

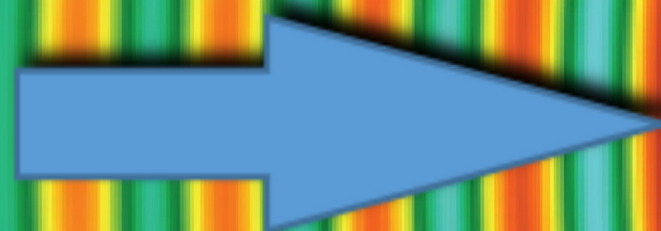
**Etkin, Valery A.  
Levin, Elizabetha  
Khmelnik, Solomon I.  
Kireev, Valery Yu.**



**Доклады Независимых Авторов № 52**

**ISSN 2225-6717  
выпуск 52 2021**

# **Доклады Независимых Авторов**



**Physics  
Medicine**

---

Series: PHYSICS

---

Khmelnik S.I.

ORCID: <https://orcid.org/0000-0002-1493-6630>

# Solving Maxwell's Equations for a Cylindrical Wave in Vacuum

## Annotation

It is proved, as a consequence of the solution of Maxwell's equations, that theoretically possible cylindrical waves of different radii and different frequencies, carrying a flux of electromagnetic energy of different magnitude. The question remains open whether there are natural processes that create such waves.

## Contents

1. Introduction
2. On the method for solving Maxwell's equations
3. About the flow of energy
4. Solving Maxwell's equations
5. Second solution of Maxwell's equations
- References

## 1. Introduction

In [1] “a cylindrically symmetric wave function  $\psi(\rho, t)$  is considered, where  $\rho = (x^2 + y^2)^{1/2}$  is the standard cylindrical coordinate. Assuming that this function satisfies the three-dimensional wave equation, which can be rewritten as

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \left( \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} \right), \quad (538)$$

it can be shown that a sinusoidal cylindrical wave with a phase angle  $\varphi$ , wave number  $k$ , and angular frequency  $\omega = kv$  has an approximate wave function

$$\psi(\rho, t) \approx \psi_0 \rho^{-1/2} \cos(\omega t - k\rho - \varphi) \quad (539)$$

in the limit  $\omega = kv$ . Here  $\psi_0 \rho^{-1/2}$  is the wave amplitude. The corresponding wavefronts (that is, surfaces with constant phase) are a set of concentric cylinders that propagate radially outward from their common

axis  $\rho = 0$  with phase velocity  $\omega/k = v$  - see Fig. 1. The wave amplitude decays as  $\rho^{-1/2}$ . This behavior can be understood as a consequence of the conservation of energy, according to which the power flowing through various surfaces  $A \propto \rho = \text{const.}$  (The areas of such surfaces are scaled as  $A \propto \rho$ . Moreover, the power flowing through them is proportional to  $\psi^2 A \psi^2$ , because the energy flux associated with the wave is usually proportional to  $\psi^2$ , and is directed perpendicular to the wave fronts.) the wave indicated in expression (539) is such that it is generated by a homogeneous linear source located at the point  $\rho = 0$ - see Fig. 1."

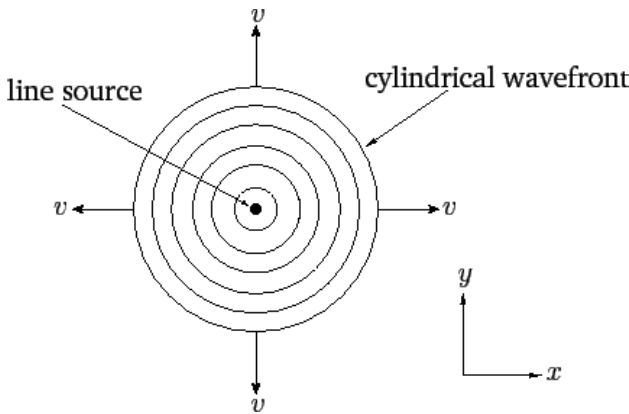


Fig. 1.

