An Investigation into Pre-Service Post-Primary Mathematics Teachers' Knowledge of Problem-Solving

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We discuss the mathematical problem-solving proficiency of pre-service post-primary mathematics teachers in an Irish University, where the participants were undertaking concurrent teacher education programmes. The conceptual framework of this study is based on the work of Chapman (2015) who outlines that problem-solving proficiency is a key component in the effective teaching of problem-solving. We describe the range of data collection exercises undertaken as part of this study, and report in detail on one whereby the participants undertook two problems following a 'Think Aloud' protocol in recorded interviews. The interviews were analysed using a general inductive approach and five main themes were identified in the participants' approaches to problem-solving. We report here on the analysis of the interviews and the role of problem-solving proficiency in the teaching of problem-solving.

Introduction and Background

In this paper, we report on a project that investigates the capacities of pre-service teachers to teach problem-solving. This project is being undertaken by the first author as her doctoral research. In this introduction, we provide the background to the project and motivate our research questions. We outline the key issue of characterising problem-solving in mathematics, outline its central role in school curricula (discussing the performance of Irish school students on international assessments of mathematical problem-solving), and link this role to the importance of teacher preparation for teaching problem solving.

Problem-Solving in Mathematics

Acknowledging Polya's (1945) efforts to put problem-solving at the centre of mathematical instruction, Schoenfeld (1992) attests that there is a wide variety of meanings for the terms "problems" and "problem solving": this has been highlighted more recently by Lester (2013). The variations in these definitions are further discussed in Owens and Nolan (2019). Recognising the need for a clear definition the following Three Key Characteristics were identified, effectively defining our perspective on problem-solving: (i) Problem-solving includes a goal; (ii) it is not immediately clear to the problem-solver how to achieve the goal; (iii) the problem-solver must organize prior knowledge to generate reasoning towards achieving the goal.

Problem-Solving in School Curricula

It is evident that problem-solving plays a key role in mathematics education nationally and internationally (Shiel & Kelleher, 2017). Mathematical problem-solving occupies a privileged position in the Irish post-primary mathematics syllabus, and is at the centre of both the Junior Cycle and Senior Cycle curricula.

Problem-Solving Capacities in Ireland and Internationally: PISA and TIMSS

In Shiel and Kelleher (2017), information from both the PISA 2012 and TIMSS 2015 reports regarding Irish students' problem-solving competencies was analysed. The PISA (2012) report showed that Irish students performed above the OECD average in applying mathematical concepts and in relating solutions back to the original problem. However, the report highlighted that Irish students were less capable in the process of translating real-world problems into mathematical representations that are productive in solving problems, relative to other problem-solving processes. The TIMSS test results showed that Irish students demonstrated most proficiency in tasks that required recall of memorized facts, carrying out learned procedures, and retrieval of information from representations such as tables or charts. Overall, this indicates a need to improve the problem-solving capacities of Irish students.

The Preparation of Mathematics Teachers

The role of the teacher plays a critical role in students' learning. According to Hattie (2012, p. 18), "teachers are among the most powerful influences in learning". Hattie suggests that while it is important what teachers do, it is most important that the teachers can effectively review the impact their actions have on their students' learning. Teacher education programmes are viewed as a critical stage in teachers' development (Teaching Council of Ireland, 2017). During which, prospective teachers' beliefs regarding teaching and learning should be considered and challenged: these beliefs will be brought forward into their professional practice (Teaching Council of Ireland, 2017).

Research Questions – A First Look

Our research questions are given in detail below, but we note at this point that in light of the discussion above, we are motivated to ask: do pre-service teachers hold the appropriate capacities to teach mathematical problem-solving in secondary schools in Ireland? How are these capacities to be developed? And what are these capacities? We now discuss the different aspects of the conceptual framework within which we ask our specific research questions.

Conceptual Framework- Scope of This Study

Our conceptual framework addresses several related aspects of the study. We revisit the concept of mathematical problem solving, and then discuss the concepts of problem-solving work and strategies; learning problem solving; teaching problem solving and teachers' capacities for teaching problem solving. As discussed above, and at more length in Owens and Nolan (2019), we have associated Three Key Characteristics with the concept of problem solving. This has implications for the selection of problems used in our study and for our interpretation of students' actions and words in their engagement with our study.

