Making Squares: Children's Responses to a Tangram Task

Siún Nic Mhuirí[1] and Denis Kelly[2]

[1]Dublin City University and [2]Adamstown Castle ETNS

This paper reports on children's responses to a tangram task. The task was designed, based on the draft specification of the primary mathematics curriculum, to facilitate children's exploration of shape properties and their engagement in mental and physical transformations of tangram pieces. The task was enacted with three cohorts of children from first and second class. Analysis shows some children's concept images of squares to be limited and their thinking dominated by prototypical geometric images. This research is pertinent in the context of the proposed changes to the Shape and Space strand of the primary curriculum.

Introduction

The lesson at the centre of this research was developed in collaboration with the Maths4All project team. Maths4All, funded by Science Foundation Ireland, develops resources for and with teachers to support high quality mathematics teaching. A lesson focusing on tangram activities was designed with reference to the draft specification of the primary mathematics curriculum (National Council for Curriculum and Assessment [NCCA], 2017). This lesson was trialled in three primary classes. This paper presents an analysis of children's responses to one of the lesson tasks. The expectations in the draft specification are quite different to existing curriculum objectives (Government of Ireland, 1999). For this reason, our analysis gives insight into the possibilities and challenges of working toward new curriculum expectations. The literature review below first presents an overview of research on children's thinking in Shape and Space then explores the Irish context.

Children's Geometrical Reasoning

The van Hiele framework, Table 1, describes progressive levels of geometric thinking, with initial levels dominated by visual imagery (Fuys et al., 1984). Increasingly sophisticated levels of description, analysis, abstraction and proof are understood to develop in response to appropriate opportunities for learning (Clements & Battista, 1992). The framework is recognised as having the potential to inform decisions around the appropriateness of tasks. This is important as primary students often experience teaching that emphasizes only the identification and naming of shapes with little offered that would develop their reasoning at higher levels (Sinclair & Bruce, 2015). van Hiele theory contends that the teacher has a crucial role in the development of children's geometric reasoning (Fuys et al., 1984). It is recommended that teaching must attend to supporting the development of rich and varied concept images of geometric shapes (Sinclair et al., 2016). *Concept image* is understood to mean the cognitive structure associated with the concept. This includes all mental images and associated properties and processes (Tall & Vinner, 1981). Children's exploration of non-prototypical examples (and non-examples) in different positions or orientations is recommended as a way to develop rich concept images (Nic Mhuirí, 2020).

Table 1

The earliest levels of the van Hiele model of geometric thinking

0. Pre-recognition	Children may attend to only a subset of a shape's visual characteristics and may be unable to identify many common shapes.
1. Visual	Children recognize shapes solely by their appearance, often by comparison with a known prototype. Limited/no awareness of shape properties.
2. Descriptive/ Analytic	Children characterise shapes by their properties but do not perceive relationships between properties. The child may be unable to identify which properties are necessary and/or sufficient to describe the object.
3. Abstract/ Relational	Children can perceive relationships between properties and between figures. They can form meaningful definitions, classify shapes and give informal justifications for their classifications.

Note. This overview draws on Clements and Battista (1992) where level 0 was added due to a perceived lack in the original model. The levels shown are those considered to be most pertinent to primary education. Reprinted from Nic Mhuirí (2020).

Composing and decomposing shapes is a key element in geometric reasoning (Clements et al., 2004). This type of reasoning can be connected to transformations and visuospatial reasoning. While different terminology and definitions are offered, at heart visuospatial reasoning is concerned with visualising objects and manipulating them mentally, for example, visualising a shape being rotated through a turn (Sinclair et al., 2016). Such reasoning is understood to be central to mathematical and other forms of thinking. The growth in attention to visuospatial reasoning in recent years is accompanied by a growing recognition that age-appropriate activities that involve explicit attention to transformations should be part of children's early learning experiences (Sinclair & Bruce, 2015).

The Irish Context

International assessments such as the Trends in International Mathematics and Science Study (TIMSS) facilitate comparison of Irish children's achievement relative to other populations. The TIMSS assessment takes place every four years and is administered at fourth class and second year level in Ireland. Measurement and Geometry form one domain of the TIMSS assessment at fourth class. Assessment tasks include solving problems involving length, mass, volume, time, perimeters of polygons, area of triangles and partial squares, lines and angles, and two- and three-dimensional shapes. For the second year TIMSS assessment, Geometry is a domain in its own right. The most recent data available is from TIMSS 2019. Though Ireland was one of the highest performing countries at both class levels, Irish students showed a relative weakness in fourth class on the Measurement and Geometry domain, and in second year Irish students showed a relative weakness in the Geometry domain (Perkins & Clerkin, 2020). Similar findings are reported in the Programme for International Student Assessment study (Perkins & Shiel, 2016). Thus, despite high achievement in most areas of mathematics, the findings of international assessments highlight the need for careful consideration of the teaching and learning of Shape and Space in this country.

