Further Results on Dual Domination in Graphs

V.Lavanya, D. S. T. Ramesh, N.Meena

Abstract: ***Let G = (V, E) be a simple graph. A set $S \subseteq V(G)$ is a dual dominating set of G (or bi-dominating set of G) if S is a dominating set of G and every vertex in S dominates exactly two vertices in V-S. The dual-domination number $\gamma_{du}(G)$ (or bi-domination number $\gamma_{bi}(G)$) of a graph G is the minimum cardinality of the minimal dual dominating set (or dual dominating set). In this paper dual domination number and relation with other graph parameters are determined.***

Keywords: ***Domination, dual-domination, chromatic number and connectivity.***

I. INTRODUCTION

Let G(V,E) be a simple, connected graph where V(G) is its vertex set and E(G) is its edge set. The degree of any vertex v in G is the number of edges incident with v and is denoted by deg v. The minimum degree of a graph is denoted by δ(G) and the maximum degree of a graph G is denoted by $\Delta(G)$. A vertex of degree 1 is called a pendent vertex. In this paper, dual domination number with other parameters are determined. For graph theoretic notations, Harary [1]and Gray chartand [2] are referred to.

II. PRELIMINARIES

Definition 2.1:[1] The chromatic number χ(G) is defined as the minimum n for which G has an n-coloring. A graph G is n-colorable if χ (G) $\leq$ n and is n-chromatic if χ(G) = n.

Definition 2.2:[1] The connectivity $\kappa = \kappa$(G) of a graph G is the minimum number of points whose removal results in a disconnected or trivial graph.

Definition2. 3:[5] A set $S \subseteq V(G)$ is a dual dominating set of G if S is a dominating set of G and every vertex in S dominates exactly two vertices in V-S.

Remark 2.4: The dual domination number γ_{du}(G) of a graph G is the minimum cardinality of all minimal dual dominating sets. The maximum cardinality of a dual dominating set of G is called the upper dual domination number of G and it is denoted by Γ_{du}(G).

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Theorem 2.5[5]: Let *G* be a connected graph, If *G* = K_n then $\gamma_{du}(G)$ = n - 2.

III. MAIN RESULT

Theorem 3.1: For any connected graph G with n $\geq$5 vertices, γ_{du}(G) + χ(G) $\leq$ 2n − 2 and the bound is sharp if and only if G $\cong$ K_n.

Proof: Let G be a connected graph with n $\geq$5 vertices. We know that χ(G)$\leq$ n and by theorem[1.5], γ_{du}(G) $\leq$ n − 2. Hence γ_{du}(G) + χ(G) $\leq$ 2n − 2. Suppose G is isomorphic to K_n. Then clearly γ_{du}(G) + χ(G) = 2n − 2. Conversely, let γ_{du}(G)+χ(G) = 2n− 2.

Case(i): Suppose χ(G) = n – r, r $\geq$ 1. Since γ_{du}(G) + χ(G) = 2n – 2, γ_{du}(G) = n + r – 2, a contradiction.

Case(ii): Suppose γ_{du}(G) = n – r, r $\geq$ 3. Since γ_{du}(G) + χ(G) = 2n – 2, χ(G) = n + r – 2, r $\geq$ 3, a contradiction.

From both cases it is observed that γ_{du}(G) + χ(G) = 2n – 2 is possible only if γ_{du}(G) = n − 2 and χ(G) = n. Hence G is isomorphic to K_n .

Theorem 3.2: For any connected graph G with n $\geq$ 3 $\gamma_{du}(G) + \Delta(G) \leq 2n - 3$

Proof: Let G be a connected graph with n $\geq$ 5 vertices. We know that for any connected graph G, $\Delta(G) \leq n - 1$. Since $\gamma_{bi}(G)$ $\leq$ n – 2, $\gamma_{du}(G)$ + Δ(G) $\leq$ 2n – 3.