As stated above in the Three Key Characteristics, the problem-solver must organize prior knowledge to generate reasoning towards achieving the goal of the problem. Mason et al. (2011) highlight that the understanding of mathematical content is one factor in mathematical thinking. This is supported by Polya (1945) who states that when problem-solving, it is essential for the problem-solver to have some knowledge of the subject matter and have the ability to select the relevant items from this pre-existing knowledge. He notes that to reach a solution, the problem-solver must recall previously solved problems, definitions and other mathematical facts. Polya explains that heuristics, the study of procedures, are independent of this subject-matter. He states that since the aim of heuristics is generality, it is therefore applicable to a variety of problems (Polya, 1945). Since heuristics can be useful in producing successful problem-solving, it is essential for teachers to be aware of the different heuristics that are accessible to their students (Lester, 2013). However, as pointed out by Lester (1994) it is not enough to teach about heuristics but heuristics should be practiced through a variety of problems. This is supported by Mason et al. (2011) who states that there is a wide acceptance that it is essential to develop an understanding of what happens during the process of attempting a problem along with developing and becoming proficient in strategies.

In addition to teachers needing a knowledge of subject matter, Polya (1945) points out that it is essential for a teacher to have a positive disposition towards problem-solving if their students are to have a positive attitude. The teaching of problem solving does not simply rely on the techniques employed but it “comes from the identity and integrity of the teacher” (Palmer, 1998 p.149) meaning “we teach who we are” (p.2). Lester and Kroll (1993) declare that the affective domain is an important contributor to problem solving behaviour. The affective domain includes attitudes, feelings and emotions. Beliefs impact on problem solving performance since beliefs contain their subjective knowledge about self, mathematics and the topics dealt with in particular mathematical tasks (Lester & Kroll, 1993). Similarly, Mason et al. (2011) identify the affective domain as an influential factor in problem-solving. Mason demonstrates the importance for teachers to understand the role that teachers play in creating an environment which promotes confidence and elements of success for their students.

Lester (2013) highlights that it is widely agreed that the development of students’ problem-solving capabilities is a main goal of mathematics instruction. The realisation of this goal involves multiple factors such as metacognition and beliefs along with factors associated with the teacher (Schoenfeld, 1992). Schoenfeld (1992) describes metacognition as one’s own knowledge about one’s own cognitive processes. He highlights that metacognitive ability plays an essential part in problem solving, and he notes that this is the structure that that allows problem solvers to dismantle more challenging problems into subtasks, prioritize and order the importance of each subtask and then complete each subtask in sequential order. Although Lester (1994) highlights the benefits of monitoring behaviours during problem-solving, he identifies that it is difficult to teach students monitoring behaviours. Mason et al. (2011) state that monitoring behaviours can be developed through practice of questions with particular focus on reflection. They note that it is success in overcoming situations of being stuck in a problem that promotes positivity in the problem-solver. Through reflection of feelings involved while problem-solving with actions, it can help the problem-solver relate these feelings when they arise again in new situations to productive actions (Mason et al., 2011). The Rubric writing approach allows the problem-solver to monitor their progress and give structure to the problem-solver through avoiding switching between different plans of attack (Mason et al., 2011).

Thus our conceptualisation of mathematical problem solving begins with the Three Key Characteristics above, and acknowledges the central role played by the employment of heuristics or other strategies, as well as the importance of metacognition and affective factors.

Our conceptualisation of teaching problem-solving draws heavily on the synthesis of research on this topic carried out by Chapman (2015). The role of the teacher is to support their students' development of problem-solving skills, and of the appropriate habits of mind (metacognitive skills) and affective factors (productive disposition) that underpin successful problem-solving. To identify what capacities teachers need to teach problem-solving effectively, Chapman (2015) conducted an extensive review of the literature with research articles dating from 1920 to 2015. Chapman identifies three main components that make up the mathematical problem-solving knowledge for teaching. These components are: 1) Problem-solving content knowledge (PSCK), 2) Pedagogical problem-solving knowledge (PPSK), and 3) Affective factors and beliefs. These three components are made up of six different capacities. PSCK is made up of the following three capacities; knowledge of problems, knowledge of problem-solving, and knowledge of problem- posing. The two capacities that make up PPSK are; the knowledge of students as problem-solvers, and the knowledge of instructional practices. Chapman's identification of these capacities align with frameworks offered by Lester (2013) and Guerin (2017).