It is strange that the author would call such waves cylindrical. They should be called conical, because in them, wave fronts are a set of concentric cylinders that propagate radially outward, and also propagate along the axis: the fronts that appeared earlier continue to expand, while the emerging cylinders begin to expand. The justification would be that really cylindrical waves, in which these cylinders retain their radius, do not exist in nature. But look at the magnifying glass with which the boy makes a fire at noon - see fig. 2. The sunbeam that enters the magnifying glass is obviously a cylindrical wave. Another example is a Fresnel loop [3] that creates a cylindrical output wave - see fig. 3 from [4]. Cylindrical lenses are also known, the feature of which is the presence of an axis in the direction of which the optical effect does not manifest itself (i.e., there is no refraction, reflection and scattering of radiation) [2].

Thus, cylindrical waves (as well as conical ones) cannot be represented by a wave function. Therefore, we will consider a new function that describes cylindrical waves and is a solution to Maxwell's equations.

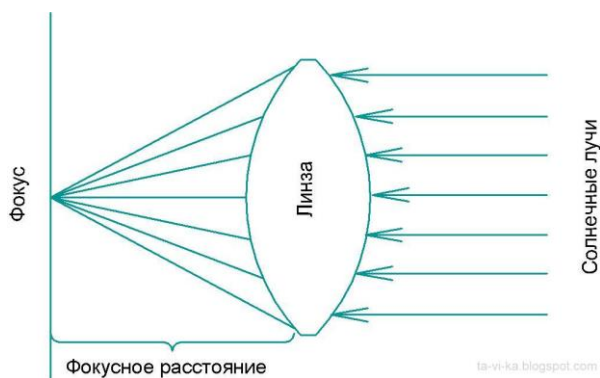


Fig. 2.

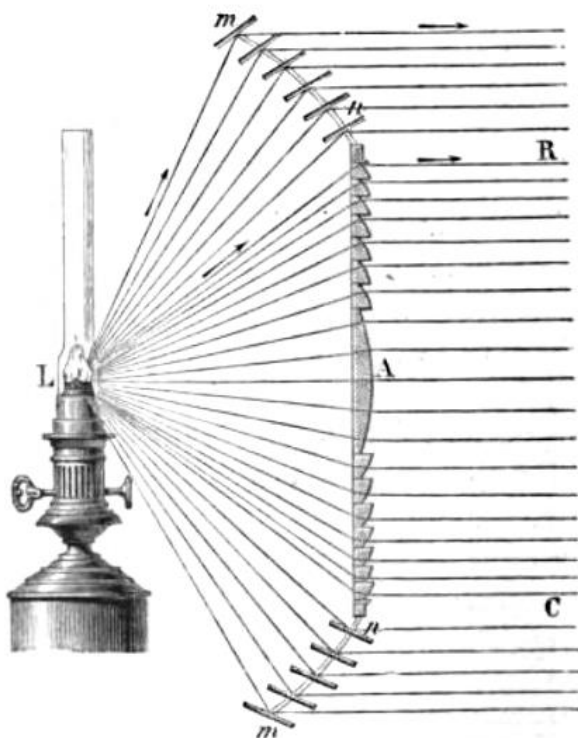


Fig. 3.

Further, it will be proved that for the system of Maxwell's equations there is a solution that describes a cylindrical wave in vacuum. This

solution maintains a constant flow of energy in such a wave and the shape of this wave.

