Modelling Division: Towards a Local Instructional Theory for the Teaching of Multi-Digit Division

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This paper reports on a teaching experiment in which a hypothetical learning trajectory was enacted. The aim of this learning trajectory was to support children to develop efficient strategies for calculating multi-digit division computations in ways that made sense to them. This paper analyses the strategies that they used and how these changed over the course of the teaching experiment. Findings indicate that the teaching approach, which emphasised trialling multiple solution strategies and selecting and justifying computation methods, allowed children to develop efficient, meaningful solution strategies.

Introduction

This paper investigates a teaching experiment focused on long division. This took place in the first author's classroom as part of the Maths4All project, a SFI funded project which aims to develop resources for, and with, teachers. In this paper, we analyse the strategies that children developed and how these changed over the course of the experiment.

Literature Review

The draft specification of the Irish primary mathematics curriculum proposes mathematical proficiency as the central goal for mathematics teaching (National Council for Curriculum and Assessment [NCCA], 2017). Mathematical proficiency is conceived as consisting of the intertwined strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Kilpatrick et al., 2001). Strategic competence is understood to mean the ability to formulate, represent and solve problems. Adaptive reasoning involves the capacity for logical thought, reflection, explanation, and justification. This vision of desirable learner outcomes has implications for teaching. In particular, five meta-practices are advocated in the research reports underpinning the draft specification (Dooley et al., 2014). While all meta-practices are relevant to the teaching of division, we highlight the proposed emphasis on mathematical modelling. Distinct from traditional understandings of teacher modelling, where a teacher might use concrete materials or other resources to model mathematical ideas, in mathematical modeling the focus is on supporting children's own modelling of problems- their use of mathematics to describe a context and develop meaningful solutions (c.f., Suh & Seshaiyer, 2017). As children develop their own models of situations, there will be conceptual and procedural components (Lesh & Harel, 2003). On a conceptual level, a model describes how elements of a system relate to each other but it may also have accompanying procedures for accomplishing goals.

There are strong implications for the teaching of mathematical operations if mathematical proficency is accepted as the ultimate goal of teaching. Traditionally, procedural fluency is a priority and many teachers understand mathematics learning as concerned primarily with memorising number facts and facility with conventional digit-based algorithms (Schulz, 2018). A contrasting approach is followed in the Netherlands, where informed by Realistic Mathematics Education, there is an early focus on supporting children's informal mental calculations. Over time, focused efforts are made to guide development from informal methods to formal algorithms using progressive schematization of informal strategies (van Putten et al., 2005). Informal methods may involve partitioning of number but generally maintain place value. This contrasts with the digit-based strategies, used in the conventional division algorithm, which operate on individual digits in a procedural way. Algorithms can minimise the demands on working memory and on reasoning processes but once introduced, may inhibit the use of number-based, informal calculation strategies (Schulz, 2018) and thereby inhibit children reasoning adaptively about the task. Amongst the relatively few studies of multi-digit division, Schulz (2018) has presented a theoretical and empirical analysis outlining the ways in which division strategies rely on two types of reasoning abilities: reasoning about relations between numbers, and reasoning about relations between operations. Repeated addition and subtraction are generally the first intuitive strategies for division. More developed strategies recognise and use the multiplicative relationships between the dividend and the divisor. Advanced strategies decompose or adapt the dividend and/or divisor to create easier calculations from which the final solution can be derived. It is also possible to categorise division strategies according to the ways in which students create multiples of the divisor (chunking) to be subtracted from the dividend (van Putten et al., 2005). *Low-level chunking* refers to using doubling or small multiples while *high-level chunking* refers to subtracting higher multiples or chunks, such as ten times the divisor.

Methodology

Hypothetical learning trajectories (HLT) are understood to involve teachers designing sequences of instructional activities that they imagine will support children in moving from their current levels of thinking to the desired goals (Simon, 1995). These trajectories are considered to be hypothetical because, until tasks are enacted, the teacher can only imagine how children might engage. The teacher-researcher, first author of this paper, developed and iteratively refined the instructional sequence described in this paper over four years of teaching at this class level, though previous iterations were not formally researched. Gravemeijer (2004) recommends the use of HLT to describe the planning of instructional activities in a classroom on a day-to-day basis. He notes that in developing HLTs teachers may draw on local instructional theory. For Gravemeijer, local instructional theories include a clear description of (i) learning goals, (ii) planned instructional activities and (iii) an empirically grounded theory of how the instructional activities might develop students' thinking. This paper presents the first empirical analysis of data related to these activities and thus is moving toward meeting part (iii) of Gravemijer's conditions.