Currently, the primary curriculum is undergoing significant reform. As mentioned above, a draft specification of the new primary mathematics curriculum (NCCA, 2017) from Junior Infants to Second Class, has been published. This draft is organised around a set of broad learning outcomes and the *strand units* of the 1999 curriculum (Government of Ireland, 1999) are reimagined as *learning outcome labels*. For the Shape and Space strand, it also suggests significant changes in terms of content. While teachers will recognise the learning outcome labels of *spatial awareness and location* and *shape* from their previous experience, it is likely that *transformation* will be more problematic. Learning outcomes for this label include, “Explore and describe the effects of shape movements” (stage 1) and “Visualise and show the effects of transformations on shapes” (stage 2) (NCCA, 2017, p. 35). The sample learning experiences described in the progression continua (p.66-67) give further insight into how it is envisaged that these outcomes might be achieved. These focus on physical and mental manipulation of shapes (visualisation) as a site for developing language to describe simple transformations, e.g., flip, turn, slide.

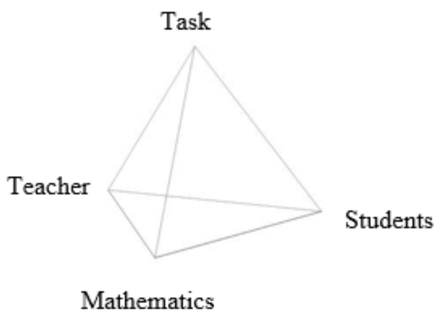
Methodology

This paper relates to a tangram lesson which was designed using the draft specification of the primary mathematics curriculum (NCCA, 2017). The goals of this lesson included that children would recognise the same shape in different orientations, that they would combine tangram pieces to form a variety of shapes and that they would name, compare and describe the properties of different tangram pieces. It was envisaged that children would also identify and discuss shape transformations in these activities, for example, physically or mentally rotate or flip tangram pieces and describe their actions or thinking. This lesson was taught on three occasions by the authors of this paper, twice at first class level and once at second class. Neither author was class teacher in any of these cases and we had limited insight into children’s previous experiences. Each lesson was recorded with three video-cameras. Two cameras were fixed, while a videographer operated the third camera. Videos were reviewed after each lesson. No changes were made to the focus task but reflection on, and refinement of, planned teacher questioning did occur. The research question which guides this paper is: What is the nature of children’s geometric thinking elicited by the focus task? Pseudonyms are used to report our findings.

We use the didactical tetrahedron (Rezat & Sträßer, 2012) as a theoretical framework. This framework, Figure 1, conceives of artefacts - alongside teachers, students, and mathematics - as fundamental constituents of any teaching situation. Drawing on Vygotsky’s notion of a psychological tool as one which impacts the mind, Rezat and Sträßer (2012) contend that all tools used in mathematics teaching can be considered as psychological tools or artefacts. Each face of the tetrahedron represents different perspectives on the teaching-learning situation. Various artefacts might be considered in relation to the research lessons, but for the purposes of this paper we focus on tasks as artefact. Notwithstanding the crucial role of the teacher in orchestrating learning opportunities, our focus in this paper is primarily on *task-mathematics-student* face as we attempt to investigate students' geometrical thinking elicited by the task. Given the constraints of the paper, we focus only on the fourth and final task in this lesson, Making Squares (details on Table 2 below).

Figure 1

Didactical Tetrahedron (Rezat & Sträßer, 2012) with Task as Artefact



First, the mathematical ideas that underpin the task were identified. While many of these ideas were discussed at the planning stage, planning for teaching tends to be focused on articulation of learning goals for children. For the purposes of analysis, we aimed to explicate the underpinning mathematics clearly. Secondly, we considered both the task as written in the planning documents and the task as implemented by the teachers of each lesson (c.f., Stein et al., 1996). Finally, each video was reviewed and relevant segments showing children's responses to the task were identified. These included occasions where children's responses were evident from visual appraisal of the video data alone, for example, evidence of a number of composed squares visible in front of an individual student. These also included occasions where video data captured extended conversations between the teacher and various children. All examples of student responses were listed and common responses to the task, including errors, were identified. Below we present an analysis of this data with reference to what we deem the most relevant or interesting examples of children's thinking.