## 2. Solving Maxwell's Equations

Consider the system of Maxwell equations for vacuum, which has the form

$$\text{rot}(E) + \frac{\mu}{c} \frac{\partial H}{\partial t} = 0, \quad (a)$$

$$\text{rot}(H) - \frac{\varepsilon}{c} \frac{\partial E}{\partial t} = 0, \quad (b)$$

$$\text{div}(E) = 0, \quad (c)$$

$$\text{div}(H) = 0. \quad (d)$$

In the system of cylindrical coordinates  $r, \varphi, z$ , these equations have the form [4]:

$$\frac{E_r}{r} + \frac{\partial E_r}{\partial r} + \frac{1}{r} \cdot \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} = 0, \quad (1)$$

$$\frac{1}{r} \cdot \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} = \frac{\mu}{c} \frac{dH_r}{dt}, \quad (2)$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = \frac{\mu}{c} \frac{dH_\phi}{dt}, \quad (3)$$

$$\frac{E_\phi}{r} + \frac{\partial E_\phi}{\partial r} - \frac{1}{r} \cdot \frac{\partial E_r}{\partial \phi} = \frac{\mu}{c} \frac{dH_z}{dt}, \quad (4)$$

$$\frac{H_r}{r} + \frac{\partial H_r}{\partial r} + \frac{1}{r} \cdot \frac{\partial H_\phi}{\partial \phi} + \frac{\partial H_z}{\partial z} = 0, \quad (5)$$

$$\frac{1}{r} \cdot \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = \frac{\varepsilon}{c} \frac{dE_r}{dt}, \quad (6)$$

$$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = \frac{\varepsilon}{c} \frac{dE_\phi}{dt}, \quad (7)$$

$$\frac{H_\phi}{r} + \frac{\partial H_\phi}{\partial r} - \frac{1}{r} \cdot \frac{\partial H_r}{\partial \phi} = \frac{\varepsilon}{c} \frac{dE_z}{dt}, \quad (8)$$

where  $E_r, E_\phi, E_z$  are electrical strengths,  $H_r, H_\phi, H_z$  are magnetic strengths.

In this chapter, we will look for these functions as follows:

$$H_r = \hat{H}_r(r) \cdot \cos, \quad (9)$$

$$H_\phi = \hat{H}_\phi(r) \cdot \sin, \quad (10)$$

$$H_z = \hat{H}_z(r) \cdot \sin, \quad (11)$$

$$E_r = \hat{E}_r(r) \cdot \sin, \quad (12)$$

$$E_\phi = \hat{E}_\phi(r) \cdot \cos, \quad (13)$$

$$E_z = \hat{E}_z(r) \cdot \cos, \quad (14)$$

where

$$\cos = \cos(\alpha\phi + \chi z + \omega t), \quad (15)$$



$$s_i = \sin(\alpha\phi + \chi z + \omega t), \quad (16)$$

and  $\alpha, \chi, \omega$  are some constants. Let us briefly consider the method for solving this system of equations [5], since further some modification of this method will be proposed

We will differentiate functions (15, 16) with respect to the arguments  $r, \phi, z$  to obtain partial derivatives of functions (9-14). These derivatives will be substituted into equations (1-8). In this case, it turns out that in each equation, all terms will have the same trigonometric function, which can be canceled. Then equations (1-8) will take the form:

$$\frac{e_r(r)}{r} + e_r'(r) - \frac{e_\phi(r)}{r} \alpha - \chi \cdot e_z(r) = 0, \quad (17)$$

$$-\frac{1}{r} \cdot e_z(r) \alpha + e_\phi(r) \chi - \frac{\mu\omega}{c} h_r = 0, \quad (18)$$

$$e_r(r) \chi - e_z'(r) + \frac{\mu\omega}{c} h_\phi = 0, \quad (19)$$

$$\frac{e_\phi(r)}{r} + e_\phi'(r) - \frac{e_r(r)}{r} \cdot \alpha + \frac{\mu\omega}{c} h_z = 0, \quad (20)$$

$$\frac{h_r(r)}{r} + h_r'(r) + \frac{h_\phi(r)}{r} \alpha + \chi \cdot h_z(r) = 0, \quad (21)$$

$$\frac{1}{r} h_z(r) \alpha - h_\phi(r) \chi - \frac{\varepsilon\omega}{c} e_r(r) = 0, \quad (22)$$

$$-h_r(r) \chi - h_z'(r) + \frac{\varepsilon\omega}{c} e_\phi(r) = 0, \quad (23)$$

$$\frac{h_\phi(r)}{r} + h_\phi'(r) + \frac{h_r(r)}{r} \alpha + \frac{\varepsilon\omega}{c} e_z(r) = 0. \quad (24)$$