Example 3.3: Consider the following graph G is given in the following figure 1

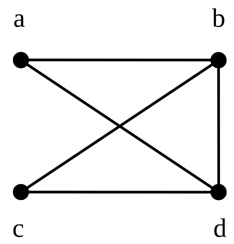

Figure1

Let S_1 = {a, c} and S_2 = {b, d}, every vertex of the set S_i , 1 $\leq$ i $\leq$ 2 dominates exactly two vertices in V - S_i. Hence S_i , 1 $\leq$ i $\leq$ 2 are the dual dominating set of G, $\gamma_{du}(G) \leq 2$. Since G is not isomorphic to either C_3 or P_3, $\gamma_{du}(G) \geq 2$. Hence $\gamma_{du}(G)$ = 2 and Δ(G) = 3, $\gamma_{du}(G)$ + Δ(G) = 5 = 2n – 3.

Theorem 3.4: Let G be a graph of order n $\geq$ 5. Then γ_{du}(G) + $\gamma_{du}(\bar{G})$ $\leq$ 2n − 6 and the bound is sharp.

Proof: Case(1): Suppose γ_{du}(G) = n – 2. Let S be a γ_{du} – set. Let V - S = {u, v}, the two vertices u and v may or may not be adjacent with both u and v in G. Let H = $\langle S \rangle$. Hence $\bar{G} = \bar{H} \cup K_2$ or $\bar{H} \cup 2K_1$. Since K_2 and K_1 do not have dual dominating set, $\bar{G}$ has no dual dominating set.

