the problem. This paper does not attempt to answer this question in 'yes' or 'no' terms. Instead its aim lies in a different direction; namely, to demonstrate that the problem may have a more sensitive nature than first thought. The main strategy is to show that $\Omega_2(t)$, the L^2-norm of the vorticity, and $\Omega_\infty(t)$, the L^∞-norm of the vorticity, both have a potentially binary nature in time. It is shown that the time-axis is divided into so-called 'plus' and 'minus' intervals on which certain inequalities involving these norms are reversed. Solutions behave differently on these intervals and their occurrence may depend on many factors that are peculiar to a particular computation suggesting that more needs to be understood regarding their occurrence and behaviour.

1.2. Background to the Euler equations

Consider the three-dimensional incompressible Euler equations [1] for the evolution of the vorticity field $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{Dt}} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} \tag{1.1}$$

acting on a domain which is a three-dimensional periodic box $\mathcal{V} = [0, L]^3$. $\boldsymbol{u}$ is the velocity field of the fluid and the material derivative is defined by

$$\frac{\mathrm{D}}{\mathrm{Dt}} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla, \qquad \operatorname{div}\boldsymbol{u} = 0\,. \tag{1.2}$$

The most important result on existence of solutions of the three-dimensional Euler equations is the theorem due to Beale *et al* [35]—see also references [35–48] and the two recent reviews [33, 34]. There are different ways of stating this theorem but its essence is:

Theorem 1 (Beale *et al* [35]). *There exists a global solution* $\boldsymbol{u} \in C([0, \infty]; H^s) \cap C^1([0, \infty]; H^{s-1})$ *of the Euler equations for* $s \geqslant 3$ *if and only if, for every* $t > 0$,

$$\int_0^t \|\boldsymbol{\omega}(\cdot, \xi)\|_\infty \,\mathrm{d}\xi < \infty. \tag{1.3}$$

The power of this result is two-fold. Firstly, it shows that only the L^∞-norm, $\|\boldsymbol{\omega}\|_\infty$, controls the regularity of solutions. Secondly, while it does not predict a singularity in $\|\boldsymbol{\omega}\|_\infty$ it restricts those that may potentially occur of the type $\|\boldsymbol{\omega}\|_\infty \sim (t_s - t)^{-p}$ to the range $p \geqslant 1$; in the case when $p < 1$ the theorem is violated.

For technical reasons it is necessary to define an additive *constant* frequency $\Omega_0 = T^{-1}$ where T is some time scale of a computation[3]. Ω_2 and Ω_∞ are defined by

$$\Omega_2(t) = L^{-3/2}\|\boldsymbol{\omega}(\cdot, t)\|_2 + \Omega_0 \qquad \Omega_\infty(t) = \|\boldsymbol{\omega}(\cdot, t)\|_\infty + \Omega_0\,. \tag{1.4}$$

Thus Ω_0, Ω_2 and Ω_∞ are ordered for all $t \geqslant 0$ such that $\Omega_0 \leqslant \Omega_2(t) \leqslant \Omega_\infty(t)$.

[3] It is not strictly necessary for Ω_0 to be defined in this way: for instance, for rotating Euler, Ω_0 could simply be twice the Earth's frequency of rotation with T chosen independently.

2. The potentially binary nature of the time-axis

Weak solutions of the Navier–Stokes equations in the Leray sense [49,50] have no counterpart for the three-dimensional Euler equations, although special weak solutions have been constructed [51,52]: for progress in this area see the recent paper by Lellis and Székelyhidi [53]. Formally, therefore, we assume that solutions exist on a chosen interval $[0, T]$ where, in practical terms, T could be taken to be the final time in a computation. Now consider two parameters $\alpha + \mu = 1$ whose inverses α^{-1} and μ^{-1} are used as the exponents in a Hölder inequality

$$\int_0^T \Omega_2^\alpha \,\mathrm{d}t \leqslant \int_0^T \Omega_\infty^\alpha \,\mathrm{d}t = \int_0^T \left(\frac{\Omega_\infty}{\Omega_2}\right)^\alpha \Omega_2^\alpha \,\mathrm{d}t \\ \leqslant \left(\int_0^T \left(\frac{\Omega_\infty}{\Omega_2}\right)^{\alpha/\mu} \mathrm{d}t\right)^\mu \left(\int_0^T \Omega_2 \,\mathrm{d}t\right)^\alpha . \tag{2.1}$$