Findings

Table 2 outlines details of the task and the underpinning mathematics. The task, sourced from *nrich.maths.org*, was selected as it has potential to develop the chosen learning outcomes. In enacting this task, we decided to make multiple tangram sets available to students. We did not want students to have to deconstruct their squares to make new ones and we intended that children would review the squares they had constructed and identify which ones were the same and different in terms of their component parts and/or the transformations needed to align orientations. The mathematical ideas underpinning this task are also listed on Table 2. It should be understood that it was not expected that students would understand all of these mathematical ideas, or indeed that all of them would become explicit through engagement with the task. That said, these details are vital as they form the background against which children's thinking is considered. Angle concepts run through all of the identified mathematical ideas and the combination of geometric and measurement reasoning involved in, c*omposing and decomposing shapes and angles* highlights the complexity and interwoven nature of these ideas. While formal measurement was not employed in this lesson, direct and visual comparison were used by students to check, for example, that the angle of a constructed square was the same as that of the single square piece. In addition, children also made judgements about whether the length of sides 'matched' or not (c.f., Clements et al., 2004). We note the gap between the mathematics described here and the expectations of the 1999 curriculum (Government of Ireland, 1999) where *transformation* does not feature at all and where the *Angles* strand unit is not introduced until second class.

Table 2

Overview of task and underpinning mathematics

Task presented on whiteboard	Orchestration of of task	Underpinning Mathematics
 *Tangram pieces are made from a square cut into seven pieces.* *Can you make other squares using some, not all, of the pieces?* *Can you make five different squares?*	The task was read to the children. Children were provided with multiple different sets of tangrams to experiment with. Teacher questions encouraged children to check their solutions and to try to make 'different' squares, for example, "I see you have lots of two-pieces squares, do you think you can make a three or four-piece square?" (Lesson 1, first class)	*Properties of the square* A square is a 2D-shape with four equal sides and four right angles. Opposite sides are parallel (and equal). *Composing and Decomposing* A square/angle/length can be composed of, or decomposed into, a number of smaller subunits. *Transformations* Shapes can be physically or mentally moved around in space by reflecting, translating and rotating.

Across all three lessons, many children's initial responses to the task involved the creation of two-piece squares using right-angled triangles of the same size, see Figure 2 (i). This sometimes evolved into larger squares made of multiple copies of a two-piece square as subunits. It appeared that, initially at least, more children were successful in combining repeated iterations of the same shape to form squares rather than combining different shapes, see figure 2 (ii). This was obvious in the relatively large number of two- and four-piece squares (made of repeated squares or triangles) compared to three- and five-piece squares. A small number of children made three- and five-piece squares relatively quickly, but the vast majority of the children continued to experiment with two- and four-piece squares until the teacher intervened to encourage experimentation with other variations. A number of children were also observed to make seven-piece squares relatively quickly after the activity was initially introduced. It seems likely that these creations were guided by the image of the complete seven-piece tangram that was shown on the interactive whiteboard.

While most children appeared to complete the initial compositions of two-piece squares with ease, one child, Síle, from first class engaged in an extended discussion with the teacher where she articulated some uncertainty. While Síle had aligned two right-angled triangles of the same size accurately to make a square, she was unconvinced that the resulting composition was actually a square. When questioned about why she did not think it was a square, the child appeared to struggle to articulate her thinking. When pressed, she stated, "Because it's not-" and pointed at the middle of the composed shape, where the edges of the two triangles met. She nodded when the teacher asked, "It's not one piece?" The same child had previously, with no observed difficulties, engaged in an activity where tangram pieces were combined to make animal shapes. It appears here that the particular concept image she had for 'square' did not include squares composed of subunits. Across all three lessons, children had repeatedly identified squares shown standing on a point, rather than sitting on a horizontal base as 'diamonds' and some children appeared to understand diamonds as quite distinct from squares. This is another example of children's limited concept images of squares and most likely arises as a result of exposure to largely prototypical representations.

A number of other different attempts to construct squares were made. For example, a number of children made quadrilaterals that were not squares. Some of these were approximations of squares, where despite small errors, attempts to construct equal sides, right angles and opposite sides parallel were obvious, for example, see Figure 2 (iii). In other cases, children constructed non-square rectangles or parallelograms. Seán, who had successfully constructed a square from four smaller square subunits, then went on to attempt to construct a square from four parallelograms, as shown in Figure 2 (ii). He claimed that this was a square and when asked why he thought it was a square, he said that, "it's got four sides but it's a bit slanted". He appeared to recognise the visual difference between the shape that he had created and 'other' squares but did not verbally identify any other properties of a square. In the lesson with second class, the teacher attempted to probe children's understanding about the properties of different quadrilaterals and the following conversation occurred.