Let's pretend that

$$h_r = k e_r, \quad (25)$$

$$h_\phi = -k e_\phi, \quad (26)$$

$$h_z = -k e_z, \quad (27)$$

where  $k$  is some constant. Let's change variables according to (25-27) in equations (17-24) and rewrite them:

$$\frac{e_r}{r} + \dot{e}_r - \frac{e_\phi}{r} \alpha - \chi e_z = 0, \quad (28)$$

$$-\frac{e_z}{r} \alpha + e_\phi \chi - \frac{\mu\omega}{c} k e_r = 0, \quad (29)$$

$$-\dot{e}_z + e_r \chi - k \frac{\mu\omega}{c} e_\phi = 0, \quad (30)$$

$$\frac{e_\phi}{r} + \dot{e}_\phi - \frac{e_r}{r} \alpha - k \frac{\mu\omega}{c} e_z = 0, \quad (31)$$

$$k \frac{e_r}{r} + k \dot{e}_r - k \frac{e_\phi}{r} \alpha - k \chi e_z = 0, \quad (32)$$

$$-k \frac{e_z}{r} \alpha + k e_\phi \chi - \frac{\varepsilon\omega}{c} e_r = 0, \quad (33)$$

$$k\dot{e}_z - ke_r\chi + \frac{\varepsilon\omega}{c}e_\varphi = 0, \quad (34)$$

$$-k\frac{e_\varphi}{r} - k\dot{e}_\varphi + k\frac{e_r}{r}\alpha + \frac{\varepsilon\omega}{c}e_z = 0. \quad (35)$$

It can be proved that this system can be reduced to a system of four equations (28-35) with unknowns  $e_z$ ,  $e_\varphi$ ,  $e_r$ ,  $k$ .

### 3. About the flow of energy

The primary challenge is to find a solution in which the energy flow remains constant over time. Let us find the conditions under which the functions (2.9-2.14) satisfy this requirement.

It is known that the flux density of electromagnetic energy is the Poynting vector

$$S = \eta E \times H, \quad (1)$$

where

$$\eta = c/4\pi. \quad (2)$$

In cylindrical coordinates  $r, \phi, z$  the electromagnetic energy flux density has three components  $S_r, S_\phi, S_z$  directed along the radius, along the circumference, along the axis, respectively. They are determined by the formula

$$S = \begin{bmatrix} S_r \\ S_\phi \\ S_z \end{bmatrix} = \eta(E \times H) = \eta \begin{bmatrix} E_\varphi H_z - E_z H_\varphi \\ E_z H_r - E_r H_z \\ E_r H_\phi - E_\phi H_r \end{bmatrix}. \quad (3)$$

We will consider the case when there are no longitudinal strengths, i.e.  $H_z(r) = 0$ ,  $E_z(r) = 0$ . Therefore,  $S_r = 0$ ,  $S_\phi = 0$ , i.e. the energy flux propagates only along the oz axis and its density is

$$S = S_z = \eta(E_r H_\phi - E_\phi H_r) \quad (4)$$

or, taking into account (9-16),

$$S = S_z = \eta(e_r h_\phi \sin^2 - e_\phi h_r \cos^2). \quad (5)$$

If

$$e_r h_\phi = -e_\phi h_r, \quad (6)$$

then

$$S_z = \eta e_r h_\phi. \quad (7)$$

From (7) we find the total energy flux through the cross section of the wave

$$\bar{S}_z = \frac{c}{4\pi} \iint_{r,\phi} (e_r h_\phi dr \cdot d\phi) = \frac{c}{2} \int_0^R (e_r h_\phi dr). \quad (8)$$

This integral is independent of time. Therefore, when condition (6) is satisfied, the energy flux of the electromagnetic wave is constant in time.