Thus we have

$$\int_0^T \left(\frac{\Omega_\infty}{\Omega_2}\right)^{\alpha/\mu} \mathrm{d}t \geqslant \left(\int_0^T (\Omega_0^{-1}\Omega_2)^\alpha \,\mathrm{d}t\right) \left(\frac{\int_0^T (\Omega_0^{-1}\Omega_2)^\alpha \,\mathrm{d}t}{\int_0^T (\Omega_0^{-1}\Omega_2) \,\mathrm{d}t}\right)^{\alpha/\mu} . \tag{2.2}$$

In the bracket on the right-hand side the numerator is bounded below by T because $\Omega_2 > \Omega_0 = T^{-1}$. In the denominator either Ω_2 can be used as it stands or it can be replaced by Ω_∞ because $\Omega_2 \leqslant \Omega_\infty$. Both options are required later so the m-label is used with the two options $m = 2$ or $m = \infty$. The m-labelling potentially extends to $\mu = \mu_m$ and $\alpha = \alpha_m$. Then (2.2) becomes (t has been exchanged for ξ in the integral in the denominator)

$$\int_0^T \left\{ \left(\frac{\Omega_\infty(t)}{\Omega_2(t)}\right)^{\alpha_m/\mu_m} - \left(\frac{[T\Omega_2(t)]^{\mu_m}}{\int_0^T \Omega_m(\xi)\,\mathrm{d}\xi}\right)^{\alpha_m/\mu_m} \right\} \mathrm{d}t \geqslant 0. \tag{2.3}$$

The construction of a positive integral of a difference was first used in [54] for the Navier–Stokes equations and again in [55–57].

On what are designated as $m^{(\pm)}$-intervals, (2.3) shows that there are potentially intervals of the time-axis[4] where the integrand is positive/negative. Because the integral is positive, the $m^{(-)}$-intervals cannot dominate. For technical reasons these are studied first.

2.1. *Behaviour on $m^{(-)}$-intervals*

Consider now $m^{(-)}$-intervals where

$$T\Omega_\infty(t)\int_0^T \Omega_m(\xi)\,\mathrm{d}\xi < [T\Omega_2(t)]^{1+\mu_m} \qquad (m^{(-)}\text{-intervals}). \tag{2.4}$$

The starting points of these are labelled as $t = t_m(\mu_m, T)$. An important question is under what circumstances, if any, are there functions $\Omega_\infty(t)$ and $\Omega_2(t)$ that satisfy (2.4)? Defining the dimensionless integral $\theta_m(t)$ as

$$\theta_m(t) = \int_0^t \Omega_m(\xi)\,\mathrm{d}\xi, \tag{2.5}$$

the inequality $\Omega_2 \leqslant \Omega_\infty$ turns (2.4) into

$$\theta_m(T) < [T\Omega_2(t)]^{\mu_m} \qquad t \geqslant t_m(\mu_m, T)\,. \tag{2.6}$$

[4] The $\pm$-intervals are also dependent on initial data but this is suppressed notationally.