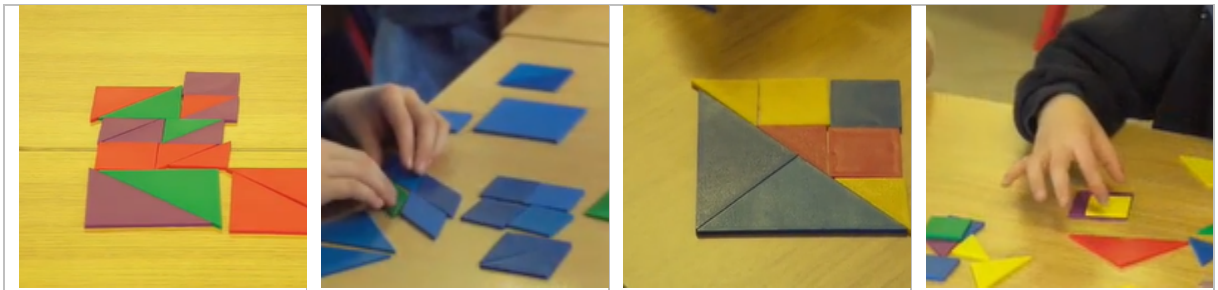
Teacher: What's the difference between squares and other types of rectangles? What's so special about squares?

Ciara : They're smaller. So like if you cut a rectangle in half, so like, you can make a square. You can make a rectangle by putting two squares.

Ciara demonstrated impressive levels of visuospatial reasoning particularly given that no singular (non-square) rectangular pieces were available to children at this time. Her response does seem to suggest though that she is drawing on the prototypical image of a 2 x 1 rectangle with width double its height. Other examples of strong visuospatial reasoning were evident in children who recognised, and described in informal language, the relationships between shapes in various different orientations and the movements necessary to change the appearance of shapes in different ways.

Figure 2

Samples of children's work



(i) Multiple iterations of 2-piece squares.	(ii) Iterations of single shapes. Child constructing non-square quadrilateral	(iii) Approximation of Square	(iv) Comparing corners of squares

Discussion

Across these lessons, we saw instances where children's thinking appeared to align with level 1 of the van Hiele framework- it was dominated by visual imagery and children had limited understanding of shape properties. While this might be expected for first and second class children, it is problematic when the imagery which guides their thinking is prototypical in nature, limiting their concept images for given geometric shapes. While much attention was given above to the limitations in children's thinking, we argue that this was a useful task for uncovering and extending that thinking. For example, teacher questions prompted students to count sides and to test and compare the size of corners on various composed shapes against the square-piece, as per Figure 2 (iv). This hands-on exploration of shape properties supported identification and naming of same and the multiple examples of composed squares that were created should enrich children's concept images. In addition, the task focused on composing shapes, a pillar of geometric reasoning, and opportunities were created for describing transformations and their effects on shapes. The fine-grained learning trajectories described by Clements et al. (2004) outline how understanding of angles is used (or not) in shape composition tasks at various stages of development depending on whether the child possesses a sense of angle as a quantitative entity. Our observations align with their research in that we observed a number of children engaging in trial and error approaches to the task, while others operated with greater intentionality and anticipation- they selected and combined shapes that they predicted would fit together to make a square based on visuospatial reasoning involving mental transformations of the selected shapes.

This paper gives some insight into Irish children's thinking about a tangram task. As per the didactical tetrahedron, we recognise the crucial role of the teacher in supporting children's mathematical exploration but did not have scope to address this here. Shape composition activities have the potential to address the proposed learning outcomes of the new primary curriculum (NCCA, 2017). The analysis of task and students' responses presented here offers insight into how the reformulated learning outcomes might be achieved.

References

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. *Handbook of research on mathematics teaching and learning,* 420-464.

Clements, D. H., Wilson, D. C., & Sarama, J. (2004). Young children's composition of geometric figures: A learning trajectory. *Mathematical Thinking and Learning, 6*(2), 163-184.

Fuys, D., Geddes, D., & Tischler, R. (1984). *English translation of selected writings of Dina van HieleGeldof and Pierre M. van Hiele.* Retrieved from https://files.eric.ed.gov/

Government of Ireland. (1999). *Primary school curriculum: Mathematics*. Stationery Office

NCCA. (2017). *Primary mathematics curriculum: Draft specification junior infants to second class.* NCCA.

Nic Mhuirí, S. (2020). *Shape and space in the senior primary classes.* NCCA.

Perkins, R., & Clerkin, A. (2020). *TIMSS 2019: Ireland's results in mathematics and science.* Educational Research Centre.

Perkins, R., & Shiel, G. (2016). *PISA in classrooms: Implications for the teaching and learning of mathematics in Ireland.* Educational Research Centre

Rezat, S., & Sträßer, R. (2012). From the didactical triangle to the socio-didactical tetrahedron: Artifacts as fundamental constituents of the didactical situation. *ZDM, 44*(5), 641-651.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.

Sinclair, N., & Bruce, C.D. (2015). New opportunities in geometry education at the primary school. *ZDM, 47*, 319–329.

Sinclair, N., Bussi, M. G. B., de Villiers, M., Jones, K., Kortenkamp, U., Leung, A., & Owens, K. (2016). Recent research on geometry education: An ICME-13 survey team report. *ZDM, 48*(5), 691- 719.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics, 12*(2), 151-169.