## 4. Solving Maxwell's equations

The next task is to find the form of the functions  $e_r, h_\varphi, e_\varphi, h_r$ , satisfying the system of equations (1-8) and condition (3.6). Before looking for solutions, it should be noted that Maxwell's equations can have many solutions (like any system of partial differential equations). Some of these solutions violate obvious physical requirements, in particular, the fulfillment of the law of conservation of energy. For example, the well-known solution in the form of a wave function, as is known, violates this one (in this solution it is preserved only on average in time, which contradicts the very spirit of this law).

In [5], a solution was found in which the amplitudes of the magnetic and electrical strengths in cylindrical coordinates  $r, \varphi, z$  have the following form:

$$|E| = Ar^\beta, \quad (1)$$

where  $A$  are some constants,  $\beta$  is a linear function of  $r$ .

In this solution, the law of conservation of energy is fulfilled. But there is another drawback - such solutions are applicable only when the parameter  $r$  is limited:

$$0 < r < \infty. \quad (2)$$

In particular, in the absence of longitudinal strengths, this solution has the following form:

$$h_z(r) = 0, \quad (3)$$

$$e_z(r) = 0, \quad (4)$$

$$e_r(r) = e_\varphi(r) = \frac{A}{2} r^{(\alpha-1)}, \quad (5)$$

$$h_\varphi(r) = -\sqrt{\frac{\varepsilon}{\mu}} e_r(r), \quad (6)$$

$$h_r(r) = \sqrt{\frac{\varepsilon}{\mu}} e_r(r), \quad (7)$$

$$\chi = \omega\sqrt{\mu\varepsilon}/c, \quad (8)$$

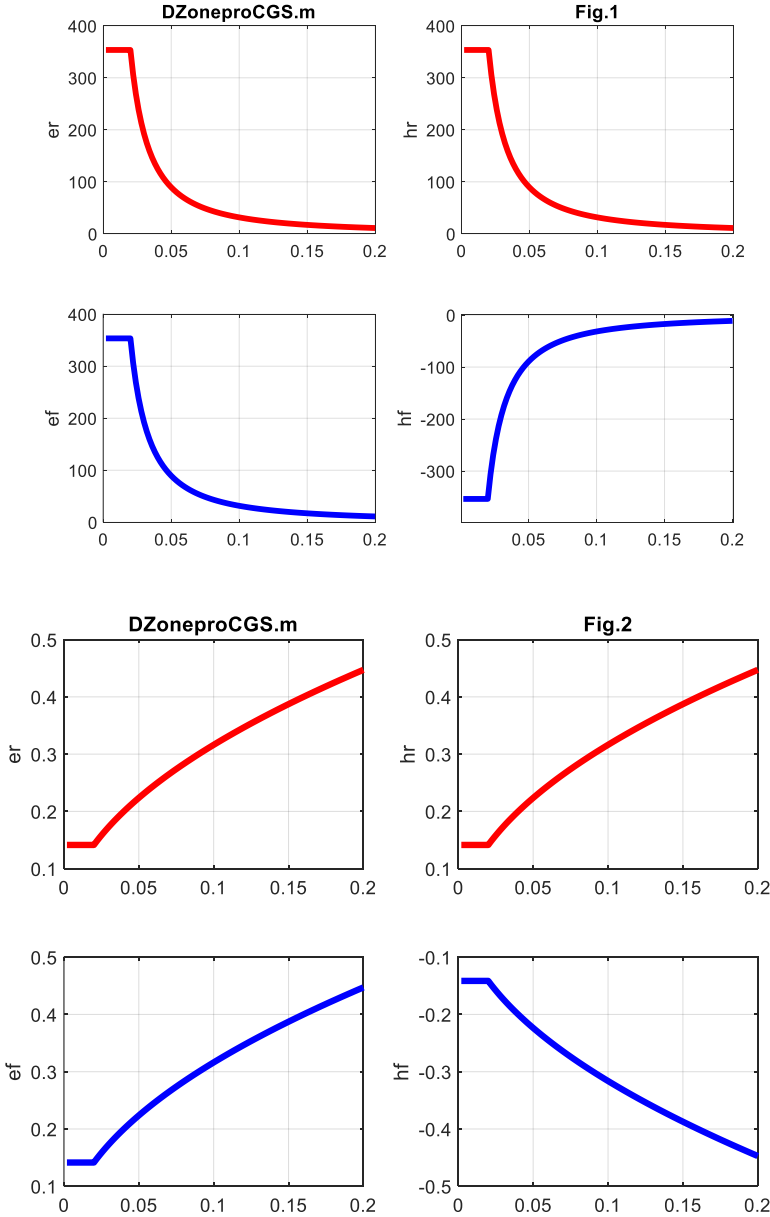
where  $A$  is some constant. In this solution, condition (3.6) is satisfied. Indeed, (5-8) implies (3.6). However, it is clear that in this solution the strengths