When $m = 2$, (2.6) is in the right form as it stands but when $m = \infty$ a modification is taken in which Ω_2 is replaced on the right-hand side by Ω_∞. Then the two options are expressed by

$$\theta_m(T) < [T\Omega_m(t)]^{\mu_m} \qquad m = 2 \text{ or } \infty\,. \tag{2.7}$$

If no singularity occurs anywhere in $[0, T]$ then $\theta_m(T) = \int_0^T \Omega_m(\xi)\,d\xi$ takes a fixed finite value for each T: it contains the whole history of $\Omega_m(t)$. Differentiation of the equality (2.5) and using new dimensionless variables based on

$$\Theta_m(\tau_m) = \theta_m(t), \qquad \tau_m = \Omega_0(t - t_m), \qquad \tau_m^* = \Omega_0(T - t_m) \tag{2.8}$$

makes (2.7) into

$$\frac{d\Theta_m}{d\tau_m} > [\Theta_m(\tau_m^*)]^{1/\mu_m} \qquad 0 \leqslant \tau_m \leqslant \tau_m^*. \tag{2.9}$$

A trivial integration gives

$$\Theta_m(\tau_m) > \Theta_m(0) + \tau_m[\Theta_m(\tau_m^*)]^{1/\mu_m}. \tag{2.10}$$

Because $\Theta_m(\tau_m) \leqslant \Theta_m(\tau_m^*)$ an increasing $\Theta_m(\tau_m)$ can only exist in this interval for a finite time $\tau_m^{\max}$ which is defined as the solution of

$$\Theta_m(\tau_m^*) = \Theta_m(0) + \tau_m^{\max}[\Theta_m(\tau_m^*)]^{1/\mu_m}. \tag{2.11}$$

Defining $f_m^* = \Theta_m(\tau_m^*)/\Theta_m(0) \geqslant 1$, $\tau_m^{\max}$ becomes

$$\tau_m^{\max} = \frac{1}{[\Theta_m(0)]^{1/\mu_m - 1}}\left(\frac{f_m^* - 1}{[f_m^*]^{1/\mu_m}}\right). \tag{2.12}$$

The effect of the initial condition (i.e. at $t = t_m$) is encoded in the first factor, while the effect of the final condition (at $t = T$) is encoded in $f_m^* \geqslant 1$. For a fixed initial condition $\Theta_m(0)$, $\tau_m^{\max}$ has a maximum attained at $f_{m,\max}^* = 1/(1 - \mu_m)$, where

$$\tau_m^{\max}\Big|_{f_{m,\max}^*} = \frac{\mu_m(1 - \mu_m)^{1/\mu_m - 1}}{[\Theta_m(0)]^{1/\mu_m - 1}} \to 1 \qquad \text{as} \qquad \mu_m \nearrow 1\,. \tag{2.13}$$

(2.10) and (2.11) show that the $m^{(-)}$– interval widths can be no greater than $\tau_m^{\max}$.

2.2. *Behaviour on $m^{(+)}$-intervals*

We now consider those regions of the time-axis, designated as $m^{(+)}$-intervals, where

$$T\Omega_\infty(t)\int_0^T \Omega_m(\xi)\,d\xi \geqslant [T\Omega_2(t)]^{1+\mu_m} \qquad (m^{(+)}\text{-intervals}). \tag{2.14}$$

(2.14) may be re-expressed as

$$\frac{\Omega_\infty(t)}{\Omega_2(t)} \geqslant [T\Omega_2(t)]^{\mu_m}\left(\int_0^T \Omega_m(\xi)\,d\xi\right)^{-1}. \tag{2.15}$$

There is a universal lower bound of unity on the ratio Ω_∞/Ω_2 but it is clear that once $\Omega_2(t)$ has increased such that

$$(T\Omega_2)^{\mu_m} > \int_0^T \Omega_m(\xi)\,d\xi \tag{2.16}$$

then the lower bound in (2.15) is raised away from unity and values of $\Omega_\infty(t)$ and $\Omega_2(t)$ must diverge rapidly, resulting in different growth rates. For $m = 2, \infty$ (having used $\Omega_2 \leqslant \Omega_\infty$), (2.16) becomes (2.7). It has already been shown in section 2.2 that a solution exists only for a finite time so this divergent behaviour can only be intermittent.