All teaching is underpinned by understandings of the overarching purpose. For both authors, this involves a commitment to developing children's agency, authority and identity through a pedagogical approach which involves enactment of the five meta-practices (Dooley et al., 2014). In practice, this involved lessons where a small number of problems were explored in great depth, with children encouraged to discuss, analyse and trial methods proposed by others as well as give justifications for their choice of strategy, i.e., numerous opportunities were created for children to engage in adaptive reasoning. The ultimate goal of the hypothetical learning trajectory was that children develop efficient strategies for calculating multi-digit division computations that make sense to them. As outlined in the overview on Table 1, the first two lessons were exploratory in nature, with no strategies presented by the teacher. The long division algorithm was introduced for the first time during Lesson 3 but it was presented as an alternative method rather than a superior method. In the final lessons, children solved division problems in a variety of ways and justified the reasonableness of their chosen approaches. Across these lessons, digital records of board work were collected as well as children's written solution strategies. At the end of each lesson, children were invited to provide a short, written reflection in response to a question posed by the teacher. These questions generally prompted children to reflect on strategies used or to select and justify their preference of strategy. All relevant ethical procedures were followed in the collection of this data and in total eighteen children participated in this study.

Table 1

Unit Overview

Lesson	*Extension*
Lesson 1 - 'Punnets of Strawberries' A punnet holds 23 strawberries. How many punnets can be filled from a basket holding 115 strawberries?	228 ÷ 38; 176 ÷ 36; 279 ÷ 54; 375 ÷ 17
Lesson 2 - 'School Library' A school library has 719 books. How many shelves, each holding 24 books, will be needed to display the books?	416 ÷ 15; 786 ÷ 19; 805 ÷ 22; 751 ÷ 45
Lesson 3 - 'The Car Transporter' A car transporter delivered 216 cars over 18 trips. How many cars were carried on each trip?	Explore 'Mandeep's Method' – The Long Division Algorithm.
Lesson 4 - 'Long Division' Solve 389 ÷ 17 using 'long division' and/or in other ways.	638 ÷ 25; 736 ÷ 34; 716 ÷ 18; 417 ÷ 16
Lesson 5 - Going Around in Circles (nrich) A railway line has 27 stations on a circular loop. If I fall asleep and travel through 312 stations, where will I end up in relation to where I started? Which station will I end up at?	If it is midday now, will it be light or dark in 539 hours? What time will it be?