- 1) take an infinite value at  $r = 0$ , if the exponent  $r$  is negative;
- 2) take an infinite value as  $r \rightarrow \infty$ , if the exponent of  $r$  is positive.



**Example 1.**

In fig. 1 shows the graphs of the functions  $e_r, e_z, h_\varphi, h_r, h_z$  at  $A = 2$ ,  $R = 0.2$ ,  $\omega = 10^5$ ,  $\frac{\varepsilon}{\mu} = 1$  in the CGS system. In fig. 1 is taken  $\alpha = -0.5$ , and in Fig. 2 it is taken  $\alpha = 1.5$ .



## 5. Second solution of Maxwell's equations

At present, the applicability of Maxwell's equations to all phenomena of electrodynamics and electrical devices is undeniable. Consequently, there must be such a solution of Maxwell's equations for an electromagnetic wave in a vacuum, where there are no infinite strengths. Further, we will show that such a solution can be obtained by a simple transformation of the solution considered above.

To do this, let's return to equations (2.1.28-2.1.37). Consider equation (2.1.28):

$$\frac{e_r(r)}{r} + e_r'(r) - \frac{e_\varphi(r)}{r} \alpha - \chi \cdot e_z(r) = 0, \quad (1)$$

If the parameter  $\alpha$  depends on  $r$ , i.e.  $\alpha = \alpha(r)$ , then this equation takes the form:

$$\frac{e_r(r)}{r} + e_r'(r) + e_r(r)\alpha(r)\alpha'(r) - \frac{e_\varphi(r)}{r} \alpha - \chi \cdot e_z(r) = 0,$$

For  $e_z(r) = 0$  and  $e_\varphi(r) = e_\varphi(r)$  it takes the form:

$$\frac{e_r}{r} + e_r' + e_r \alpha \alpha' - \frac{e_r}{r} \alpha = 0, \quad (2)$$

or

$$\left(\frac{1}{r}(1 - \alpha) + \alpha \alpha'\right) e_r + e_r' = 0, \quad (3)$$

Let's denote:

$$y(x) = e_r(x) \quad (4)$$

and from (3, 4) we get:

$$\frac{dy}{dx} = \left(\frac{1}{x}(\alpha - 1) - \alpha \alpha'\right) y. \quad (5)$$

The solution to this equation is known and has the form:

$$y = A e^F,$$

$$F = \int \left(\frac{1}{r}(\alpha - 1) - \alpha \alpha'\right) dr = (\alpha - 1) \log(r) - \frac{\alpha^2}{2},$$

$$\begin{aligned} e^F &= \exp \left( (\alpha - 1) \log(r) - \frac{\alpha^2}{2} \right) = x^{(\alpha-1)} \exp \left( -\frac{\alpha^2}{2} \right) \\ &= \left( \exp \left( -\frac{\alpha^2}{2} \right) x^{(\alpha-1)} \right) \end{aligned}$$

Returning to the previous notation, we find

$$e_r(r) = A \left( \exp \left( -\frac{\alpha(r)^2}{2} \right) r^{(\alpha(r)-1)} \right). \quad (6)$$