The nature of the integral over T in (2.3) makes it plain that the $2^{(\pm)}$-intervals do not necessarily fall in precisely the same positions as the $\infty^{(\pm)}$-intervals although it might be expected that they are related. Comparing (2.14) and (2.4), and using the inequality $\Omega_2(t) \leqslant \Omega_\infty(t)$, it is easy to prove the following conditional result

Lemma 1. *If $\mu_2 \leqslant \mu_\infty$ then:*

(1) Every $\infty^{(-)}$-interval is contained in some $2^{(-)}$-interval.
(2) Every $2^{(+)}$-interval is contained in some $\infty^{(+)}$-interval.

2.3. Singular behaviour?

A final question is whether (2.4) and (2.14) are formally consistent with the occurrence of singular solutions if T is chosen large enough? (2.14) is consistent with

1. $\Omega_\infty(t)$ blowing up ($\theta_\infty(T) = \infty$) with $\Omega_2(t)$ remaining finite ($\theta_2(T) < \infty$) on both $2^{(+)}$ and $\infty^{(+)}$-intervals.
2. $\Omega_\infty(t)$ and $\Omega_2(t)$ both blowing up simultaneously on both $2^{(+)}$ and $\infty^{(+)}$-intervals.

3. Conclusion

The brief observations in this paper may have some bearing on the subtle yet contradictory differences appearing in the different computations referenced in section 1. While few definitive conclusions have been drawn and detailed conclusions have been left for further discussion, the analysis has shown that the potential occurrence of $m^{\pm}$-intervals in computed solutions for Ω_m ($m = 2, \infty$) may have unintended consequences, as it is likely that their existence, width and distribution will be dependent upon initial data, the nature of the numerical scheme employed and any other processes that may differ with each computation. Numerically it may be useful to compare the time evolution of $\Omega_\infty(t)$ and $\Omega_2(t)$ in a computation over a chosen $[0, T]$ to check which type of intervals appear. If $m^{(-)}$-intervals exist then the subset $m^{(+)}$ is closed which potentially could make it a fractal set, thereby adding to the difficulties inherent in the problem. However, given the positivity of the integral it is possible that no $m^{(-)}$-intervals may ever be seen, although this in itself could be an interesting result.

Our conclusion is that these results call for caution in how three-dimensional Euler computations are interpreted, although we agree that a geometric understanding of solutions may ultimately be the way to make progress with the 'yes' or 'no' question posed in section 1 [44, 46, 47].

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References

[1] Majda A J and Bertozzi A L 2001 *Vorticity and Incompressible Flow* (Cambridge: Cambridge University Press)
[2] Constantin P 2008 *Proc. Conf. "Euler Equations 250 Years On" (Aussois, France, June 2007) Physica* D to appear August 2008
[3] Morf R H, Orszag S A and Frisch U 1980 Spontaneous singularity in three-dimensional, inviscid incompressible flow *Phys. Rev. Lett.* **44** 572–5
[4] Bardos C, Benachour S and Zerner M 1976 Analyticité des solutions périodiques de l'équation d'Euler en deux dimensions *C. R. Acad. Sci. Paris* **282A** 995–8
[5] Bardos C and Benachour S 1977 Domaine d'analyticité des solutions de l'équation d'Euler dans un ouvert de R^n *Ann. Sc. Norm. Super. Pisa, Cl. Sci. IV Ser.* **4** 647–87
[6] Pauls W, Matsumoto T, Frisch U and Bec J 2006 Nature of complex singularities for the 2D Euler equation *Physica* D **219** 40–59
[7] Chorin A J 1982 The evolution of a turbulent vortex *Commun. Math. Phys.* **83** 517–35

[8] Brachet M E, Meiron D I, Orszag S A, Nickel B G, Morf R H and Frisch U 1983 Small-scale structure of the Taylor–Green vortex *J. Fluid Mech.* **130** 411–52

[9] Siggia E D 1984 Collapse and amplification of a vortex filament *Phys. Fluids* **28** 794–805

[10] Ashurst W and Meiron D 1987 Numerical study of vortex reconnection *Phys. Rev. Lett.* **58** 1632–5