This paper focuses on the capacity *knowledge of problem-solving*. This capacity entails teachers' proficiency in problem-solving and in understanding the nature of approaches to problem solving. Chapman (2015) outlines that teachers' own proficiency in problem-solving is essential for them to be able to understand students' approaches and predict the implications of these approaches. Problem-solving proficiency is defined as "what is necessary for one to learn and do genuine PS successfully" (Chapman, 2015, p.9). Kilpatrick et al. (2001) state that the components of mathematical proficiency are not one-dimensional and are interdependent. Chapman proposes that since mathematical proficiency is interwoven, then problem-solving proficiency is too. She suggests that to support students in developing their problem-solving proficiency, teachers must be able to solve the problems and also understand the elements associated with the development of problem-solving proficiency.

Research Questions

In the context described in our introduction and in the setting of the conceptual framework just described, we now state our full set of research questions. Question 1:What do pre-service teachers understand a mathematical problem to be? Question 2 (a):Are pre-service teachers proficient in problem-solving? Question 2(b): Are taught strategies implemented while problem-solving? Question 3:What are pre-service teachers' capacities in relation to problem posing? Question 4:What beliefs do pre-service teachers hold regarding problem-solving? The research question addressed in the present study is research question 2 a).

Methodology

Participants were recruited on a voluntary basis and were pre-service mathematics teachers (PSMTs) undertaking a concurrent initial teacher education programme. The participants were taking a module that includes the study (and practice) of mathematical problem-solving. This module adopted the Rubric Writing approach to problem-solving (Mason et al., 2011).

The PSMTs were interviewed on a one-to-one basis by one of the researchers (EO). The interview consisted of the PSMTs being given two mathematical problems and asked to solve them, following a 'Think Aloud' protocol. Working on problem solving often involves strategies and involves metacognitive and affective aspects. We used think-aloud to create a space for students to display these. Cowan (2019, p. 1) describes the 'Think Aloud' process as "a voluntary activity in which learners having been asked to tackle a relevant task, talk their thoughts out aloud, while engaging with the task". The interviews came to an end when the participants had nothing further to add to their attempt.

All the problems used in the interviews were taken from the NRICH website (NRich, 2019). The problems dealt with the topics of probability, geometry, trigonometry, number, and proportion and ratio. To categorise the tasks, both researchers independently compared the task to the following two criteria of a problem: 1) there is a goal, 2) it is not clear how to reach the goal.

The interviews were recorded and transcribed. Cohort One completed one interview during the module while participants in Cohort Two and Cohort Three both conducted two interviews. These were conducted near the beginning of the module when problem-solving had been introduced in the course content and where limited instruction in the Rubric Writing method (Mason et al, 2011) had been received. The other interview was post-module. Nine participants in Cohort One completed the single interview for that cohort. Five participants in Cohort Two completed the first interview and three of these five completed the second interview. Five participants from Cohort Three completed both interviews.

The data were analysed using a general inductive approach in order to account for both the different strategies that participants may employ and affective utterances that would occur while problem-solving. The data analysis of the interview transcripts involved the repetitive process of coding, comparing, and grouping the data with similarities to construct categories (Jones and Alony, 2011).

Results

The analysis described above led to the identification of five main themes (or categories) in all three cohorts. These categories are; *Introduce, Productive reasoning, Unproductive reasoning, Resilience,* and *Identity.* Analysis of the interviews from Cohort Two and Cohort Three found that there was evidence of participants questioning themselves. This is referred to as *Productive Questioning* and is viewed as a sub-category of *Productive Reasoning.* Revision of the transcripts of Cohort One were done in order to identify if

Productive Questioning was evident, and it was not found to be so. Participants 1-9 were in Cohort One, participants 10-14 were in Cohort Two, and participant 15-19 were in Cohort Three. The excerpts below exemplify each category: in these, Px/Cy refers to Participant x of Cohort y. Table 1 shows the number of occurrences of each theme in the interviews. Columns 1 and 2 indicate the relevant cohort interview and problem respectively.

Table 1

Occurrence of themes count for every problem by the three different cohorts.