We need to find a decreasing function in which the condition

$$\alpha(r = 0) = 1 \quad (7)$$

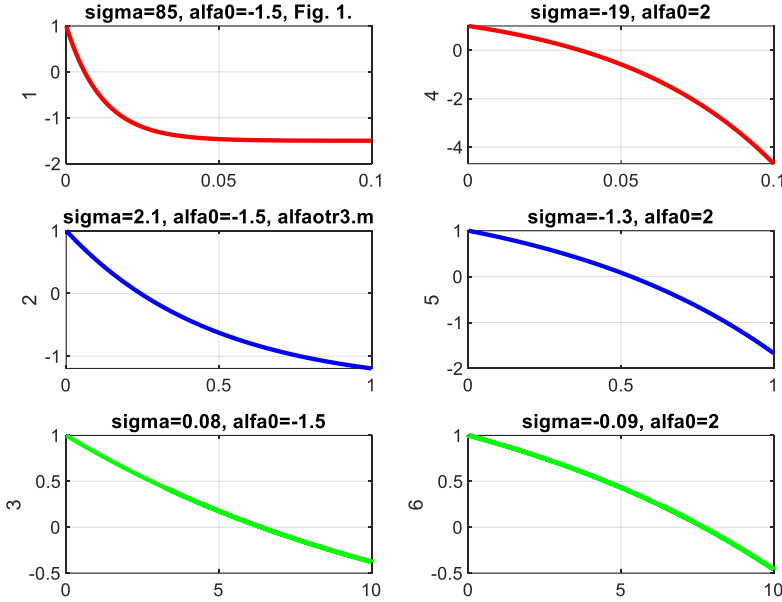
so that for  $r = 0$  we get  $r^{(\alpha(r)-1)} = 0$ . We will use the following function

$$\alpha(r) = 1 + (\alpha_o - 1)(1 - \exp(-\sigma r)). \quad (8)$$

In this function

$$\begin{cases} \sigma > 0, \text{ if } \alpha_o < 0 \\ \sigma < 0, \text{ if } \alpha_o > 0 \end{cases} \quad (9)$$

Fig. 1 shows functions (8) for various values of  $\alpha_o, \sigma, R$  (the value of  $R$  is the maximum value on the abscissa).



All equations (2.1.28-2.1.35) can be transformed in the same way. In addition, here we will not consider longitudinal strengths. Then equations (2.1.28-2.1.31) will take the form:

$$\frac{e_r}{r} + \dot{e}_r - \frac{e_\varphi}{r} \alpha - \chi e_z = 0, \quad (10)$$

$$-\frac{e_z}{r} \alpha + e_\varphi \chi - \frac{\mu\omega}{c} k e_r = 0, \quad (11)$$

$$-\dot{e}_z + e_r \chi - k \frac{\mu\omega}{c} e_\varphi = 0, \quad (12)$$

$$\frac{e_\varphi}{r} + \dot{e}_\varphi - \frac{e_r}{r} \alpha - k \frac{\mu\omega}{c} e_z = 0, \quad (13)$$

The solution of these equations will differ from formulas (4.3-4.8) only in the form of equation (4.5), which in this case will take the form (6).

From (3.7, 6, 4.6) we find the density of the electromagnetic energy flux at a given radius  $r$

$$S(r) = \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} e_r^2(r). \quad (14)$$

and then, according to (3.8), the energy flux in the cylinder with radius  $R$ :