[11] Pumir A and Kerr R M 1987 Numerical simulation of interacting vortex tubes *Phys. Rev. Lett.* **58** 1636–39

[12] Pumir A and Siggia E 1990 Collapsing solutions to the 3D Euler equations, *Phys. Fluids* A **2** 220–41

[13] Bell J B and Marcus D L 1992 Vorticity intensification and transition to turbulence in the three-dimensional Euler equations *Commun. Math. Phys.* **147** 371–94

[14] Brachet M E, Meneguzzi V, Vincent A, Politano H and Sulem P-L 1992 Numerical evidence of smooth self-similar dynamics and the possibility of subsequent collapse for ideal flows *Phys. Fluids* A **4** 2845–54

[15] Kerr R M 1993 Evidence for a singularity of the three-dimensional incompressible Euler equations *Phys. Fluids* A **5** 1725–46

[16] Kerr R M 2005 Vorticity and scaling of collapsing Euler vortices *Phys. Fluids* A **17** 075103–114

[17] Bustamante M D and Kerr R M 2008 3D Euler about a 2D symmetry plane *Proc. of "Euler Equations 250 Years On" (Aussois, France, June 2007) Physica* D to appear August 2008 (doi:10.1016/j.physd.2008.02.007)

[18] Hou T Y and Li R 2006 Dynamic depletion of vortex stretching and non-blowup of the 3-D incompressible Euler Equations *J. Nonlinear Sci.* **16** 639–64

[19] Hou T Y and Li R 2008 Blowup or no blowup? The interplay between theory and numerics *Proc. "Euler Equations 250 Years On" (Aussois, France, June 2007) Physica* D to appear August 2008

[20] Orlandi P and Carnevale G 2007 Nonlinear amplification of vorticity in inviscid interaction of orthogonal Lamb dipoles *Phys. Fluids* **19** 057106

[21] Grauer R, Marliani C and Germaschewski K 1998 Adaptive mesh refinement for singular solutions of the incompressible Euler equations *Phys. Rev. Lett.* **80** 4177–80

[22] Grafke T, Homann H, Dreher J and Grauer R 2008 Numerical simulations of possible finite time singularities in the incompressible Euler equations: comparison of numerical methods *Proc. "Euler Equations 250 Years On" (Aussois, France, June 2007) Physica* D to appear August 2008

[23] Kida S 1985 Three-dimensional periodic flows with high-symmetry *J. Phys. Soc. Japan* **54** 2132–6

[24] Kerr R M 2005 Vortex collapse and turbulence *Fluid Dyn. Res.* **36** 249–60

[25] Boratav O N and Pelz R B 1994 Direct numerical simulation of transition to turbulence from a high-symmetry initial condition *Phys. Fluids* **6** 2757–84

[26] Boratav O N and Pelz R B 1995 On the local topology evolution of a high-symmetry flow *Phys. Fluids* **7** 1712–31

[27] Pelz R B 1997 Locally self-similar, finite-time collapse in a high-symmetry vortex filament model *Phys. Rev.* E **55** 1617–26

[28] Pelz R B and Gulak Y 1997 Evidence for a real-time singularity in hydrodynamics from time series analysis *Phys. Rev. Lett.* **79** 4998–5001

[29] Pelz R B 2001 Symmetry and the hydrodynamic blow-up problem *J. Fluid Mech.* **444** 299–320

[30] Cichowlas C and Brachet M-E 2005 Evolution of complex singularities in Kida–Pelz and Taylor–Green inviscid flows *Fluid Dyn. Res. 36* 239–48

[31] Gulak Y and Pelz R B 2005 High-symmetry Kida flow: time series analysis and resummation *Fluid Dyn. Res.* **36** 211–20

[32] Pelz R B and Ohkitani K 2005 Linearly strained flows with and without boundaries—the regularizing effect of the pressure term *Fluid Dyn. Res.* **36** 193–210

