Space-Time in the Time-Dependent Schrodinger Equation and Classical Mechanics

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The time-independent Schrodinger equation: En= -1/2m [d/dx dWn/dx] / Wn(x) + V(x) ((1)) matches a classical conservation of energy equation where -1/2m [d/dx dWn/dx] / Wn(x) = KEclassical(x). In classical physics, however, one has x(t) so each x point is associated with a specific t for a given energy En. In the quantum time-independent equation, one has exp(-iEnt) as the time dependent factor and does not consider any link with x although if one is comparing ((1)) to a classical conservation equation then a link between x and t could be made to complete the analogy. This is usually not considered, but we examine this point as we argue it becomes important in the time-dependent situation.

For the time-dependent Schrodinger case we consider two approaches. First, a linear combination of time independent solutions is taken i.e.

W(x,t) = Sum over n an(En) exp(-iEnt) Wn(x) ((2))

Again each Wn(x) has an association with KEclassical(x) and so one may think in terms of (x,t) classical pairs for a given En.

The second approach is to consider the function: exp(-iEnt + m Integral (0,x) v(y) dy) ((3)). This satisfies the time-dependent Schrodinger equation and yields a classical conservation of energy equation with v(x) being the classical velocity. Noting that En= .5m v(x)v(x) + V(x), ((3)) may be rewritten as: exp(i Integral L dt) where L = T-V is the Langrangian with T=.5mv(x)v(x). At this point En has been removed from the picture, but is replaced with:

x(t)= X g(t)/g(T) ((4)) where (X,T) are a space-time pair corresponding to a particular energy E value.

Using exp(i Action) ((5)) in the time-dependent Schrodinger equation yields a conservation equation which holds for a particular classical energy as long as (x,t) values in the equation are classically linked to the appropriate classical (X,T) values i.e. for a classical trajectory there is a specific family of X at time T for a given energy. For another energy, there is another family of (X,T) pairs. Thus the equation obtained represents energy conservation for many different energy values depending on which simultaneous values one uses for x and t. There is, however, no overlap as a specific (X,T) family corresponds to an energy E and another nonoverlapping family of (X,T) to another energy.

This, however, is the same situation as ((2)) for which there are also families of (X,T) corresponding to each En. The only difference is that the En are quantized i.e. discrete. ((3)) and ((5)) are both solutions to the time-dependent Schrodinger equation which represent a variety of energies and families of (X,T) associated to the classical solution of the energy in question. We thus suggest that ((2)) and ((4)) are equivalent for this reason without using the idea of a complete set of basis functions represented by Wn(x).

We consider these ideas as the Schrodinger propagator is given by ((5)), but with ((4)) generalized to:

x(t)= Xi g(tf-t)/g(tf-ti)   + Xf g(t-ti)/g(tf-ti) for the case that g(0)=0 (2) so that Xi,Xf, tf and ti enter the picture.

This then should be equivalent to:

Sum over n   exp(-iEntf) Wn(xf) Wn* (xi)   ((6))

In this note, we suggest that one may link ((5)) and ((6)) based on ideas of space-time families associated with different energies as well as the two being solutions of the time-dependent Schrodinger equation.

Stochastic Nature of the Time-Independent Schrodinger Case

We have shown in previous notes (1) that one may obtain for either the relativistic or nonrelativistic classical free particle Action where v=X/T:

dAction/dX = p   and dAction/dt = -E   ((7a))(((7b))

One may combine ((7)) with a nonrelativistic energy conservation equation to obtain the time-dependent Schrodinger equation for a free particle. Normally one would not vary X and T independently as one main idea of classical mechanics is that they are linked through x(t), but we argue that ((7)) shows the stochastic nature of quantum mechanics. It may be noted, however, that even though one uses d/dx and d/dt in the quantum case, the solution exp(ipx) in free space represents the probability of a particle which moves with constant p i.e. on average x=p/m t. Thus there is a family of (x,t) pairs associated with the motion.

One may use the free particle probability exp(ipx) to build a statistical distribution of interfering complex probabilities which are used to create averages which are subject to the classical conservation of energy equation i.e.

E =   -1/2m [Sum over p a(p) pp/2m exp(ipx)] / [Sum over p a(p)exp(ipx)] + V(x)   ((8))

This may be rewritten as a differential equation which only has solutions for a discrete set of energies i.e. En.

((8)) has the form of a classical energy conservation equation with:

KEclassical(x) = -1/2m [Sum over p a(p) pp/2m exp(ipx)] / [Sum over p a(p)exp(ipx)]

If one makes the analogy of a vrms (rms velocity) behaving as a classical particle in time-independent Schrodinger theory, then this vrms for a specific energy En should be associated with (X,T) classical space time pairs corresponding to the classical motion. One may mathematically think of these T values as appearing in exp(-iEnt), the time dependent factor of the wavefunction. Thus, formally one may think in terms of a family of (X,T) pairs for a given energy.