Subcase(1a): Suppose $\gamma_{du}(G) = n - 3$. Since $\gamma_{du}(\bar{G}) \neq n - 1$, $\gamma_{du}(G) + \gamma_{du}(\bar{G}) \neq 2n - 4$.
Subcase(1b): Suppose $\gamma_{du}(G) = n - 4$. Since $\gamma_{du}(\bar{G}) \neq n$, $\gamma_{du}(G) + \gamma_{du}(\bar{G}) \neq 2n - 4$.
Subcase(1c): Suppose $\gamma_{du}(G) = n - r, r \geq 5$. Since $\gamma_{du}(\bar{G}) \neq n + s, s \geq 1$, $\gamma_{du}(G) + \gamma_{du}(\bar{G}) \neq 2n - 4$.
From the cases (1), (1a) and (1b), $\gamma_{du}(G) + \gamma_{du}(\bar{G}) \neq 2n - 4$.
Case(2): Suppose either $\gamma_{du}(G)$ or $\gamma_{du}(\bar{G})$ is equal to $n - 2$. As in case(1) dual dominating set doesnot exist for G or $\bar{G}$.
Subcase(2a): Suppose $\gamma_{du}(G) = n - 4$. Since $\gamma_{du}(\bar{G}) \neq n - 1$. Hence $\gamma_{du}(G) + \gamma_{du}(\bar{G}) \neq 2n - 5$.
Subcase(2b): Suppose $\gamma_{du}(G) = n - r, r \geq 5$. Since $\gamma_{du}(\bar{G}) \neq n + s, s \geq 0$. Hence $\gamma_{du}(G) + \gamma_{du}(\bar{G}) \neq 2n - 5$.
case(3): Suppose $\gamma_{du}(G) = n - 5$ and $\gamma_{du}(\bar{G}) \neq n - 1$. Hence $\gamma_{du}(G) + \gamma_{du}(\bar{G}) \neq 2n - 6$.
Subcase(3a): Suppose $\gamma_{du}(G) = n - r, r \geq 6$ and $\gamma_{du}(\bar{G}) \neq n + s, s \geq 0$. Hence $\gamma_{du}(G) + \gamma_{du}(\bar{G}) \neq 2n - 6$.
Case(4): Let $G = C_5$ and $\bar{G}$ is also C_5 and $\gamma_{du}(C_5) = 2$. Hence $\gamma_{du}(G) + \gamma_{du}(\bar{G}) = 4 = 2n - 6$.
From all the cases $\gamma_{du}(G) + \gamma_{du}(\bar{G}) \leq 2n - 6$.
Remark 3.5: Let $|V(G)| = 4$. G has a dual dominating set if and if G is isomorphic to C_4, K_4 and $K_4 - e$. Hence $\bar{G} = 2K_2$, $4K_1$ and $K_2 \cup 2K_1$ respectively. Hence $\bar{G}$ has no dual dominating set.
Remark 3.6: Let $|V(G)| = 3$. G has a dual dominating set if and if G is isomorphic to P_3 and C_3. Hence $\bar{G} = 3K_1$ and $K_2 \cup K_1$ respectively. Hence $\bar{G}$ has no dual dominating set.
Theorem 3.7: Let G be a connected graph with $n \geq 3$ vertices, $\gamma_{du}(G) + \kappa(G) \leq 2n - 3$ and the bound is sharp if and only if G is isomorphic to K_n.
Proof: Let G be a connected graph with $n \geq 3$. We know that $\kappa(G) \leq n - 1$ and $\gamma_{du}(G) \leq n - 2$. Hence $\gamma_{du}(G) + \kappa(G) \leq 2n - 3$. Suppose G is isomorphic to K_n. Then clearly $\gamma_{du}(G) + \kappa(G) = 2n - 3$. Conversly, Let $\gamma_{du}(G) + \kappa(G) = 2n - 3$. This is possible only if $\gamma_{du}(G) = n - 2$ and $\kappa(G) = n - 1$. Hence G is isomorphic to K_n.
Theorem 3.8: Let G be a connected graph with $n \geq 4$ vertices. Let S be a minimum dual dominating set of G . If $\kappa(G) = n - 2$ or $n - 1$, $\Delta(G) = n-1$, $\chi(G) = n-1$ or n, and $diam(G) = 2$ or 1 iff $|S| = n - 2$ and $< S >$ is complete graph.
Proof: Let G be a connected graph with $n \geq 4$ vertices.
$S = \{ v_1, v_2, \ldots, v_{n-2}\}$ is the dual dominating set of G and $< S >$ is complete graph.
Case(i): Suppose the vertices v_{n-1} and v_n belong to $V - S$ is adjacent with each other. Then clearly $\kappa(G) = n - 1$, $\Delta(G) = n-1$, $\chi(G) = n$, and $diam(G) = 1$.
Case(ii): Suppose the vertices v_{n-1} and v_n belong to $V - S$ not adjacent with each other. Then clearly $\kappa(G) = n - 2$, $\Delta(G) = n-1$, $\chi(G) = n - 1$, and $diam(G) = 2$.
Conversely,
Case(i): Suppose $\kappa(G) = n - 1$ then G is isomorphic to K_n. Clearly $\Delta(G) = n-1$, $\chi(G) = n$, and $diam(G) = 1$.
Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. $S = \{v_1, v_2, \ldots, v_{n-2}\}$ is the minimum dual dominating set of G. $|S| = n - 2$ and $< S >$ is complete graph.
Case(ii): Suppose $\chi(G) = n - 1$ then G is isomorphic to $K_n - e$. Clearly $\Delta(G) = n-1$, $\kappa(G) = n - 2$, and $diam(G) = 2$.
Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. $S = \{v_1, v_2, \ldots, v_{n-2}\}$ is the minimum dual dominating set of G. The vertices v_{n-1} and v_n not adjacent with each other. $|S| = n - 2$ and $< S >$ is complete graph.

IV. CONCLUSION

In this paper, dual domination number with chromatic number, connectivity and Nordhaus-Gaddum type result are discussed .

REFERENCES

1. Harary.F, Graph theory, Addison-Wesley Publishing Company Reading, Mass (1972).
2. Gary Chartand, Ping Zhang, Introduction to Graph theory, McGraw Hill Education (India) Private Limited , New Delhi (2006)
3. Teresa W.Haynes, Routlege , Domination in graphs: Volume 2: Advanced Topics,(2017)
4. V.Lavanya[1] , D. S. T. Ramesh[2] and N. Meena[3], Bi-domination in Graphs, International Journal of Emerging Technologies and Innovative Research(2019), 424-427
5. V.Lavanya[1] , D. S. T. Ramesh[2] and N. Meena[3], Bi-domination in Corona Related Graphs, Journal of Computer and Mathematical Sciences, Vol.10(11),1650-1653 November 2019(An International Research Journal), www.compmath-journal.org ISSN 0976-5727 (Print) ISSN 2319-8133 (Online).

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