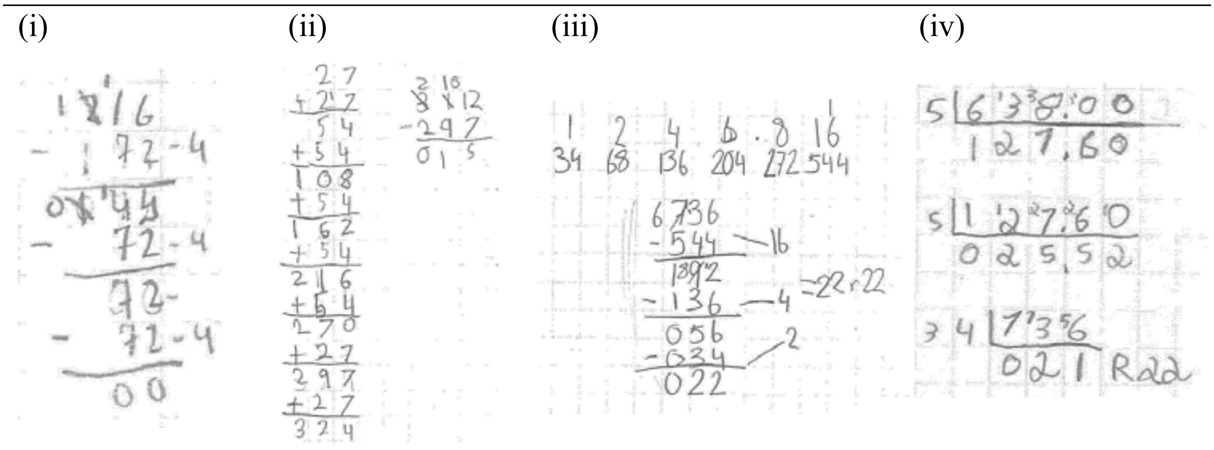
We are interested in understanding how children's models of division, their conceptual and procedural understandings of division situations (c.f., Lesh & Harel, 2003), developed across the teaching sequence. We are guided by the research questions: (i) what strategies did children use as they engaged in this sequence of activities? (ii) how did these change over the course of the teaching experiment? Our focus on 'strategies' gives explicit attention to the procedural aspects of children's solutions. Some inferences about conceptual understandings are also possible. Given the constraints of this paper, only a subset of the data was analysed. Seven students were selected from the larger group and their reflections and recordings of their strategies across all lessons were analysed. Descriptors for the division strategy used in each problem were assigned. These strategies, described in more detail below, align with those identified in the literature (c.f., Schulz, 2018; van Putten et al., 2005).

Findings

These findings offer insight into the division strategies that were developed by seven, fifth class participants as they engaged with the tasks of the HLT. The approaches emerged as the children engaged in lessons which aimed to foster adaptive reasoning (Kilpatrick et al., 2001), where the teacher used the meta-practices advocated in Dooley et al. (2014). Analysis of the strategies that different children employed, at different points of the HLT, offers some insight into their trajectories of learning and the reasoning that they were engaged in. These findings share the variety of division strategies that emerged from the participants and provide some insight to their learning trajectories. Findings are of relevance given the pedagogical alignment with the advocated emphasis on children's mathematical modeling in the draft curriculum specification. The children's strategies are discussed under six categories below.

Figure 1

Division Strategies Developed by Fifth Class Children



Repeated Addition and Repeated Subtraction

Strategies involving repeated addition and repeated subtraction were used repeatedly by the participants in this study. During Lesson One 'the strawberry problem' was presented for the children to solve collaboratively, using an approach that they could justify. All participants used repeated addition or subtraction, as one of their approaches to solving this problem. Strategies, such as this, are indicative of reasoning about relations between operations (Schulz, 2018). In the case of 'the strawberry problem' the children applied their understanding of subtraction to the unfamiliar division context of *115 ÷ 23.* Formatting the task as a word problem offered flexibility for the development of various approaches and the numbers were chosen to accommodate repeated addition and subtraction. The frequent use of repeated addition and subtraction during initial lessons correlates with Schulz's (2018) suggestion that these are often the first intuitive strategies for division. The numbers chosen, as part of the HLT, were organised with the quotient increasing as the children progressed through the tasks. As the quotient increased, repeated addition and subtraction strategies became cumbersome and all participants adapted their initial approaches to solve the tasks more efficiently, through low-level chunking. Some of the participants developed this strategy independently while others adopted it following the sharing of strategies. Repeated addition, repeated subtraction and low-level chunking remained the preferred solution strategies for a number of the participants throughout this study.

Low-Level Chunking

Figure 1 (i) and Figure 1 (ii) illustrate how two child adapted their initial strategies to improve efficiency. All of the participants adapted their repeated addition and subtraction strategies but some were more methodical in their approach than others. Some children appeared to make decisions before beginning their calculation, such as deciding to quadruple the divisor, as shown in Figure 1 (i). The strategies adopted by others appeared to emerge as they worked on a problem. Figure 1 (ii) depicts a child's low-level chunking, taken from Lesson Five. This child's strategy appeared to emerge as they worked on the task, using a combination of adding 27, and adding double 27, to reach 312. We can see that when the child surpassed 312, they returned to the previous step of the calculation and used that sum, 297, to calculate the remainder for their solution. The fact that the strategy emerges as the child enacts it, indicates engagement in reasoning and decision making throughout the process.