Time-Dependent Schrodinger Case from the Time-Independent Case

The time-independent Schrodinger case may be used to create linear combinations which are a solution to the time-dependent equation i.e.

W(x,t) = Sum over n an(En)exp(-iEnt) Wn(x) ((9))

Continuing with the classical analogy, Wn(x) is linked to classical kinetic energy through -1/2m d/dx d/dx so for a given En, there is a family of (X,T) classical pairs linked to the classical motion. Each En has its own family of these pairs. Furthermore, these pairs do not overlap so ((9)) is interesting in that each summand represents a certain portion of space-time i.e. that of its family of (X,T) values. Thus space-time contains nonoverlapping families of (X,T) regions, each representing a specific En.

We wish to extend this idea to a different approach to finding a solution of the time-dependent Schrodinger equation.

Time-Dependent Schrodinger Equation From exp(i Action)

It is possible to create a classical conservation of energy equation from the time dependent Schrodinger equation by postulating a function:

exp(-iEnt + i Integral (0,x) m v(y) dy) ((10))

The conservation equation resulting holds for a specific En. One may next rewrite ((10)) using En=.5mv(x)v(x) + V(x) to obtain:

exp(i Action) where Action= Integral (0,t) dt (KE-V) ((11))

To use ((11)) one needs x(t) and v(t). Using:

x= X/g(T) g(t) where (X,T) are a classical pair for a given energy En allows one to remove En from the picture explicitly, but implicitly it is present in the family of (X,T) pairs associated with En's classical motion. The resulting equation (from the time-dependent Schrodinger equation) contains X and T. One may use a family of (X,T) pairs for one energy or another (X,T) family for another energy. Thus all energies are present, but spacetime contains into (X,T) family regions associated with different energies.

This appears to be very similar to the situation of ((9)) where one also has a solution to the time-dependent Schrodinger equation with space time containing (X,T) families. ((9)) however, consists of discrete En solutions linked to boundary conditions, so we retain this idea, but suggest that ((11)) and ((9)) essentially contain the same physics. We thus argue that one may link a linear combination of time-independent Schrodinger solutions (with exp(-iEnt) factors) to a classical type of exp(i Action) solution.

The reason we are concerned about this link is that one may generalize ((9)) and ((11) to the propagator approach to quantum mechanics. The propagator is given in the two schemes by:

K(T, Xf, Xi) = Sum over n exp(-iEnT) Wn(Xf) Wn*(Xi) ((12))

And exp(i Action) where Action = Integral (0,T) L dt with

x(t)= Xi g(tf-t)/g(tf-ti) + Xf g(t-ti)/g(tf-ti) for the case that g(0)=0 (see (2)) ((13))

At first the two approaches seem very different, but both are solutions of the time-dependent Schrodinger equation representing a range of different energies, each with their respective nonoverlapping families of (X,T) classical pairs.

Conclusion

In conclusion, we argue that the time-independent Schrodinger equation represents a classical conservation of energy equation with KEclassical(x)= [-1/2m d/dx dW/dx]/W and exp(-iEnt) represents the time dependent factor. If one carries this analogy with classical physics further, there is a family of (X,T) pairs associated with the classical motion for a given En. If one creates a linear combination of time-independent solutions i.e. W(x,t)=Sum over n an(En)exp(-iEnt) Wn(x), then each summand is associated with a nonoverlapping family of (X,T) space points associated with the classical motion.

One may alternatively use the time-dependent Schrodinger equation together with the solution exp(-iEnt + i Integral (0,x) m v(y) dy). This may be rewritten as exp (i Integral dt L) where L is the Lagrangian = KE-V. En has disappeared, but one uses x(t)= X/g(T) g(t) with X and T being any pair of a family corresponding to the classical motion for this energy. The resulting equation following from the time-dependent Schrodinger equation represents conservation of energy for the family (X,T) associated with En. If one uses other (X,T) families, they are associated with other energy values. This is very similar to the linear combination of time-independent solutions with exp(-iEnt) factors we argue. Thus, given that both approaches yield a solution to the time-dependent Schrodinger equation with a range of energies and (X,T) nonoverlapping families, we suggest they are the same. This we argue may be a useful way to consider the propagator exp(i Action) which follows by using:

x(t)= Xi g(tf-t)/g(tf-ti) + Xf g(t-ti)/g(tf-ti) for the case that g(0)=0 with exp(iAction) or is alternatively written as:

K(T,Xf,Xi) = Sum over n exp(-iEnt) Wn(Xf)Wn*(Xi)

References

1.Ruggeri, Francesco R. The Klein Gordon Equation, exp(i Action) and Fluctuations (preprint, zenodo, 2021)

2. https://en.wikipedia.org/wiki/Path_integral_formulation