[33] Gibbon J D 2008 The three dimensional Euler equations: how much do we know? *Proc. "Euler Equations 250 Years On" (Aussois, France, June 2007) Physica* D to appear August 2008 (doi:10.1016/j.physd.2007.10.014)

[34] Bardos C and Titi E S 2007 Euler equations of incompressible ideal fluids *Russ. Math. Surv.* **62** 409–51

[35] Beale J T, Kato T and Majda A J 1984 Remarks on the breakdown of smooth solutions for the 3D Euler equations *Commun. Math. Phys.* **94** 61–6

[36] Ferrari A 1993 On the blow-up of solutions of the 3D Euler equations in a bounded domain *Commun. Math. Phys.* **155** 277–94

[37] Kozono H and Taniuchi Y 2000 Limiting case of the Sobolev inequality in BMO, with applications to the Euler equations *Commun. Math. Phys.* **214** 191–200

[38] Ponce G 1985 Remarks on a paper by J. T. Beale, T. Kato and A. Majda *Commun. Math. Phys.* **98** 349–53

[39] Chae D 2003 Remarks on the blow-up of the Euler equations and the related equations *Commun. Math. Phys.* **245** 539–50

[40] Chae D 2004 Local existence and blow-up criterion for the Euler equations in the Besov spaces *Asymptotic Analysis* **38** 339–58

[41] Chae D 2005 Remarks on the blow-up criterion of the 3D Euler equations *Nonlinearity* **18** 1021–9

[42] Chae D 2007 On the finite time singularities of the 3D incompressible Euler equations *Commun. Pure Appl. Math.* **60** 597–617

[43] Constantin P 1994 Geometric statistics in turbulence *SIAM Rev.* **36** 73–98

[44] Constantin P, Fefferman Ch and Majda A J 1996 Geometric constraints on potentially singular solutions for the 3D Euler equation *Commun. Partial Diff. Eqns* **21** 559–71

[45] Cordoba D and Fefferman Ch 2001 On the collapse of tubes carried by 3D incompressible flows *Commun. Math. Phys.* **222** 293–8

[46] Deng J, Hou T Y and Yu X 2005 Geometric properties and non-blowup of 3D incompressible Euler flow *Commun. Partial Diff. Eqns* **30** 225–43

[47] Deng J, Hou T Y and Yu X 2003 Improved geometric condition for non-blowup of the $3D$ incompressible Euler equation *Commun. Partial Diff. Eqns* **31** 293–306

[48] Gibbon J D 2007 Ortho-normal quaternion frames, Lagrangian evolution equations and the three-dimensional Euler equations *Russ. Math. Surv.* **62** 1–26
Gibbon J D 2007 *Uspekhi Mat. Nauk* **62** 47–72

[49] Constantin P and Foias C 1988 *Navier–Stokes Equations* (Chicago, IL: The University of Chicago Press)

[50] Foias C, Manley O, Rosa R and Temam R 2001 *Navier–Stokes equations & Turbulence* (Cambridge: Cambridge University Press)

[51] Brenier Y 1999 Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations *Commun. Pure Appl. Math.* **52** 411–52

[52] Shnirelman A 1997 On the non-uniqueness of weak solution of the Euler equation *Commun. Pure Appl. Math.* **50** 1260–86

[53] De Lellis C and Székelyhidi L 2007 The Euler equations as a differential inclusion *Ann. Math.* to be published

[54] Gibbon J D and Doering C R 2003 Intermittency is solutions of the three-dimensional Navier–Stokes equations *J. Fluid Mech.* **478** 227–35

[55] Gibbon J D and Doering 2005 Intermittency & regularity issues in three-dimensional Navier–Stokes turbulence *Arch. Rat. Mech. Anal.* **177** 115–50

[56] Gibbon J D and Titi E S 2005 Cluster formation in complex multi-scale systems *Proc. R. Soc.* **461** 3089–97

[57] Gibbon J D and Pavliotis G A 2007 Estimates for the two-dimensional Navier–Stokes equations in terms of the Reynolds number *J. Math. Phys.* **48** 065202