In developing low-level chunking, which involves using basic multiples, e.g., doubles, of the divisor in conjunction with repeated addition or subtraction, the participants demonstrated both reasoning about the relations between operations and reasoning about the relations between numbers (Schulz, 2018). The children made connections between the division problem context and the operations of addition and subtraction. They also engaged in reasoning about number through their comparison of the dividend and the divisor and their endeavour to manipulate the divisor to expediate their calculations.

High-Level Chunking

A number of participants adapted and extended low-level chunking to invent a new, more efficient strategy, which we have coded as high-level chunking. High-level chunking involves a deeper engagement with the multiples of the divisor and the comparison of these multiples to the dividend. Figure 1 (iii) depicts the use of this strategy to solve, *736 ÷ 34*. While there is an error in this child's recording, it appears that they were involved in considerable reasoning about the relationship between the divisor and the dividend. The participant created a list of multiples, of 34, and appeared to take cognisance of the dividend, scaling up their multiples until they found one that was sufficiently close to the dividend to make their strategy efficient. Some children used a calculator to help them establish the multiples of the divisor as part of this strategy.

The work in Figure 1 (iii) was completed during Lesson Four, after the children were introduced to the formal, long division algorithm. At this point, some students were using the formal algorithm and some were using their preferred invented strategies. The child's work in Figure 1 (iii) demonstrates deeper reasoning about the relations between numbers and operations (Schulz, 2018), compared to that demonstrated with low-level chunking. The adoption of a high-level chunking approach is indicative of reasoning associated with number sense, multiplication, subtraction and division and of the connections that can be utilised to solve problems in novel contexts. While all the participants in the study utilised low-level chunking, only some adopted a high-level chunking approach. In respecting the different trajectories of learning of the children, a broad variety of student invented strategies were shared, discussed and praised during whole-class discussions at all stages of the unit of work.

Missing Factor

The Missing Factor approach emerged during Lesson One and many of the participating children utilised it repeatedly in subsequent lessons. The strategy involved estimation and trial-and-improvement as a child aimed to determine the missing factor in multiplication sentences to solve division problems e.g., determine the missing factor in *24 x __ = 720* to identify the solution to *720 ÷ 24.* This strategy suggests reasoning about relations between operations (Schulz, 2018), as the children utilise the inverse relationship between multiplication and division. Some participants appeared to experience greater success with this strategy than others. Those who engaged in deeper reasoning about the relations between numbers (Schulz, 2018) and who were able to make accurate estimates could use this strategy efficiently whereas those who found estimation more difficult engaged in a longer series of trial-and-improvement cycles and tended to prefer other strategies. Schulz (2018) identifies strategies that use the multiplicative relationships between the dividend and the divisor, such as this, as being more advanced than chunking strategies.

Decompose the Divisor and Divide Stepwise

Figure 1 (iv) shows two different solution strategies that one participant employed during Lesson Four. The first involved decomposing the divisor and dividing stepwise, to solve *638 ÷ 25*. This child was the first participant to propose the decomposition of the divisor. They utilised many different approaches throughout the unit but they employed this strategy during each of the five lessons. When they first developed the approach they recorded, "I came up with my own way. An example of my way is, instead of dividing 657 by 9 I divide 657 by 3 and that equals 219. Then I divide 219 by 3" (Child C, Reflections). The child appeared to take ownership of this solution strategy, using it repeatedly and sharing it with their classmates during whole-class discussions. Schulz (2018) views strategies that involve the decomposition of the divisor, in this way, as advanced strategies of division.

This participant encountered a difficulty, in employing their strategy, when the dividend was not divisible by the decomposed divisor, as occurred in the case of Figure 1 (iv). However, they appeared eager to learn about the relationship between remainders and decimals and continued to choose this division strategy. During the unit of work there were many opportunities for the children to share strategies with their classmates and to use each other's ideas. After this child shared their approach with the class a number of students adopted it, perhaps because of its contrast to the strategies discussed thus far. This strategy demonstrates reasoning about relations between operations (Schulz, 2018) as the child manipulated the divisor to apply the short division algorithm that was familiar to them. It also demonstrates reasoning about the relations between numbers (Schulz, 2018) as they explored divisibility and the relationship between remainders and decimals.