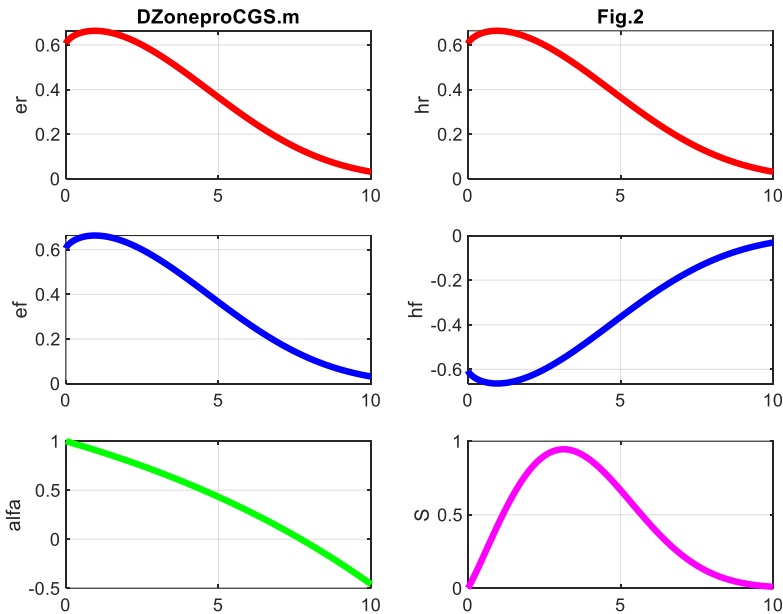
$$\bar{\bar{S}}_z = \frac{c}{4\pi} \iint_{r,\varphi} (S(r) dr \cdot d\varphi) = \frac{c}{2} \sqrt{\frac{\varepsilon}{\mu}} \int_0^R (r e_r^2 dr). \quad (15)$$

or

$$\bar{\bar{S}}_z = \frac{c}{4\pi} \iint_{r,\varphi} (S(r) dr \cdot d\varphi) = \frac{c}{2} \sqrt{\frac{\varepsilon}{\mu}} \int_0^R (r e_r^2 dr). \quad (16)$$

where

$$S_0(r) = (r e_r^2). \quad (17)$$



In fig. 2 shows the functions  $\alpha(r), e_r(r), S_0(r)$  - see (8, 6, 17) for  $R = 10, \sigma = 0.09, \alpha_0 = 2, A = 1$ . It is seen that the functions  $e_r(r), S_0(r)$  take zero values for  $r > R$ . This means that an electromagnetic wave exists in a cylinder with a certain radius. It makes sense to call such a wave a cylindrical wave.

*Some reader will exclaim: "Ha! The wave function is known, in which there are no additional parameters. And here a solution is proposed*

*in which you need to select some parameters. Who is looking for and installing them? "*

*My answer is this. The wave function arises as a result of processes unknown to us so far, and this result is described as a solution to Maxwell's equations. The reason for this state of affairs is still unknown to us. But our belief that the wave function exists is justified by the fact that it is a solution to Maxwell's equations and nothing more. Consequently, it can be argued that any solution of Maxwell's equations is realized physically under condition (3.6) - whatever this solution may be.*

*It turns out that the wave function cannot exist physically, but there are various other functions, less elegant, but existing physically.*

Thus, there is a solution to Maxwell's equations that describe a cylindrical wave.

## References

1. Richard Fitzpatrick, Professor of Physics, Oscillations and Waves, <http://farside.ph.utexas.edu/teaching/315/Waves/Waveshtml.html>.
2. Cylindrical lenses: basic characteristics and applications, [https://in-science.ru/library/article\\_post/cilindricheskie-linzy](https://in-science.ru/library/article_post/cilindricheskie-linzy) (in Russian)
3. Fresnel\_lens, [https://en.wikipedia.org/wiki/Fresnel\\_lens](https://en.wikipedia.org/wiki/Fresnel_lens);
4. Fig 2. see in [https://upload.wikimedia.org/wikipedia/commons/a/ab/Fresnel\\_lighthouse\\_lens\\_diagram.png](https://upload.wikimedia.org/wikipedia/commons/a/ab/Fresnel_lighthouse_lens_diagram.png)
5. S.I. Khmelnik. Inconsistency Solution of Maxwell's Equations. 16th edition, 2020, ISBN 978-1-365-23941-0. Printed in USA, Lulu Inc., ID 19043222, <http://doi.org/10.5281/zenodo.3833821>