Cohort	Problem Number	Introduce	Productive reasoning	Unproductive reasoning	Resilience	Identity	Productive Questioning
1	1	0	23	8	2	8	0
[N=9]	2	34	38	19	25	22	0
2 Pre	3	2	21	4	3	2	7
[N=5]	2	24	26	1	5	14	31
2 Post	4	4	16	0	2	3	6
[N=3]	5	9	12	4	2	5	8
3 Pre	3	3	10	4	0	2	0
[N=5]	2	3	8	8	3	5	4
3 Post	4	5	9	12	5	6	1
[N=5]	5	9	12	8	4	3	5
	Total	93	175	68	51	70	62

Introduce refers to the introduction by the problem-solver of diagrams, constructions within given diagrams, and notation. Mason et al (2011) highlights that the introduction of diagrams and appropriate notation plays a key role in organizing information when problem-solving. Examples of participants' use of *Introduce* include:

P10/ C2: "ok if I set x as time, told travels, x +20 +y"
P11/C2: "So I am going to start by drawing a picture."

The *Productive Reasoning* category includes statements made or actions taken by the participants that promote progress towards a solution of the problem. This category includes the interpreting of information given in the question, use of prior knowledge, specializing and generalising.

P13/C2: "well we know that ¼ is more than ⅕ and less than ⅓."
P15/C3: ""So then I would use Pythagoras to look at the top triangle."

As stated above, *Productive Questioning* was evident in the interviews of both Cohort Two and Cohort Three. This category refers to the participant questioning themselves on their work towards a solution, their chosen strategy, or how to proceed. This *Productive* *questioning* is seen as a sub-category of *Productive reasoning* as the questioning helped participants towards achieving a solution.

> *P13/C2: "Can I find the distances from the courtyard that would be helpful? … Is there a way to make right angled triangles to help?"*
>
> *P12/C2: "So his average speed overall was 93.5km/h. How does that help?"*

Unproductive reasoning involves actions or statements which do not help (or even constrict) the problem-solver from progressing or being successful. This includes procedural errors, making assumptions, misconceptions, and persisting with a line of reasoning despite previously stating that it is incorrect.

> *P16/C3: "we assume that he's at his average speed for almost an hour".*
>
> *P18/C3: "I'm just going to have to guess 3.37 and I don't even know why."*

Resilience includes statements that reflect a participant learning from mistakes, demonstrating a willingness to restart or try a new strategy, and demonstrating a positive response when faced with difficulty.

> *P13/C2: "So what is some other ways?"*
>
> *P18/C3: ""I'm just writing down I'm stuck. I'm writing down where I'm stuck. I'm trying to, I don't know how to find a formula to find A the time after."*

Statements that indicate a participants' self-belief and confidence make up the *Identity* category. This involves the affective domain which is seen as an important influence on problem-solving behaviour (Lester & Kroll, 1993).

> *P12/C2: "I just hope I'm on the right path here. […] I'll see where it goes."*

Conclusion and Next Steps

From the analysis of the interviews, five main themes were identified. With the exception of Problem One, *Introduce* appeared to be a starting point for *Productive Reasoning*. This was evident through the introduction of diagrams, notation, and constructions within given diagrams. This use of *Introduction* as a starting point of a strategy indicates proficiency amongst the PSMTs in both problem-solving and the implementation of the taught strategy provided by Mason *et al.* (2011). However, the exception mentioned indicates that use of this strategy is tied to the problem under consideration. The use of different elements of *Introduction* signal towards proficiency as outlined in the conceptual framework through the use of heuristics. In Cohort Two and Cohort Three, *Productive Questioning*, as described above, was evident in the PSMTs' problem-solving attempts. This questioning was not evident in Cohort One. *Productive Questioning* statements were particularly prominent in both Problem 2 and Problem 5 which were both trigonometry problems. It is encouraging to note that statements categorised as *Productive Reasoning* were not only the most common categorisation but also outnumbered *Unproductive Reasoning* statements by a factor of four. This indicates problem-solving proficiency amongst the PSMTs through demonstrations of procedural fluency, strategic competence, and conceptual understanding. However, this view must be tempered by the fact that Unproductive Reasoning statements outnumber Resilience statements in all but two of the rows of Table 1. Resilience, including the ability to re-start, reflect, and identify misconceptions is vital in problem-solving as identified by Mason et al. (2011). While we have not coded Identity statements as indicating a positive or negative disposition towards problem-solving, it is noteworthy that this important element of our overall problem-solving framework emerges in the interviews. It may be of concern that there an increase in the *Unproductive Reasoning* statements of Cohort Three participants between the pre- and post-module interviews. There did not appear to be any other increase in the four other categories between the pre- and post-module interviews. Future work will involve the analysis of the interviews in terms of explicit implementation of taught strategies by the PSMTs when problem-solving.

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