Standard Algorithms

During Lesson Three the standard long division algorithm was introduced to the children. Initially, they solved a problem using invented methods and subsequently the new strategy, which was dubbed 'Mandeep's Method,' was presented. The children were encouraged to make sense of 'Mandeep's Method,' to compare it to their own invented strategies and to use it themselves. In previous lessons they were asked to use one another's methods and, in a similar way, they were now being asked to try a new method. Memorisation of the procedure was not encouraged and there was an understanding that the children could choose to use their preferred, invented methods or the long division algorithm.

All participants utilised the long division algorithm during the final two lessons. Some moved towards using it exclusively, some used it in conjunction with an invented method as a way of self-correcting and some tried it but then reverted to invented strategies. One particular participant used the long division algorithm in conjunction with an invented strategy during Lesson Four but then move towards using the algorithm exclusively during the final lesson. In their reflection at the end of Lesson Four they noted, "I would use the long way/new way. I would use it because, to me, it's easier and quicker" (Child E, Reflections). This reflection appears to support the idea that the use of an algorithm can minimise demands on working memory, making finding a solution easier and quicker (Schulz, 2018). Cognisance was taken of the multiple trajectories of learning present in the classroom and emphasis was placed on progressing each child's understanding and on developing their conceptual understandings of the division of multi-digit numbers.

In another case during Lesson Four, a participant engaged in reasoning about the relations between operations by making connections between the long division and the short division algorithms, adapting the latter to accommodate multi-digit division. The second solution strategy, depicted in Figure 1 (iv) and solving $736 \div 34$, depicts how a participant adapted the short division algorithm, that they would have encountered during the previous school year, to take account of the multi-digit divisor. The child developed this solution strategy after being introduced to the long division algorithm. The solution strategy follows the same general procedure as the long division algorithm but the child demonstrated strong mental arithmetic skills as many steps were completed mentally rather than symbolically.

Conclusion

The findings discussed in this paper highlight one aspect of the collected data, with this paper presenting the first empirical analysis thereof. This analysis offers insight into the reasoning of the participants in this study, as they engaged with the division problems, and highlights the varying strategies that they developed for solving said problems, many of which stretched beyond the anticipated student responses predicted in the planning of the HLT. The findings highlight the participants' capacity to develop efficient and meaningful solution strategies and to engage meaningfully with the formal algorithm. The previous discussion focuses on the children's solutions and samples of their work but this is just one facet of the situation. The importance of the pedagogical approach, in developing adaptive reasoning, cannot be understated and merits further exploration. This paper aimed to offer insight into the broad variety of strategies that the participants utilised, but it is important to note that each child followed an individual trajectory of learning. The individual trajectories that the participants followed also merit further investigation and study.

References

Dooley, T., Dunphy, E., Shiel, G., Butler, D., Corcoran, D., Farrell, T., NicMhuirí, S., O'Connor, M., Travers, J., & Perry, B. (2014). *Mathematics in early childhood and primary education (3-8 years). Teaching and learning. National Council for Curriculum and Assessment (NCCA). Research report no. 18*. NCCA

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical thinking and learning*, *6*(2), 105-128.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academy Press

Lesh, R., & Harel, G. (2003). Problem solving, modeling, and local conceptual development. *Mathematical thinking and learning*, *5*(2-3), 157-189.

NCCA (2017). *Primary mathematics curriculum: Draft specification junior infants to second class*. NCCA.

Schulz, A. (2018). Relational reasoning about numbers and operations–Foundation for calculation strategy use in multi-digit multiplication and division. *Mathematical Thinking and Learning*, *20*(2), 108-141.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for research in mathematics education*, *26*(2), 114-145.

Suh, J. M., & Seshaiyer, P. (2017). *Modeling mathematical ideas: Developing strategic competence in elementary and middle school*. Rowman & Littlefield Publishers.

van Putten, C. M., van den Brom-Snijders, P. A., & Beishuizen, M. (2005). Progressive mathematization of long division strategies in Dutch primary schools. *Journal for Research in Mathematics Education*, *36*(1), 44-73.